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# GATES' BIDDING MODEL

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## ABSTRACT

In evaluating closed-bid competitive procurement auctions, the most crucial issue is to determine the probability of placing a winning bid for a given mark-up level. There has long been disagreement on how this should be done due to the absence of a mathematical derivation of one of the main evaluation techniques – Gates' method. Gates' method is shown in this paper to be valid if, and only if, bids can be described using the proportional hazards family of statistical distributions. When mark-up values are included in Gates' method, it is seen that the underlying statistical distribution required for the method to work is closely related to the Weibull distribution. Likelihood based methods are suggested for parameter estimation and an illustrative example is provided by analysis of Shaffer and Micheau's (1971) construction contract bidding data.

**Keywords:** Bidding models, Gates, proportional hazards.

## INTRODUCTION

As Crowley (2000) has observed, "there has been a lingering, unresolved debate concerning two contrasting bid models introduced in articles by Friedman (1956) and Gates (1967) …

each model provides an approach to evaluating closed-bid competitive [procurement auctions] amongst known competitors … the distinction between the models being in the determination of the probability of placing a winning bid". The main reason for the disagreement has been the lack of a convincing proof of the Gates' formula, described by Gates as being based on a "balls in the urn" model. Immediately upon publication of the Gates paper, Stark (1968) questioned the validity of his model while Benjamin (1969), recognising that Gates provides no mathematical proof, attempted to rectify the situation but was unable to do so. Rosenshine's (1972) later attempt met with equal lack of success as did Gates himself (Gates 1976), leading Englebrecht-Wiggans (1980) to reflect that Gates' model appears to be an empirical fit formula that is still without "mathematical justification".

Most of the bidding literature is concerned with setting a mark-up, *m*, so that the probability, *Pr(m)*, of entering the winning bid reaches some desired level. Several models have been proposed for calculating *Pr(m)*. Of these, Gates' (1967) model appears unique in enabling *Pr(m)* to be calculated directly, without the need to specify any underlying probability density functions (pdfs). However, it is argued below that in applying Gates' method to determine *Pr(m)* one is essentially assuming a Weibull distribution for the bids.

This paper is organized as follows: First, Gates' formulation is shown to be correct if, and only if, the distributions involved are from the proportional hazards family. Furthermore, for this to hold with the application of a mark-up, the probability density function must be specifically Weibull. Maximum and Rank Likelihood equations are then presented. Finally, an illustrative example is provided using Shaffer and Micheau's (1971) data.

**GATES' MODEL**

Let $X_1, X_2, \ldots, X_k$ be independently distributed random variables. If we generate one value, i.e., $x_1, x_2, \ldots, x_k$ from each variable, the probability of $x_i$ being the lowest is, according to Gates (1967)

$$P_{i.} = \frac{1}{1 + \sum_{\substack{j=1 \\ j \neq i}}^{k} \frac{1 - P_{ij}}{P_{ij}}} \quad \textbf{(1)}$$

where $P_{ij}$ ($0 < P_{ij} < 1$) is the probability that $x_i < x_j$. To simplify the notation, we work with $O_{ij}$ ($0 < O_{ij} < \infty$), where $O_{ji} = P_{ji} / P_{ij} = (1 - P_{ij}) / P_{ij}$, that is, the odds on $x_j < x_i$. Now, as $O_{ii} = 1$ by definition and the probabilities must add to unity,

$$\sum_{i=1}^{k} \frac{1}{\sum_{j=1}^{k} O_{ji}} = 1$$

Selecting *any* three from the *k* variables also gives

$$\frac{1}{1 + O_{ji} + O_{li}} + \frac{1}{1 + O_{ij} + O_{lj}} + \frac{1}{1 + O_{il} + O_{jl}} = 1$$

which leads to the useful result that

$$O_{lj} = \frac{O_{li}}{O_{ji}} \quad \textbf{(2)}$$

so that all the $O_{lj}$ can be defined in terms of $O_{q1}(q = 2, \ldots, k)$. Now, for any two from *k* variables, from **(1)**

$$P_{1.} = \frac{1}{1 + O_{21} + O_{31} + \ldots + O_{k1}} = \frac{1}{\sum_{i=1}^{k} O_{i1}}$$

and

$$P_{2.} = \frac{1}{1 + O_{12} + O_{32} + \ldots + O_{k2}}$$

which becomes, from **(2)**

$$P_{2.} = \frac{1}{O_{11}/O_{21} + O_{21}/O_{21} + O_{31}/O_{21} + \ldots + O_{k1}/O_{21}} = \frac{1}{O_{12}\sum_{i=1}^{k} O_{i1}} = \frac{O_{21}}{\sum_{i=1}^{k} O_{i1}}$$

and hence

$$\frac{P_{2.}}{P_{1.}} = O_{21} \tag{3}$$

The probability $P_{i.}$ may be expressed in terms of pdfs (see for example, Skitmore and Pemberton, 1994),

$$\begin{aligned} P_{i.} &= \int_0^\infty f_i(x) \prod_{\substack{j=1 \\ j \neq i}}^{k} S_j(x) dx \\ &= \int_0^\infty h_i(x) \prod_{j=1}^{k} S_j(x) dx \end{aligned} \tag{4}$$

with survival functions $S_j(x) = \int_x^\infty f_j(u) du$, pdfs $f_j(x)$ and hazard functions $h_j(x) = f_j(x)/S_j(x)$, *j=1,...,k.* Now a collection of distributions are said to form a *proportional hazards (PH) family* of distributions if

$$h_j(x) = c_j h_0(x), \tag{5}$$

for all *j=1,...k* with some baseline hazard function $h_0(x)$ and constants $c_j > 0$. If one assumes a PH family of distributions for construction bids then from **(4)** and **(5)** we see that **(3)** is satisfied with $O_{ij} = c_i / c_j$. Therefore, Gates' method correctly determines the probability of placing the lowest bid if the collection of bids in a given auction forms a proportional hazards family. In the appendix it will be shown that in a mildly restricted setting, Gates' method correctly determines the probability of placing the lowest bid *only* for PH families.

The above arguments show that Gates' method is justified for PH families. One common application of Gates' method is to the calculation of a mark-up multiplier *m* so that instead of placing a bid $X_1$ the bid $mX_1$ is placed. The hazard function of the bid $mX_1$ is $m \cdot h_1(x/m)$. As we are using Gates' method to determine the probability of lowest bid, the bids $mX_1, X_2, \ldots, X_k$ must also form a PH family for all possible mark-up values. This would require

$$h_1(x/m) = c_m h_1(x) \quad \textbf{(6)}$$

for all possible mark-up *m*. Letting $x = 1$, $c_m$ can be expressed in terms of $h_1(\cdot)$ so that

$$h_1(x/m)h_1(1) = h_1(1/m)h_1(x). \quad \textbf{(7)}$$

Finally, setting $s(x) = h_1(x)/h_1(1)$ we have

$$s(x/m) = s(1/m)s(x), \quad \textbf{(8)}$$

which is one of Cauchy's functional equations. Assuming that $h_1(\cdot)$ is continuous at some point greater than zero, the unique solution to **(8)** is known to be $s(x) = x^a$ for some *a* (see proposition 56 of Klambauer (1975) for details). This corresponds to a Weibull distribution. For a Weibull distribution with parameter $\omega > 0$ the hazard function is $h(x) = h(1)x^{\omega-1}$. Therefore noting the form of hazard for bids $mX_1$ and equations **(5)** and **(1)** the probability of bidder 1 placing the lowest bid with mark-up *m* is

$$P_{1.} = \frac{1}{1 + m^{\omega}(O_{21} + \cdots + O_{k1})}. \quad \textbf{(9)}$$

