ESTIMATION AND SIMULATION OF THE RIESZ-BESSEL DISTRIBUTION

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Key Words: Riesz-Bessel distribution; Lévy motion; GCMED.

ABSTRACT

In this paper, further properties of the Riesz-Bessel distribution are provided. These properties allow for the simulation of random variables from the Riesz-Bessel distribution. Estimation is addressed by nonlinear generalized least squares regression on the empirical characteristic function. The estimator is seen to approximate the maximum likelihood estimator. The distribution is illustrated with financial data.

1. INTRODUCTION

Bochner (1949) and Feller (1952) demonstrated the connection between the stable distribution and fractional calculus by proposing a Cauchy problem whose solution is the class of stable distributions. Specifically, the Cauchy problem studied by Bochner (1949) was

$$\frac{\partial p}{\partial t} = -\left(-\Delta\right)^{\alpha} p\left(t,x\right), \quad p\left(0,x\right) = \delta\left(x\right),$$

where $\alpha \in (0,1]$, $\delta(x)$ is the Dirac delta function and the operator $(-\Delta)^{\alpha}$ is understood as the inverse of the Riesz potential defined by the kernel

$$J_{\alpha}\left(x\right) = \frac{\Gamma\left(n/2-\alpha\right)}{\pi^{n/2}4^{\alpha}\Gamma\left(\alpha\right)}\left|x\right|^{2\alpha-n}.$$

The solution is the symmetric 2α-stable distribution. The operator $-(-\Delta)^{\alpha}$ and its generalization by Feller (1952) are part of general theory concerning infinitesimal generators of Lévy semigroups, that is, the transition probability density functions of Lévy motions. Despite a large number of fractional operators (Samko, Kilbas and Marichev, 1993) and the connection established by Bochner (1949) their remains few specific examples of Cauchy problems generating Lévy semigroups. Most of the work in this direction remains concentrated on the stable distribution (Gorenflo and Mainardi, 1998, 1999). In Anh and McVinish (2004), the Riesz-Bessel distribution is proposed as the solution to the Cauchy problem

$$\frac{\partial p}{\partial t} = -\left(-\Delta\right)^{\alpha}\left(I-\Delta\right)^{\gamma} p\left(t,x\right), \quad p\left(0,x\right) = \delta\left(x\right), \tag{1.1}$$

where the operator $(I-\Delta)^{\gamma}$ is understood as the inverse of the Bessel potential defined by the kernel

$$I_{\gamma}\left(x\right) = \frac{\left(4\pi\right)^{\gamma}}{\Gamma\left(\gamma\right)} \int_0^{\infty} e^{-\pi|x|^2/s - s/4\pi} s^{\gamma - n/2} \frac{ds}{s}.$$

The solution of (1.1) is given in terms of its spatial Fourier transform

$$\hat{p}\left(t,\lambda\right) = \exp\left[-t\left|\lambda\right|^{2\alpha}\left(1+|\lambda|^2\right)^{\gamma}\right], \quad \lambda \in R^d, \tag{1.2}$$

and $\hat{p}\left(t,\lambda\right)$ is a characteristic under certain conditions on α and γ.

As with the stable and Linnik distributions, despite the simple form of the characteristic function, there is no closed form expression for the probability density function of the Riesz Bessel distribution. When there is no closed form expression for the density, the problem of simulating random variables is sometimes addressed via special representations. An example of this is simulation algorithm proposed by Kozubowski (2000) which makes use of the mixture representation of the Linnik distribution derived by Kotz and Ostrovskii (1996). Also, the method of simulating stable random variables proposed by Chambers, Mallow and Stuck (1976) (see also Weron (1996)) is based on an integral representation due to Zolotarev (1966).

In the estimation problem, the lack of a closed form for the density means direct maximum likelihood estimation is usually abandoned. Numerous methods have been proposed for the stable and Linnik distributions, though they can be applied more generally. An incomplete list of these methods include the fractional moment estimation (Kozubowski, 2001 and Nikias and Shao, 1995), method of moment type (Anderson, 1992 and Press, 1972), minimal distance method (Anderson and Arnold, 1993 and Paulson, Holocomb and Leitch, 1975), log-log regression of characteristic function (Koutrouvelis, 1980) and the k–L procedure of Feuerverger and McDunnough (1981). The use of these methods is usually supported by some asymptotic results together with a simulation study to suggest their accuracy on small samples.