## LIKELIHOOD METHODS OF PARAMETER ESTIMATION

### Two parameter Weibull

The pdf. of the 2 parameter Weibull distribution is

$$f(x) = \lambda\omega(\lambda x)^{\omega-1}\exp\left[-(\lambda x)^{\omega}\right] \quad \textbf{(10)}$$

with

$$S(x) = \exp\left[-(\lambda x)^{\omega}\right] \quad \textbf{(11)}$$

where $\omega > 0$ and $\lambda > 0$ are shape and scale parameters. This includes the exponential distribution $(\omega = 1)$ as a special case. With this parameterization of the Weibull distribution the odds are given by $O_{ij} = (\lambda_i / \lambda_j)^{\omega}$.

To perform maximum likelihood estimation we prefer to work with the log transformation of the Weibull distribution which is the extreme value distribution, with pdf

$$f(y;\theta) = \exp\left(\tfrac{y-\theta}{\sigma} - \exp\left(\tfrac{y-\theta}{\sigma}\right)\right) \qquad -\infty < y < \infty$$

with $$S(y) = \exp\left(-e^{(y-\alpha)/\sigma}\right) \qquad -\infty < y < \infty$$

where $\theta(-\infty < \theta < \infty)$ and $\sigma(\sigma > 0)$ are parameters, with $\theta = \log\lambda^{-1}$ and $\sigma = \omega^{-1}$.

If, for a series of construction contract auctions, $x_{jl}$ is the value of a bid entered by bidder $j$ ($j$=1,…,$r$) for contract $l$ ($l$=1,…,$c$) then, for $y_{jl} = \log(x_{jl})$, we let $\theta_{jl} = \alpha_j + \beta_l$ where $\alpha_j$ denotes the scale parameter for each bidder and, following Skitmore (1991), $\beta_l$ denotes a contract datum (nuisance) parameter. The parameter $\sigma$ is assumed common for all bidders and contracts. Therefore, $y_{jl}$ has the pdf.

$$\frac{1}{\sigma}\exp\left(z_{jl} - e^{z_{jl}}\right) \qquad -\infty < y_{jl} < \infty \tag{12}$$

where $z_{jl} = \left(y_{jl} - \alpha_j - \beta_l\right)/\sigma$. So the log-likelihood is:

$$\log L = -N\log\sigma + \sum_j^r\sum_l^c \delta_{jl} z_{jl} - \sum_j^r\sum_l^c \delta_{jl} z_{jl}$$

where $\delta_{jl} = 1$ if bidder *j* bids for contract *l*

$= 0$ if bidder *j* does not bid for contract *l*

$N = \sum_j^r\sum_l^c \delta_{jl}$ = total number of bids

The maximum likelihood equations over $\alpha$'s, $\beta$'s and $\sigma$ are:

$$\frac{\partial LogL}{\partial\alpha_j} = -\frac{n_j}{\sigma} + \frac{1}{\sigma}\sum_l^c \delta_{lj} e^{z_{jl}}$$

$$\Rightarrow e^{\hat{\alpha}_j} = \left[\frac{1}{n_j}\sum \delta_{lj}\exp\left(\frac{y_{lj} - \beta_l}{\hat{\sigma}}\right)\right]^{\hat{\sigma}} \tag{13}$$

$$\frac{\partial LogL}{\partial\beta_l} = -\frac{n_l}{\sigma} + \frac{1}{\sigma}\sum_j^r \delta_{lj} e^{z_{jl}}$$

$$\Rightarrow e^{\hat{\beta}_l} = \left[\frac{1}{n_l}\sum \delta_{lj}\exp\left(\frac{y_{lj} - \alpha_j}{\sigma}\right)\right]^{\sigma} \tag{14}$$

$$\frac{\partial LogL}{\partial\sigma} = -N - \sum\sum \delta_{lj} z_{lj} + \sum\sum \delta_{lj} z_{lj} e^{z_{jl}}$$

$$\propto \sum\sum \delta_{lj}\left\{z_{lj}\left[e^{z_{jl}} - 1\right] - 1\right\} = 0 \tag{15}$$

where $n_j = \sum_l^c \delta_{jl}$, the number of bids by bidder *j*, and $n_l = \sum_j^r \delta_{jl}$, the number of bids for contract *l*. By setting $\sigma = 1$ and solving the $\alpha$'s and $\beta$'s by iteration of **(13)** and **(14)** provides the required maximum likelihood estimates for the exponential distribution . For the Weibull pdf., the parameter estimates can be obtained by solving the $\alpha$'s and $\beta$'s by iteration of **(13)** and **(14)** for trial $\sigma$ values - finding the best $\sigma$ using the Newton-Raphson method for **(15)**. On completion, the maximum likelihood estimates of the original parameters are then $\omega = \sigma^{-1}$ and $O_{ij} = \exp(\omega(\alpha_j - \alpha_i))$.

**Rank likelihood**

Consider the random variable $Y_j$ having the pdf $f(y;\theta_j) = \exp(y - \theta_j - \exp(y - \theta_j))$. Let $X_j$ be the random variable such that $G(X_j) = Y_j$ for some monotone increasing transformation $G$. The hazard function of $X_j$ is

$$G'(x)\exp(G(x)) \cdot \exp(-\theta_j) \qquad \textbf{(16)}$$

It is known that for any given hazard function a function $G$ can be determined so that all PH families may be obtained from transformations of families of extreme value distributions. In particular, note that for $G(x) = \omega \log(x)$ one obtains the family of Weibull distribution with shape parameter $\omega$. Now suppose that the transformation $G$ is completely unknown. In this case all information in the data is contained in the ranks, i.e. precise values of bids do not contain information on the parameters of interest, only their ordering contain information. The probability of a particular ordering of the bids is given by (Pettitt, 1986)

$$\Pr(X_1 < \ldots < X_k) = \exp\left(-\sum_{j=1}^{k}\theta_j\right)\left\{\prod_{j=1}^{k}\left[\sum_{i=j}^{k}\exp(-\theta_j)\right]\right\}^{-1} \qquad \textbf{(17)}$$

The rank likelihood for the parameter vector $\theta$ can then be formed

$$L(\theta; X) = \prod_{l=1}^{m}\exp\left(-\sum_{j=1}^{k}\theta_j\delta_{jl}\right)\left\{\prod_{j=1}^{k}\sum_{i=j}^{k}\exp\left(-\delta_{r(l,j)l}\theta_{r(l,j)}\right)\right\}^{-1} \qquad \textbf{(18)}$$

where *r*(*l*,*j*) is the bidder label of the *j*-th ranked bid on contract *l* and $\delta_{jl}$ is the indicator variable taking the value one if the *j* bidder bids on contract *l* and is zero otherwise. Note that from the estimate of $\theta$ Gates formula can be used with $O_{ij} = \exp(\theta_j - \theta_i)$.

The rank likelihood function can be maximized using a standard numerical method such as the Nelder-Mead simplex algorithm. Alternatively standard statistical software can be used to fit a proportional hazards model stratifying the data according to contract so that for each contract a separate baseline hazard function is fitted. It is important to note that by fitting a separate baseline hazard function for each contract there is no need to account for the effect of the contract size on the bid distribution.

**WORKED EXAMPLE**

For a worked example, we refer to the data published in a previous paper by Shaffer and Micheau (1971). This comprises all identified bids for a series of 50 sealed bid building contract auctions over the period 1965 to 1969. For the purposes of the example, it is assumed that the bids for first 49 contracts are known but only the identity of the bidders are known for the 50$^{th}$ contract – Ours, Dick, Leon, Les, Lyle and Rob. The task, therefore, is to estimate the theoretical probabilities associated with each of these being the lowest bidder for the 50$^{th}$ contract. From the above, there are 3 main contending models, comprising (1) the exponential, (2) the Weibull and (3) the general PH family.

As Gates has pointed out, a reasonable number of bids for previous auctions are required from each bidder for the analysis to be carried out. Here a reasonable number is arbitrarily chosen as five – bidders entering less than five bids being assigned to a single pooled group. For the Shaffer and Micheau data, only eight bidders have entered five or more bids – 'Ours' (48 bids), 'Abel' (6 bids), 'Adam' (6 bids), 'Alan' (8 bids), 'Alex' (5 bids), 'Ben' (6 bids), 'Dave' (5 bids) and 'Dick' (5 bids).

**Gates' empirical method**

For comparative purposes, Gates' empirical method **(1)** was used by counting the number of winning bids made by a contractor and dividing by the number of attempts. This approach immediately encountered problems with bidder 'Dick', who had been the lowest bidder on all previous attempts, thus making **(1)** undefined with at least one $P_{ji} = 0$. For this method, therefore, the 50th auction was included. Solving **(1)** for each bidder then gives a probability of 0.132605, 0.631579, and 0.056902 for Ours, Dick and the other bidders respectively.

**Exponential model**

For the exponential model, $\sigma = 1$ in **(13)** and **(14)**. Taking the log values of all the bids and using starting values of all $\alpha_j = 0$ (j=Pooled, Ours, Abel, Adam, Alan, Alex, Ben, Dave, Dick), the $\beta$ value is calculated for the first auction from **(14)** as

$$e^{\beta_1} = \frac{1}{8}(2936000 + 3155000 + 3192660 + 3197000 + 3205828 + 3224800 + 3259413 + 3456000)$$
$$= 3203337.6$$

$$\therefore \beta_1 = 14.979704$$

Repeating this for all $l$=2, …, 49 auctions produces the results shown in the second column of Table 1. These $\beta$ values are now inserted into **(13)** to produce a set of revised $\alpha_j$ as shown in iteration 2 in Table 2. These two operations are now repeated until the results of the nth iteration are close to the results for the n-1th iteration. As the Tables indicate, this occurs at the start of the 12th iteration. The $\lambda$ values are calculated from the final $\alpha$ values, using the formula $\lambda_j = \exp(-\alpha_j)$. The probability of each bidder entering the lowest bid for the 50th contract can then be calculated from **(1)** using $O_{ij} = \exp(\alpha_j - \alpha_i)$, i.e.,

$$P = \frac{1}{1 + 1.012383 + 1.12310 + 1 + 1 + 1} = 0.162981$$

for each of the pooled bidders – Leon, Les, Lyle and Rob, with

$$P = \frac{1}{1 + \frac{1.123310}{1.012383} + \frac{1}{1.012383} + \frac{1}{1.012383} + \frac{1}{1.012383} + \frac{1}{1.012383}} = 0.164999$$

for 'Ours' and

$$P = \frac{1}{1 + \frac{1.012383}{1.123310} + \frac{1}{1.123310} + \frac{1}{1.123310} + \frac{1}{1.123310} + \frac{1}{1.123310}} = 0.183078$$

for Dick.

**Weibull model**

The Weibull model, leaves $\sigma$ to be estimated by **(15)**. Using starting values of all $\alpha_j = 0$ and $\sigma = 1$ the contract $\beta$ values are calculated as before. These are then inserted into **(15)** and the RHS calculated as 227.578864. $\sigma$ is now set to slightly below unity, the $\alpha$ and $\beta$ recalculated, inserted into **(15)** and the RHS recalculated. The improvement is then used for the next trial $\sigma$ value of 0.248537, which produces a **(15)** RHS of 206.460859. This procedure is continued until the RHS becomes zero at which point $\sigma$ is 0.067572, i.e., $\omega$ 14.799046 (Table 3).

Again, the probability of each bidder entering the lowest bid for the 50th contract can then be calculated from **(1)** using $O_{ij} = \exp(\omega(\alpha_j - \alpha_i))$.This gives a probability of 0.098473, 0.621487, and 0.070010 for Ours, Dick and the other bidders respectively.

**Rank likelihood method**

The rank likelihood approach is a statistically more efficient method of obtaining the estimates without introducing additional assumptions on the distribution of bids. The parameters were estimated using the coxph function from the Survival package in the open source statistical software R which maximizes the same likelihood function. The rank likelihood approach still has difficulty with the bidder 'Dick' being the lowest bidder in all auctions except the 50th. Optimizing the rank likelihood and solving **(1)** the bidder probability are estimated as 0.097031, 0.641756 and 0.065303 for Ours, Dick and the other bidders respectively.

**Probabilities with mark ups**

To be realistic in observing the effects of applying a mark up, it is first necessary to reestimate the parameters of the models using the reference bidder's cost estimates instead of bids in the analysis. Table 4 summarizes the results of this. **(9)** is then applied for a series of mark-up values using the Weibull and exponential models. Fig 1 provides the results.

## DISCUSSION

The Weibull distribution was invented by Waloddi Weibull – a Swedish engineer/scientist – in 1937 for use as an alternative to the normal distribution in the analysis of life data and is particularly adept at modelling the distribution of failures and failure times. From its early days, however, it was apparent to its originator that the Weibull distribution "… may sometimes render good service" to a wide range of problems and this has proved to be the case today where, in addition to being the leading method in the world of fitting and analysing life data, it is used in medical and dental implants, warrant analysis, life cycle cost, materials properties and production process control, etc (Abernathy 2000).

The Weibull is also known as the extreme value type III distribution and has been proved to be the best model for the time of the first failure of a part with multiple failure modes – sometimes known as the 'weakest-link-in-the-chain' concept (Gumbel 1958). It approximates the normal distribution and the Raleigh. The lognormal distribution, however, is not a member of the Weibull family and is by far the most significant competitor – the lognormal being the best choice for some material characteristics, for crack growth rate and for non-linear accelerating system deterioration.