The paper is organized as follows: In section 2, properties of the Riesz-Bessel distribution are reviewed and two new properties are presented. In Section 3, based on one of the new properties, a method for simulating a random variable from the Riesz-Bessel distribution is proposed. In Section 4, the estimation problem for the Riesz-Bessel distribution is studied within the quasi-likelihood framework (see Heyde (1997) for details). This enables us to see the k–L procedure as an approximate maximum likelihood approach. The paper concludes with an illustration of the fitting method by application to Japanese Yen returns data.

2. THE RIESZ-BESSEL DISTRIBUTION

A Lévy motion such that the characteristic function of its distribution at time t is given by (1.2) is called a Riesz-Bessel-Lévy motion (RBLm) and will be denoted by $RB\left(t\right)$. As stated in the Introduction, (1.2) is a characteristic function, but only for a specific range of values of α and γ. The conditions for $p\left(t,x\right)$ to be a probability distribution are given in the following theorem.

Theorem 2.1 The function $\hat{p}\left(t,z\right)$ is the characteristic function of a distribution for all $t \geq 0$ if and only if $\alpha \in \left(0,1\right]$, $\alpha+\gamma \in [0,1]$.

This class of distributions can be made strictly type equivalent by setting

$$\hat{p}\left(t,\lambda\right) = \exp\left[-t\left|\lambda\right|^{2\alpha}\left(c^2+\left|\lambda\right|^2\right)^{\gamma}\right], \quad \lambda \in R^d, \tag{2.1}$$

with $c > 0$. However, it will be assumed throughout that $c = 1$, unless stated otherwise. Theorem 2.1 was proved in Anh and McVinish (2004) by first showing that

$$\phi\left(\lambda\right) = \exp\left[-t\lambda^{\alpha}(1+\lambda)^{\gamma}\right], \quad \lambda > 0. \tag{2.2}$$

is the Laplace-Stieltjes transformation of a probability distribution for all $t > 0$, $\alpha \in$

$(0,1]$, $\alpha+\gamma \in [0,1]$. The Lévy motion whose distribution at time t has Laplace-Stieltjes transform (2.2) is called the Riesz-Bessel-Lévy subordinator (RBLs) and will be denoted by $RBS\left(t\right)$. Simple conditioning arguments then show that

$$RB\left(t\right) \stackrel{d}{=} W\left(RBS\left(t\right)\right) \tag{2.3}$$

where $W\left(t\right)$ is a Brownian motion with variance $2t$ and equality is in the sense of finite dimensional distributions. A distribution whose Lévy motion can be written in the form (2.3) is said to be of Type-G. Type-G distributions were introduced in Marcus (1987) and defined on R^1 as being the distribution of a random variable that is equal in law to σZ, where Z is a standard normal and σ^2 is a non-negative infinitely divisible random variable. An extension to R^d is given in Bandorff-Nielsen and Pérez-Abreu (2002). It should be noted that representation (2.3) can also be interpreted in terms of a transformation of the heat (Gaussian) semigroup to a new semigroup.

The role of the parameters α, γ in RBLm is clear from (1.2): The parameter α determines which moments are finite and so, as $t \to \infty$, the distribution can be re-scaled to converge to a symmetric 2α-stable distribution. The parameter γ acts together with α to determine the small time behaviour, that is, as $t \to 0$ the distribution can be re-scaled to converge to a symmetric $2\left(\alpha+\gamma\right)$-stable distribution. A similar interpretation of the parameters can be applied to RBLs. This type of behaviour is consistent with experience in applying stable distributions to financial returns data. Taylor (1986) notes that the index of stability estimated from returns data tend to increase with time horizon from ~ 1.6 to near 2. By taking $\alpha = 1$ and $\gamma < 0$, the RBLm is able to incorporate this observation in a parsimonious manner. Simulated sample paths demonstrating this property are given in Section 3.