There are many instances of the Weibull being used to model auction bids, including those for timber sales (Athey *et al* 2004; Haile 2001; Paarsch 1992, 1997; Donald *et al* 1997), road construction contracts (Marion 2004a, 2004b; Jofre-Bonet and Pesendorfer 2003), oil/oil tracts sales (Sareen 1999; Smiley 1980), on-line auctions (Bapna *et al* 2006), milk delivery contracts (Marshall et al 2001), road marking contracts (Elköf and Lunander 1998), stock market shares (Wilson 1979) and procurement auctions in general (e.g., Lunander 2002; Rothkopf 1969, 1980a, 1980b; Rothkopf *et al* 2003). In some cases, the use of Weibull is justified solely on the grounds of its flexibility (Marshall *et al* 2001; Keefer *et al* 1991; Haile 2001) and computational convenience (Keefer *et al* 1991). In several others, the choice occurs as a result of empirical analysis (Athey *et al* 2004; Bapna *et al* 2006; Elköf and Lunander 1998; Jofre-Bonet and Pesendorfer 2003) sometimes in explicit preference to other distributions such as the lognormal and Gumbel (Smiley 1980).

To date, there has been little interest in accounting for these empirical results. An exception is Bapna *et al* (2006), who reason that the 'weakest-link' principle applies in their on-line consumer surplus study. ".. we have a set of bids in an auction. Among the differences between each of the bids and the winning price, the surplus is the only positive value, and this is the minimum of these differences. In this sense, the winning/highest bid is the 'weakest link' because it has the smallest distance from the price compared to all the other bids that were placed in auction" adding that "the surplus divided by price also follows a Weibull distribution" (p.25).

In fact, little has been said to account for any of the models using in bidding in terms of statistical bid generating mechanisms. A *normal* distribution, for example, is likely to be appropriate if the difference between firms' total cost estimates is a summation of a large fixed number of individually small item estimation differences. Similarly, two possibilities are immediately obvious for the Weibull:

1. The minimum of any number of independent draws from a Weibull distribution itself has a Weibull distribution, and the minimum of draws from any distribution bounded below approaches a Weibull distribution as the number of draws becomes large (Rothkopf and Harstad 1994, citing Gumbel 1958). Therefore, in the (usual) situation where the bid is based on *m* subcontracts, for which each have been priced in the main bid at the value of the lowest of $n_i \left(i = 1,2,\ldots,m\right)$ quotes then, if $n_i$ is large, the resulting price for subcontract *i* will be approximately distributed as a Weibull random variable. When one of these subcontracts (e.g., the mechanical and electrical installation) accounts for the majority of the value of the contract then the resulting main bid may still be well approximated by a Weibull distribution.
2. If the number of items, *N*, contained in the cost estimate is a random variable with a geometric distribution having large mean *m*, and if each of the item estimates have a similar distribution with any mean then, from the geometric stability of the exponential distribution, the resulting bid formed by $m^{-1}\sum_{j=1}^{N} X_j$ approximates to the exponential distribution (see Gnedenko, 1970).

There are possibly other mechanisms which would result in the bid for a contract to be, at least approximately, distributed according to a Weibull distribution. Here we have simply suggested two such mechanisms.

## CONCLUSIONS

This paper provides the mathematical basis of the Gates bidding model, showing it to be uniquely associated with the proportional hazards family of pdfs. When the introduction of mark-up values is allowed, only the Weibull distribution holds. Likelihood based methods are proposed for parameter estimation and an illustrative example is provided by analysis of Shaffer and Micheau's (1971) construction contract bidding data. It should be noted, however, that the estimation of the maximum likelihood parameters is rather beyond those of hand calculation. For this example, a computer program was written in Fortran and one of the Nag library subroutines used (c05nbf) for the Newton-Raphson iterations. Though seemingly demanding computationally, the actual run time on a standard PC to produce the whole of Fig 1 was around 20 seconds in total. No special software is needed for the rank likelihood method, as many standard statistical packages, such as R, are available to perform the necessary calculations.

**APPENDIX**

In this appendix we shall show that, under certain mildly restrictive conditions, Gates' method is correct *if, and only if,* the contract bids are drawn from the PH family of distributions. The following simplifying assumptions are introduced:

A1. There is an infinite collection of bid distributions such that for any subset of $k$ distributions, relationship given by equation **(3)** holds.

A2. The hazard function of each bid distribution is piecewise constant. For a given bid distribution the minimal distance between discontinuity points of a hazard function is bounded away from zero.

A3. The bid distributions have finite mean.

A4. For every $x > 0$, $\lim_{k\to\infty}\sum_{j=1}^{k} h_j(x) = \infty$.

Assumption A1 is essentially that Gates' method correctly determines the probability of a bidder placing the lowest bid. That we are assuming an infinite collection of possible bidders may be considered as an approximation to reality were there are a large number of possible bidders although only a small number will bid on any given contract. Assumption A2 restricts attention to piecewise constant hazard function, but it should be noted that any distribution can be closely approximated by a distribution with piecewise constant hazard function. Assumption A3 is made to simplify the proof and could be removed. The final assumption A4 does not constrain the behaviour of any individual hazard function but is made to exclude certain degenerate behaviour in the collection of distributions.

*Theorem: Under assumptions* (A1) – (A4)*, Gates' method implies the collection of bid distributions forms a PH family.*

*Proof:* To prove the theorem first we re-write equation **(3)** as

$$\int_0^\infty [f_2(x)S_1(x) - O_{21}f_1(x)S_2(x)]\left\{\prod_{j=3}^{k} S_j(x)\right\}dx = 0. \qquad \textbf{(19)}$$

From A3 it follows that $\int_0^\infty |f_2(x)S_1(x) - O_{21}f_1(x)S_2(x)|dx < \infty$. Let $S_k^*(x) = \prod_{j=3}^{k} S_j(x)$. Noting that $S_j(x) = \exp\left[-\int_0^x h_j(u)du\right]$ it follows from assumption A4 that

$$\lim_{k\to\infty} \frac{S_k^*(x)}{S_k^*(\varepsilon)} = 0 \qquad \textbf{(20)}$$

for any $x > \varepsilon$. As the survival functions are decreasing it follows that

$$\int_0^\varepsilon S_k^*(x)dx \geq \varepsilon S_k^*(\varepsilon). \qquad \textbf{(21)}$$

From equations **(20)**, **(21)** and applying the Lebesgue dominated convergence theorem it follows that for any $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{\int_\varepsilon^\infty [f_2(x)S_1(x) - O_{21}f_1(x)S_2(x)]S_k^*(x)dx}{\int_0^\infty S_k^*(x)dx} = 0. \qquad \textbf{(22)}$$

From the mean-value theorem

$$\frac{\int_0^\varepsilon [f_2(x)S_1(x) - O_{21}f_1(x)S_2(x)]S_k^*(x)dx}{\int_0^\infty S_k^*(x)dx} = \left[f_2(x_k^*)S_1(x_k^*) - O_{21}f_1(x_k^*)S_2(x_k^*)\right]\frac{\int_0^\varepsilon S_k^*(x)dx}{\int_0^\infty S_k^*(x)dx} \qquad \textbf{(23)}$$

where $x_k^* \in [0, \varepsilon]$. Taking limits as *k* goes to infinity we have

$$\lim_{k\to\infty}\frac{\int_0^{\varepsilon}[f_2(x)S_1(x)-O_{21}f_1(x)S_2(x)]S_k^*(x)dx}{\int_0^{\infty}S_k^*(x)dx}=\lim_{k\to\infty}\left[f_2(x_k^*)S_1(x_k^*)-O_{21}f_1(x_k^*)S_2(x_k^*)\right] \quad \textbf{(24)}$$

As $\varepsilon$ is arbitrary and $f_2(x)S_1(x)-O_{21}f_1(x)S_2(x)$ is continuous in a neighbourhood of zero it follows that

$$\lim_{k\to\infty}\frac{\int_0^{\varepsilon}[f_2(x)S_1(x)-O_{21}f_1(x)S_2(x)]S_k^*(x)dx}{\int_0^{\infty}S_k^*(x)dx}=f_2(0)S_1(0)-O_{21}f_1(0)S_2(0). \quad \textbf{(25)}$$

Combining equations **(19)** and **(25)** it is seen that $f_2(0)S_1(0)-O_{21}f_1(0)S_2(0)=0$. As the hazard functions are piecewise constant $f_2(x)S_1(x)-O_{21}f_1(x)S_2(x)=0$ for all $x\in[0,\delta_1]$, where $\delta_1$ is the first point of discontinuity of the hazard functions $h_1$ and $h_2$. Equation **(19)** may now be written as

$$\int_{\delta_1}^{\infty}[f_2(x)S_1(x)-O_{21}f_1(x)S_2(x)]\left\{\prod_{j=3}^{k}S_j(x)\right\}dx=0. \quad \textbf{(26)}$$

The above argument is repeated to show that $f_2(x)S_1(x)-O_{21}f_1(x)S_2(x)=0$ for all $x\in[0,\delta_2]$, where $\delta_2$ is the second point of discontinuity of the hazard functions $h_1$ and $h_2$. The argument continues for all subsequent points of discontinuity and so $f_2(x)S_1(x)-O_{21}f_1(x)S_2(x)=0$ for all *x*. This last equation can only be true for all *x* if the hazard functions are proportional. This completes the proof.

**REFERENCES**

Abernathy, R.B. (2000) *The new Weibull handbook.* 4th ed. Published by the author. ISBN 0-9653062-1-6.

Athey, S., Levin, J., Seira, E. (2004) Comparing open and sealed bid auctions: theory and evidence from timber auctions. Mimeo, Stanford University.

Bapna, R., Jank, W., Shmueli, G. (2006) Consumer surplus in on-line auctions. Mimeo, University of Minesota.

Benjamin, N.B.H. (1969). "Competitive bidding for building construction contracts." *Technical report 106*, Department of Civil Engineering, Stanford University, California.

Crowley, L.G. (2000). "Friedman and Gates – another look." *Journal of Construction Engineering and Management, Proceedings of the American Society of Civil Engineers* **126**(4) 306-12.

Donald, S.G., Paarsch, H.J., Robert, J. (1997) Identification, estimation and testing in empirical models of sequential, ascending price auctions with multi-unit demand: an application to Siberian timber export permits. Working Paper, University of Iowa.

Elköf, M., Lunander, A. (1998) Contract bidding under a first-price sealed bid auction – an empirical analysis of competition in procurement of road markings in Sweden. Mimeo, Department of Economics, Uppsala University, Sweden.

Englebrecht-Wiggans, R. (1980). "Auctions and bidding models: a survey." *Journal of Management Science* **26**(2) 119-42.

Friedman, L. (1956). "A competitive bidding strategy." *Operations Research* **1**(4): 104-12.

Gates, M. (1967). "Bidding strategies and probabilities." *Journal of the Construction Division, Proceedings of the American Society of Civil Engineers*, **93** (CO1), 75-107.

Gates, M. (1976). "Discussion of bidding models: 'Truths and comments'" by Michael Fuerst. *Journal of the Construction Division*, *Proceedings of the American Society of Civil Engineers*, **102**(4) 696-9.

Gnedenko, B.V. (1970) Limit theorems for sums of random number of positive independent random variables. *Proc. 6th Berkeley Symp. On Math. Statist. Probab.*, Vol 2, 537 – 549.

Gumbel, E.J. (1958) *Statistics of extremes*. Columbia University Press: New York.

Haile, P.A. (2001) Auctions with resale markets: an application of USA forest timber sales. *The American Economic Review* **91**(3) 399-427.

Jofre-Bonet, M., Pesendorfer, M. (2003) Estimation of a dynamic auction game. *Econometrica* **71**(5) 1443-89.

Keefer, D.L., Beckley Smith Jr, F., Back, H.B. (1991) Development and use of a modeling system to aid a major oil company in allocating bid capital. *Operations Research* **39**(1) 28-41.

Klambauer, G. (1975). *Mathematical Analysis.* Marcel Dekker: New York.

Lundander, A. (2002) Procurement bidding in first-price and second-price sealed bid auctions within the common value paradigm. *Computational Economics* **19**(2) 227-244.

Marion, J. (2004a) Redistribution through public purchases: competition, procurement cost and efficiency of a bid program. Mimeo, University of Chicago

Marion, J. (2004b) Are bid preferences benign? The effect of small business subsidies in highway procurement auctions. Mimeo, Department of Economics, University of California, 15 Nov.

Marshall, R.C., Rai, M.E., Richard, J-F., Schulenberg, S.P. (2001) The impact of cost synergies on bidding in the Georgia school milk market. Mimeo, Penn State University.

Paarsch, H.J. (1992) Empirical methods of auctions and an application to British Columbian timber sales. University of Western Ontario Research Report No 9212, Oct.

Paarsch, H.J. (1997) Deriving an estimate of the optimal reserve price: an application to British Columbian timber sales. *Journal of Econometrics* **78**(2) 333-57.

Pettitt, A. N. (1986). “Rank likelihood.” In *Encyclopedia of Statistical Sciences, volume 7*, *Kotz, S. and Johnson, N. L. (eds.)*, John Wiley and Sons.

Rosenshine, M. (1972). “Bidding models: resolution of a controversy.” *Journal of the Construction Division*, *Proceedings of the American Society of Civil Engineers*, **98**(1) 143-8.

Rothkopf, M.H. (1969) A Model of Rational Competitive Bidding. *Management Science* **15** 362-73.

Rothkopf, M.H. (1980a) On Multiplicative Bidding Strategies," *Operations Research* **28** 570-75.

Rothkopf, M.H. (1980b) Equilibrium Linear Bidding Strategies," *Operations Research* **28** 576-83.

Rothkopf, M.H., Harstad, R.M. (1994) Modeling competitive bidding: a critical essay. *Management Science* **40**(3) 364-84.

Rothkopf, M.H., Harstad, R.M., Yuhong Fu (2003) Is Subsidizing Inefficient Bidders Actually Costly? *Management Science* **49** 71-84.

Sareen, S. (1999) Posterior odds comparison of symmetric low-price, sealed-bid auctions within the common-value and the independent-private-values paradigm. *Journal of Applied Econometrics* **14** 651-76.

Shaffer, L.R., Micheau, T.W. (1971). "Bidding with competitive strategy models." *Journal of the Construction Division, Proceedings of the American Society of Civil Engineers* **97**(CO1) Jul 113-26.