The Laplace-Stieltjes transform of the RBLs has the Lévy representation

$$E\left(e^{-\lambda RBS(t)}\right) = \exp\left[-at\lambda - t\int_0^\infty \left(1-e^{-x\lambda}\right)\nu_S\left(dx\right)\right],$$

where $\nu_S\left(dx\right)$ is called the Lévy measure. The Lévy measure can be expressed in terms of Kummer's confluent hypergeometric function,

$${}_1F_1\left(a;b;x\right) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{\left(a\right)_k}{\left(b\right)_k}$$

and

$$(a)_k = \begin{cases} 1 & k = 0 \\ a\,(a+1)\ldots(a+k) & k \geq 1 \end{cases}$$

see Andrews, Askey and Roy (1999) for details. For $\alpha + \gamma \in [0, 1)\,, a = 0$ and

$$\nu_S\,(dx) = \left[\frac{\alpha\ {}_1F_1\,(1-\gamma; 2-\alpha-\gamma; -x)}{\Gamma\,(2-\alpha-\gamma)\,x^{\alpha+\gamma}} + \frac{(\alpha+\gamma)\ {}_1F_1\,(1-\gamma; 1-\alpha-\gamma; -x)}{\Gamma\,(1-\alpha-\gamma)\,x^{1+\alpha+\gamma}}\right] dx, \quad (2.4)$$

and for $\alpha + \gamma = 1$, $a = 1$ and

$$\nu_S\,(dx) = \alpha x^{-1}\left[\,{}_1F_1\,(\alpha; 1; -x) - {}_1F_1\,(\alpha+1; 2; -x)\right] dx. \quad (2.5)$$

The qualitative behaviour of the paths of RBLs changes with the value of $\alpha + \gamma$; when $\alpha + \gamma = 0$, the process is a compound Poisson process; when $\alpha + \gamma \in (0, 1)$, the process is a pure jump process with jumping times dense in $(0, \infty)$; when $\alpha + \gamma = 1$, the process is a compound Poisson process with drift.

The characteristic function of RBLm has Lévy representation

$$E\left(e^{i\lambda RB(t)}\right) = \exp\left[-at\lambda^2 - t\int_R (\cos(\lambda x) - 1)\,\nu\,(dx)\right]$$

where $\nu\,(d\lambda)$ is called the Lévy measure. As RBLm is a subordinated Brownian motion, it's Lévy measure is of the form

$$\nu\,(dx) = \int_0^\infty (4\pi s)^{-1/2} \exp\left(\frac{-x^2}{4s}\right) \nu_S\,(ds)\,dx, \quad (2.6)$$

with $\nu_S\,(dx)$ given by either (2.4) or (2.5). As with RBLs, the qualitative behaviour of the paths of RBLm changes with the value of $\alpha + \gamma$. For $\alpha + \gamma < 1$ RBLs and RBLm display similar behaviour. For $\alpha + \gamma = 1$ RBLm is the sum of a compound Poisson process and an independent Brownian motion.

Despite there being no closed form for the density function of the Riesz-Bessel distribution, it is still possible to visualize the density by numerical inversion of the characteristic function. In Figures 1 and 2, the density of the Riesz-Bessel distribution is plotted for $t = 1$, $\alpha = 1$ and γ varying. The graphs were generated using the method described in Mittnik, Doganoglu and Chen (1999) for the stable distributions. Note that the variance in these plots is held constant.

Figure 1: The Riesz-Bessel density calculated by numerical inversion of the Fourier transform, $t = 1, \alpha = 1; \gamma = -0.8, -0.6, \ldots, 0$. As $\gamma \downarrow -1$ the density becomes more peaked at $x = 0$. Note that the case $\alpha + \gamma = 0$ is excluded, as the corresponding distribution has an atom at $x = 0$.

Figure 2: Densities from Figure 1 plotted on the semi-log scale. As $\gamma \downarrow -1$ the tails of the distribution become heavier which can also be seen from the Lévy density.