Skitmore, R.M. (1991). "The contract bidder homogeneity assumption: an empirical analysis." *Construction Management and Economics* **9**(5) 403-29.

Skitmore, R.M., Pemberton, J. (1994). "A multivariate approach to contract bidding mark-up strategies." *J Opl Res Soc*, **45**(12) 1263-72.

Smiley, A.K. (1980) *Competitive bidding under uncertainty: the case of off-shore oil*.

Stark, R.M. (1968). "Discussion of 'Bidding strategies and probabilities'." *Journal of the Construction Division, Proceedings of the American Society of Civil Engineers* **94**(1) 110-2.

Wilson, R. (1979) Auctions of shares. *The Quarterly Journal of Economics* **93**(4) 675-89.

**FIGURE CAPTIONS:**

Fig 1: Effects of mark-up

Table 1: $\beta$ values per iteration (exponential model)

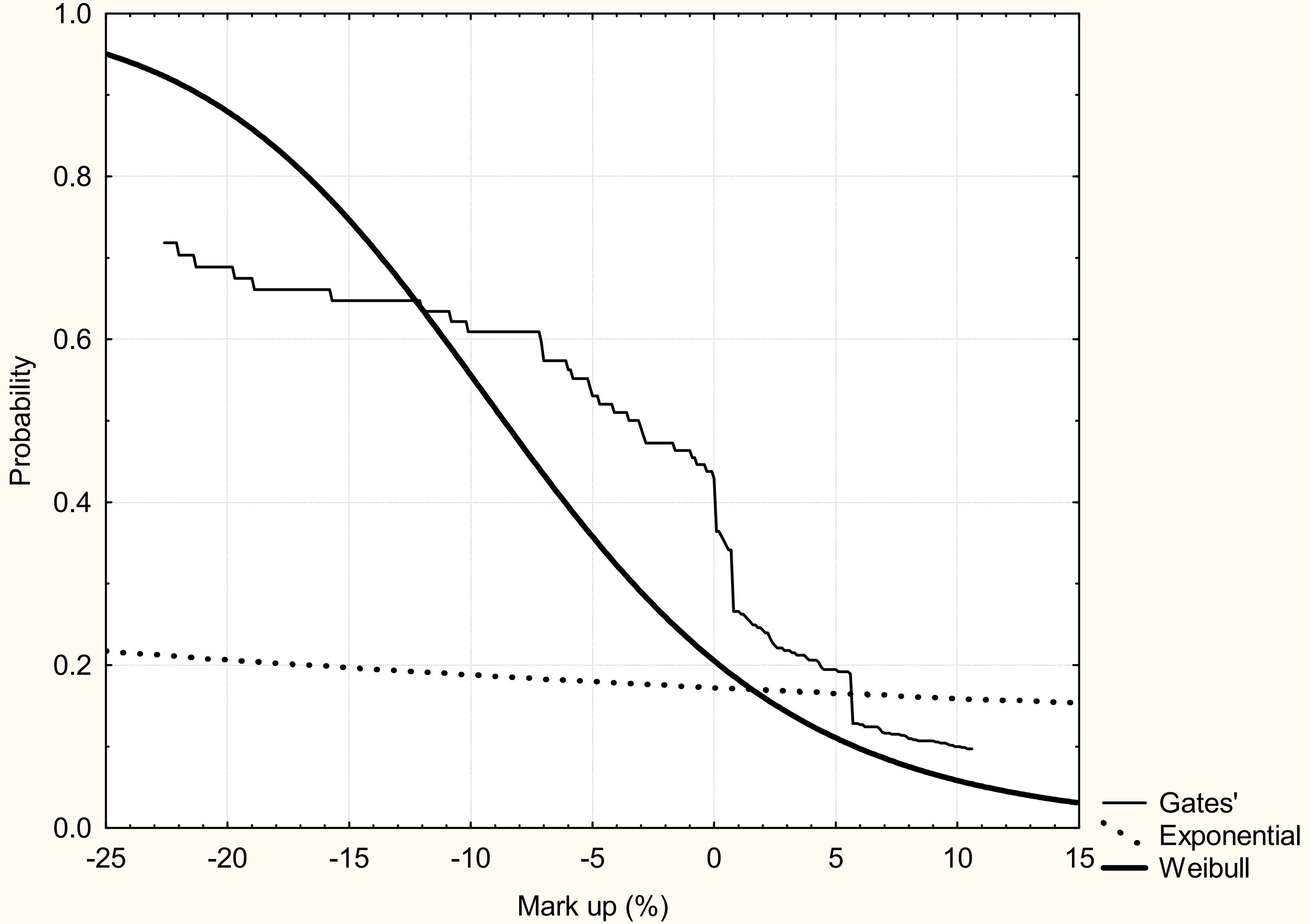
Table 2: $\alpha$ values per iteration (exponential model)

Table 3: $\alpha$ values per iteration (Weibull model)

Table 4: Parameters for mark up

.