We now consider the problem of determining if RBLs is a member of the class of generalized convolutions of mixtures of exponentials (GCMED), that is, can RBLs be obtained as a weak limit of sums of random variables with completely monotone densities. These results rely on chapter nine of Bondesson (1992). A distribution of the class of GCMED is a distribution on $[0, \infty)$ with Laplace-Stieltjes transform

$$\phi(\lambda) = \exp\left[-a\lambda + \int_{(0,\infty)} \left(\frac{1}{x+\lambda} - \frac{1}{x}\right) Q(dx)\right], \quad \lambda \geq 0,$$

where $a \geq 0$ and the non-negative measure Q on $(0, \infty)$ satisfies

$$\int_{(0,\infty)} \frac{1}{x(1+x)} Q(dx) < \infty.$$

This class of distributions can be characterized as those infinitely divisible distributions whose Lévy measure has a completely monotone derivative, in which case the Lévy measure is given by $\nu_S(x)\,dx = \int e^{-xy} Q(dy)\,dx$, (Theorem 9.1.2 of Bondesson (1992)). It is noted that this class is closed under weak limits (Theorem 9.1.1 of Bondesson (1992)). A special subclass is obtained by restricting $Q(dx)$ to have an increasing density. The resulting class is the generalized Gamma convolutions (GGC), that is the class of distributions obtained as weak limits of sums of Gamma random variables. As GGC contain a large number of interesting distributions, such as positive stable, Mittag-Leffler, log-normal and generalized inverse Gaussian to name a few, it is also of interest to establish if RBLs is a member of GGC.

Proposition 2.1 The Riesz-Bessel-Lévy subordinator is a member of the generalized convolutions of mixtures of exponentials.

Proof: Assume $\alpha+\gamma \in (0,1)$ and $\gamma > 0$. The inverse Laplace transform of the Lévy density can be obtained from Equation 3.33.1.3 of Prudnikov, Brychkov and Marichev (1990) and basic properties of the Laplace transform as

$$\frac{1}{\pi}\int_0^x \sin(\alpha\pi)\, u^\alpha (1-u)_+^{\gamma-1}\left(\frac{\alpha}{u}-\alpha-\gamma\right)+\sin(\pi(\alpha+\gamma))\, u^\alpha (u-1)_+^{\gamma-1}\left(\alpha+\gamma-\frac{\alpha}{u}\right) du. \tag{2.7}$$

The first term of the integrand is positive for $u \in [0, \alpha/(\alpha+\gamma))$ and negative for $u \in (\alpha/(\alpha+\gamma), 1]$ and zero for $u > 1$. The second term of the integrand is zero for $u \in [0,1]$ and positive for all $u > 1$. It follows that if the integral is non-negative at $x = 1$ then the integral is non-negative for all $x > 0$. From elementary properties of the Gamma function, the integral at $x = 1$ is zero and hence the Lévy density is completely monotone on $(0,\infty)$.

Now we assume that $\alpha+\gamma \in (0,1)$ and $\gamma < 0$. The inverse Laplace transform of the Lévy density is given in equation 3.33.1.2 of Prudnikov et al. (1990) as

$$\begin{aligned}
& \frac{x^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\left[{}_2F_1(1-\gamma;\alpha;1+\alpha;x)-\frac{\alpha+\gamma}{1+\alpha}\,{}_2F_1(1-\gamma;1+\alpha;2+\alpha;x)\right]1_{(0<x\le 1)} \\
+ & \frac{x^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)\Gamma(2-\alpha-\gamma)}\left[\alpha\,{}_2F_1(1-\gamma;1-\alpha-\gamma;2-\alpha-\gamma;1/x)\right. \\
+ & \left.(1-\alpha-\gamma)\,x\,{}_2F_1(1-\gamma;-\alpha-\gamma;1-\alpha-\gamma;1/x)\right]1_{(x>1)}.
\end{aligned}$$