Fig 1: Effects of mark up



| Contract | Iteration | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 14.979704 | 14.989777 | 14.991014 | 14.991058 | 14.990991 | 14.990947 | 14.990926 | 14.990917 | 14.990913 | 14.990911 | 14.990911 |
| 2 | 13.024402 | 13.026222 | 13.025708 | 13.025345 | 13.025171 | 13.025095 | 13.025063 | 13.025050 | 13.025045 | 13.025043 | 13.025042 |
| 3 | 12.072437 | 12.063954 | 12.061773 | 12.061209 | 12.061043 | 12.060987 | 12.060966 | 12.060958 | 12.060955 | 12.060953 | 12.060953 |
| 4 | 13.666118 | 13.670915 | 13.671441 | 13.671477 | 13.671459 | 13.671445 | 13.671437 | 13.671434 | 13.671432 | 13.671432 | 13.671431 |
| 5 | 12.697690 | 12.709588 | 12.711320 | 12.711457 | 12.711395 | 12.711345 | 12.711319 | 12.711307 | 12.711303 | 12.711301 | 12.711300 |
| 6 | 13.301764 | 13.304697 | 13.304987 | 13.305052 | 13.305070 | 13.305075 | 13.305076 | 13.305076 | 13.305076 | 13.305076 | 13.305076 |
| 7 | 13.438633 | 13.442452 | 13.442829 | 13.442913 | 13.442937 | 13.442943 | 13.442944 | 13.442944 | 13.442944 | 13.442944 | 13.442944 |
| 8 | 13.714978 | 13.718876 | 13.719811 | 13.720008 | 13.720040 | 13.720042 | 13.720040 | 13.720038 | 13.720038 | 13.720037 | 13.720037 |
| 9 | 12.362480 | 12.374589 | 12.377908 | 12.378856 | 12.379132 | 12.379213 | 12.379237 | 12.379244 | 12.379246 | 12.379247 | 12.379247 |
| 10 | 14.360775 | 14.362468 | 14.362635 | 14.362672 | 14.362683 | 14.362685 | 14.362686 | 14.362686 | 14.362686 | 14.362686 | 14.362686 |
| 11 | 13.780691 | 13.784458 | 13.785189 | 13.785329 | 13.785353 | 13.785355 | 13.785355 | 13.785354 | 13.785354 | 13.785354 | 13.785354 |
| 12 | 12.991719 | 12.994382 | 12.994645 | 12.994704 | 12.994721 | 12.994725 | 12.994726 | 12.994726 | 12.994726 | 12.994726 | 12.994726 |
| 13 | 14.015155 | 14.017306 | 14.017727 | 14.017835 | 14.017857 | 14.017859 | 14.017858 | 14.017857 | 14.017857 | 14.017856 | 14.017856 |
| 14 | 12.427134 | 12.432395 | 12.432913 | 12.433030 | 12.433062 | 12.433070 | 12.433072 | 12.433072 | 12.433072 | 12.433072 | 12.433072 |
| 15 | 13.194684 | 13.241208 | 13.253648 | 13.257183 | 13.258208 | 13.258509 | 13.258598 | 13.258624 | 13.258632 | 13.258635 | 13.258635 |
| 16 | 9.915778 | 9.919473 | 9.919837 | 9.919919 | 9.919942 | 9.919948 | 9.919949 | 9.919949 | 9.919949 | 9.919949 | 9.919949 |
| 17 | 14.307393 | 14.309586 | 14.309803 | 14.309852 | 14.309865 | 14.309868 | 14.309869 | 14.309869 | 14.309869 | 14.309869 | 14.309869 |
| 18 | 12.014410 | 12.016428 | 12.016627 | 12.016672 | 12.016685 | 12.016688 | 12.016688 | 12.016689 | 12.016688 | 12.016688 | 12.016688 |
| 19 | 13.995416 | 14.006493 | 14.007892 | 14.007958 | 14.007891 | 14.007845 | 14.007823 | 14.007814 | 14.007810 | 14.007809 | 14.007808 |
| 20 | 13.166613 | 13.180062 | 13.181561 | 13.181659 | 13.181613 | 13.181578 | 13.181562 | 13.181555 | 13.181552 | 13.181551 | 13.181550 |
| 21 | 14.134182 | 14.135287 | 14.135396 | 14.135421 | 14.135428 | 14.135430 | 14.135430 | 14.135430 | 14.135430 | 14.135430 | 14.135430 |
| 22 | 14.606690 | 14.609900 | 14.610217 | 14.610288 | 14.610308 | 14.610313 | 14.610314 | 14.610314 | 14.610314 | 14.610314 | 14.610314 |
| 23 | 12.067715 | 12.073471 | 12.074038 | 12.074165 | 12.074201 | 12.074210 | 12.074211 | 12.074212 | 12.074212 | 12.074212 | 12.074212 |
| 24 | 14.089404 | 14.091394 | 14.091591 | 14.091635 | 14.091647 | 14.091650 | 14.091651 | 14.091651 | 14.091651 | 14.091651 | 14.091651 |
| 25 | 13.639737 | 13.639737 | 13.639737 | 13.639737 | 13.639737 | 13.639737 | 13.639737 | 13.639737 | 13.639737 | 13.639737 | 13.639737 |
| 26 | 11.899547 | 11.884402 | 11.881669 | 11.881129 | 11.880994 | 11.880950 | 11.880933 | 11.880926 | 11.880924 | 11.880922 | 11.880922 |
| 27 | 13.850155 | 13.853915 | 13.854285 | 13.854369 | 13.854392 | 13.854398 | 13.854399 | 13.854399 | 13.854399 | 13.854399 | 13.854399 |
| 28 | 11.891026 | 11.893092 | 11.893296 | 11.893342 | 11.893354 | 11.893357 | 11.893358 | 11.893358 | 11.893358 | 11.893358 | 11.893358 |
| 29 | 13.597796 | 13.601341 | 13.601691 | 13.601770 | 13.601791 | 13.601797 | 13.601798 | 13.601798 | 13.601798 | 13.601798 | 13.601798 |
| 30 | 12.630523 | 12.636434 | 12.637593 | 12.637815 | 12.637852 | 12.637856 | 12.637855 | 12.637855 | 12.637854 | 12.637854 | 12.637854 |
| 31 | 11.502493 | 11.504441 | 11.504633 | 11.504677 | 11.504689 | 11.504692 | 11.504692 | 11.504692 | 11.504692 | 11.504692 | 11.504692 |
| 32 | 14.460705 | 14.463527 | 14.463805 | 14.463868 | 14.463886 | 14.463890 | 14.463891 | 14.463891 | 14.463891 | 14.463891 | 14.463891 |
| 33 | 15.659474 | 15.644288 | 15.641299 | 15.640687 | 15.640540 | 15.640498 | 15.640483 | 15.640477 | 15.640475 | 15.640474 | 15.640474 |
| 34 | 14.307414 | 14.308964 | 14.309324 | 14.309346 | 14.309328 | 14.309316 | 14.309309 | 14.309307 | 14.309306 | 14.309305 | 14.309305 |
| 35 | 12.118272 | 12.119536 | 12.119660 | 12.119688 | 12.119696 | 12.119698 | 12.119699 | 12.119699 | 12.119699 | 12.119699 | 12.119699 |
| 36 | 12.451380 | 12.457028 | 12.458128 | 12.458338 | 12.458373 | 12.458377 | 12.458377 | 12.458376 | 12.458376 | 12.458375 | 12.458375 |
| 37 | 11.785075 | 11.788804 | 11.789172 | 11.789255 | 11.789277 | 11.789283 | 11.789284 | 11.789285 | 11.789285 | 11.789284 | 11.789284 |
| 38 | 13.550709 | 13.581909 | 13.590342 | 13.592745 | 13.593443 | 13.593647 | 13.593708 | 13.593726 | 13.593731 | 13.593733 | 13.593734 |
| 39 | 14.638640 | 14.644218 | 14.644767 | 14.644891 | 14.644925 | 14.644934 | 14.644936 | 14.644936 | 14.644936 | 14.644936 | 14.644936 |
| 40 | 12.536359 | 12.540246 | 12.540995 | 12.541187 | 12.541227 | 12.541230 | 12.541228 | 12.541226 | 12.541225 | 12.541225 | 12.541225 |
| 41 | 14.158697 | 14.161332 | 14.161592 | 14.161650 | 14.161667 | 14.161671 | 14.161672 | 14.161672 | 14.161672 | 14.161672 | 14.161672 |
| 42 | 15.168579 | 15.148914 | 15.145014 | 15.144215 | 15.144024 | 15.143969 | 15.143949 | 15.143942 | 15.143939 | 15.143938 | 15.143938 |
| 43 | 14.239199 | 14.240952 | 14.241125 | 14.241164 | 14.241175 | 14.241178 | 14.241178 | 14.241178 | 14.241178 | 14.241178 | 14.241178 |
| 44 | 11.898539 | 11.900588 | 11.900790 | 11.900835 | 11.900848 | 11.900851 | 11.900852 | 11.900852 | 11.900852 | 11.900852 | 11.900852 |
| 45 | 12.897211 | 12.912247 | 12.916410 | 12.917597 | 12.917935 | 12.918031 | 12.918058 | 12.918065 | 12.918068 | 12.918068 | 12.918068 |
| 46 | 13.577178 | 13.577178 | 13.577178 | 13.577178 | 13.577178 | 13.577178 | 13.577178 | 13.577178 | 13.577178 | 13.577178 | 13.577178 |
| 47 | 12.070635 | 12.074061 | 12.074399 | 12.074475 | 12.074496 | 12.074502 | 12.074503 | 12.074503 | 12.074503 | 12.074503 | 12.074503 |
| 48 | 14.634901 | 14.630307 | 14.628711 | 14.628089 | 14.627829 | 14.627719 | 14.627673 | 14.627655 | 14.627647 | 14.627644 | 14.627643 |
| 49 | 11.241270 | 11.244918 | 11.245278 | 11.245359 | 11.245381 | 11.245387 | 11.245388 | 11.245388 | 11.245388 | 11.245388 | 11.245388 |