For $x \in [0,1]$, taking the series expansion of the Gaussian hypergeometric function yields

$$\begin{aligned}
& {}_2F_1(1-\gamma,\alpha;1+\alpha;x)-\frac{\alpha+\gamma}{1+\alpha}\,{}_2F_1(1-\gamma,1+\alpha;2+\alpha;x) \\
= & \sum_{k=0}^\infty \frac{(1-\gamma)_k(\alpha)_k}{(1+\alpha)_k}\frac{x^k}{k!}-\frac{\alpha+\gamma}{1+\alpha}\sum_{k=0}^\infty\frac{(1-\gamma)_k(1+\alpha)_k}{(2+\alpha)_k}\frac{x^k}{k!} \\
= & 1-\frac{\alpha+\gamma}{1+\alpha}+\sum_{k=1}^\infty\frac{\alpha(1-\gamma)_k}{k+\alpha}\frac{x^k}{k!}-\sum_{k=1}^\infty\frac{(\alpha+\gamma)(1-\gamma)_k}{k+1+\alpha}\frac{x^k}{k!} \\
= & \frac{1-\gamma}{1+\alpha}+\sum_{k=1}^\infty(1-\gamma)_k\left(\frac{\alpha}{\alpha+k}-\frac{\alpha+\gamma}{k+1+\alpha}\right)\frac{x^k}{k!}.
\end{aligned}$$

As $\gamma < 0$ it follows that (2.8) is non-negative for $x \in [0,1]$. For $x > 1$, the series expansion of the Gaussian hypergeometric function yields,

$$\alpha\,{}_2F_1(1-\gamma,1-\alpha-\gamma;2-\alpha-\gamma;x)-(1-\alpha-\gamma)\,x\,{}_2F_1(1-\gamma,-\alpha-\gamma;1-\alpha-\gamma;x)$$

$$
\begin{aligned}
&= \alpha + (1-\alpha-\gamma)\,x + \alpha \sum_{k=1}^{\infty} \frac{(1-\alpha-\gamma)\,(1-\gamma)_k}{k+1-\alpha-\gamma} \frac{x^{-k}}{k!} \\
&\quad + (1-\alpha-\gamma) \sum_{k=1}^{\infty} \frac{(-\alpha-\gamma)\,(1-\gamma)_k}{k-\alpha-\gamma} \frac{x^{1-k}}{k!} \\
&= \alpha + (1-\gamma)\,(-\alpha-\gamma) + (1-\alpha-\gamma)\,x \\
&\quad + \sum_{k=1}^{\infty} \left\{ \alpha + \frac{(-\alpha-\gamma)\,(k+1-\gamma)}{(k+1)} \right\} \frac{(1-\alpha-\gamma)\,(1-\gamma)_k\, x^{-k}}{k!\,(k+1-\alpha-\gamma)} \\
&= (-\gamma)\,(1-\alpha-\gamma) + (1-\alpha-\gamma)\,x \\
&\quad + (-\gamma)\,(1-\alpha-\gamma) \sum_{k=1}^{\infty} \frac{(1-\gamma)_k\, x^{-k}}{k!\,(k+1-\alpha-\gamma)} \left(1 - \frac{\alpha+\gamma}{k+1}\right).
\end{aligned}
$$

As $\gamma < 0$ and $\alpha + \gamma < 1$ it follows that (2.8) is positive for $x > 1$. Hence, the Lévy density is completely monotone for this range of parameters. The remaining cases, $\alpha + \gamma = 1$ and $\alpha + \gamma = 0$, are members of GCMED as this class is closed under weak limits. This completes the proof.

The functions (2.7) and (2.8) give the density q of the measure Q in the respective parameter ranges. For $\gamma > 0$, $\alpha + \gamma < 1$ it is clear that q is not increasing and from theorem 9.1.4 of Bondesson (1992) it follows that RBLs is not a member of the class GGC. Furthermore, q is not bounded as $x \to \infty$ and hence from theorem 9.1.5 of Bondesson (1992) it follows that the distribution of RBLs does not have a completely monotone derivative for any $t > 0$. These statements also hold for $\gamma < 0$. Recall the following property due to Gauss of the hypergeometric function. If $\Re\,(c-a-b) < 0$, then

$$
\lim_{x \to 1^-} \frac{{}_2F_1\,(a;b;c;x)}{(1-x)^{c-a-b}} = \frac{\Gamma\,(c)\,\Gamma\,(a+b-c)}{\Gamma\,(a)\,\Gamma\,(b)},
$$

see Andrews et al. (1999) for details. For $\gamma < 0$, q satisfies

$$
\lim_{x \to 1^-} \frac{q\,(x)}{(1-x)^{\gamma}} = \frac{1}{\Gamma\,(\alpha)\,\Gamma\,(1-\alpha)},
$$

thus, q cannot be increasing and so RBLs is not a member of GGC. Furthermore, as q is unbounded, the distribution of RBLs does not have a completely monotone derivative for any $t > 0$. The remaining case of $\alpha + \gamma = 1$ cannot be a member of GGC as RBLs is a compound Poisson process with drift in this case and hence is not self-decomposable. However, the distribution of RBLs does have a completely monotone derivative for some $t > 0$ in this case.