Table 1: $\beta$ values per iteration (exponential model)

| Iteration | Pooled | Ours | Able | Adam | Alan | Alex | Ben | Dave | Dick |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 2 | 0.000000 | -0.010908 | -0.045415 | -0.003798 | -0.000638 | -0.022287 | 0.071586 | -0.012209 | -0.083188 |
| 3 | 0.000000 | -0.011979 | -0.049588 | -0.005993 | -0.001777 | -0.023779 | 0.085072 | -0.015619 | -0.106789 |
| 4 | 0.000000 | -0.012220 | -0.049735 | -0.005965 | -0.002101 | -0.023468 | 0.087877 | -0.016235 | -0.113511 |
| 5 | 0.000000 | -0.012287 | -0.049530 | -0.005751 | -0.002152 | -0.023229 | 0.088557 | -0.016312 | -0.115460 |
| 6 | 0.000000 | -0.012304 | -0.049398 | -0.005625 | -0.002146 | -0.023119 | 0.088752 | -0.016311 | -0.116034 |
| 7 | 0.000000 | -0.012307 | -0.049337 | -0.005568 | -0.002136 | -0.023072 | 0.088818 | -0.016305 | -0.116205 |
| 8 | 0.000000 | -0.012308 | -0.049310 | -0.005543 | -0.002130 | -0.023054 | 0.088841 | -0.016301 | -0.116257 |
| 9 | 0.000000 | -0.012308 | -0.049300 | -0.005533 | -0.002128 | -0.023047 | 0.088850 | -0.016300 | -0.116272 |
| 10 | 0.000000 | -0.012308 | -0.049296 | -0.005529 | -0.002126 | -0.023044 | 0.088854 | -0.016300 | -0.116277 |
| 11 | 0.000000 | -0.012307 | -0.049294 | -0.005528 | -0.002126 | -0.023043 | 0.088855 | -0.016300 | -0.116279 |
| 12 | 0.000000 | -0.012307 | -0.049293 | -0.005527 | -0.002126 | -0.023042 | 0.088856 | -0.016300 | -0.116280 |
| $O_{j1} = e^{-\alpha_j}$ | 1.000000 | 1.012383 | 1.050528 | 1.005543 | 1.002128 | 1.023310 | 0.914978 | 1.016433 | 1.123310 |

Table 2: $\alpha$ values per iteration (exponential model)

| Iteration | $\sigma$ | RHS eqn (15) | Pooled | Ours | Able | Adam | Alan | Alex | Ben | Dave | Dick |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.000000 | 227.578864 | 0.000000 | -0.012307 | -0.049293 | -0.005527 | -0.002126 | -0.023042 | 0.088856 | -0.016300 | -0.116280 |
| 2 | 1.000000 | 227.578864 | 0.000000 | -0.012307 | -0.049293 | -0.005527 | -0.002126 | -0.023042 | 0.088856 | -0.016300 | -0.116280 |
| 3 | 0.248537 | 206.460859 | 0.000000 | -0.014370 | -0.053313 | -0.010270 | -0.009501 | -0.028486 | 0.082385 | -0.021031 | -0.126274 |
| 4 | 0.092470 | 91.031818 | 0.000000 | -0.019673 | -0.063378 | -0.017700 | -0.021727 | -0.038827 | 0.068389 | -0.030814 | -0.142810 |
| 5 | 0.071303 | 18.360293 | 0.000000 | -0.022416 | -0.067405 | -0.019369 | -0.024963 | -0.042198 | 0.065011 | -0.034672 | -0.146931 |
| 6 | 0.065956 | -8.674014 | 0.000000 | -0.023344 | -0.068632 | -0.019744 | -0.025862 | -0.043183 | 0.064255 | -0.035891 | -0.147777 |
| 7 | 0.067671 | 0.518674 | 0.000000 | -0.023034 | -0.068229 | -0.019628 | -0.025570 | -0.042861 | 0.064489 | -0.035487 | -0.147526 |
| 8 | 0.067575 | 0.013740 | 0.000000 | -0.023051 | -0.068252 | -0.019634 | -0.025586 | -0.042879 | 0.064476 | -0.035510 | -0.147541 |
| 9 | 0.067572 | -0.000022 | 0.000000 | -0.023052 | -0.068253 | -0.019634 | -0.025587 | -0.042879 | 0.064475 | -0.035510 | -0.147542 |
| 10 | 0.067572 | 0.000000 | 0.000000 | -0.023052 | -0.068253 | -0.019634 | -0.025587 | -0.042879 | 0.064475 | -0.035510 | -0.147542 |
| 11 | 0.067572 | 0.000000 | 0.000000 | -0.023052 | -0.068253 | -0.019634 | -0.025587 | -0.042879 | 0.064475 | -0.035510 | -0.147542 |
| 12 | 0.067572 | 0.000000 | 0.000000 | -0.023052 | -0.068253 | -0.019634 | -0.025587 | -0.042879 | 0.064475 | -0.035510 | -0.147542 |
| $\lambda = e^{-\alpha_j}$ | | | 1.000000 | 1.023319 | 1.070636 | 1.019828 | 1.025917 | 1.043812 | 0.937559 | 1.036148 | 1.158981 |
| $\omega = 1/\sigma$ | | | 14.799046 | 14.799046 | 14.799046 | 14.799046 | 14.799046 | 14.799046 | 14.799046 | 14.799046 | 14.799046 |
| $O_{j1} = \lambda^{\omega}$ | | | 1.0000 | 1.4065 | 2.7458 | 1.3372 | 1.4603 | 1.8862 | 0.3851 | 1.6913 | 8.8770 |

Table 3: $\alpha$ values per iteration (Weibull model)

| Model | Params | Pooled | Ours | Able | Adam | Alan | Alex | Ben | Dave | Dick |
|---|---|---|---|---|---|---|---|---|---|---|
| exponential | $\lambda$ | 1.000000 | 1.064605 | 1.052568 | 1.007183 | 1.004139 | 1.025822 | 0.915839 | 1.018468 | 1.110959 |
| | $\omega$ | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 | 1.000000 |
| Weibull | $\lambda$ | 1.000000 | 1.074534 | 1.072812 | 1.020919 | 1.027598 | 1.046816 | 0.937673 | 1.037368 | 1.142513 |
| | $\omega$ | 14.978274 | 14.978274 | 14.978274 | 14.978274 | 14.978274 | 14.978274 | 14.978274 | 14.978274 | 14.978274 |

Table 4: Parameters for mark up