Proposition 2.2 Let $Y_t = RBS(t) - t$ and assume $\alpha + \gamma = 1$. The distribution function of Y_t has an atom at zero with mass $e^{-t(1-\alpha)}$. The absolutely continuous component of the distribution has a density given by

$$\frac{1}{\pi}\int_0^1 \exp\left[-ux - t\left(u^\alpha (1-u)^{1-\alpha}\cos(\alpha\pi) + u\right)\right]\sin\left(t\sin(\alpha\pi)\,u^\alpha (1-u)^{1-\alpha}\right)du, \quad x \geq 0, \tag{2.9}$$

provided $t \leq \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\pi/\sin(\alpha\pi)$.

Proof: The Laplace-Stieltjes transform of the distribution of Y_t is given by

$$\phi(\lambda) = \exp\left\{-t\left[\lambda^\alpha (1+\lambda)^{1-\alpha} - \lambda\right]\right\}. \tag{2.10}$$

From proposition 2.1 it is known that Y_t is a member of the class of GCMED. By application of some elementary properties of the Laplace transform to (2.5), the density of the Q measure in this case is given by

$$q(x) = \frac{t}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^x (1-u)_+^{-\alpha} u^{\alpha-1}(\alpha - u)\,du.$$

The above integral is bounded by $t\alpha^\alpha(1-\alpha)^{1-\alpha}\pi/\sin(\alpha\pi)$ for all $\alpha \in (0,1)$ and hence, the density $q(x)$ is finite for all $x > 0$. Application of Theorem 9.1.5 of Bondesson (1992) gives that the density of RBLs is completely monotone on $x > t, t \in \left(0, \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\pi/\sin(\alpha\pi)\right]$ for $\alpha + \gamma = 1$. It follows that (2.10) is the Stieltjes transform of a Borel measure $\mu(du)$ which may be obtained by application of the Stieltjes complex inversion formula (Widder (1941) - chapter VIII, theorem 7a).

$$\lim_{\eta\to 0^+}\frac{1}{2\pi i}\int_0^u [\phi(-\sigma - i\eta) - \phi(-\sigma + i\eta)]\,d\sigma = \frac{\mu(u+) + \mu(u-)}{2} - \frac{\mu(0+) + \mu(0)}{2} \tag{2.11}$$

for $\lambda > 0$. The integrand in (2.11) remains bounded as $\eta \to 0$. Applying the Lebesgue dominated convergence theorem we see that μ is absolutely continuous and hence has representation (2.9). This completes the proof.

From (2.6) and Proposition 2.1 it follows that the density of the Lévy measure of RBLm is completely monotone on $(0,\infty)$. This implies that RBLm for $\alpha + \gamma < 1/2$ (this condition ensures RBLm has paths of bounded variation) can be written as the difference of two subordinators whose distribution belongs to the class of GCMED. Geman, Madan and Yor (2001) provide the following interpretation of these processes in a financial setting. Let S_t be the price of some traded asset. The log price process is given by

$$\log\left(S_t/S_0\right) = U\left(t\right) - V\left(t\right)$$

where $U\left(t\right), V\left(t\right)$ are the prevailing buy/sell orders which are modeled as independent subordinators. The representation of a Lévy measure with a completely monotone derivative as the Laplace transform of some measure is interpreted as an economy populated by individuals who submit prevailing price buy or sell orders with an exponential distribution. The measure from the representation of the Lévy measure relates to the number of orders per unit time at a particular mean level, with exponential size distribution. A further property of subordinators with GCMED distributions is that by solely observing small jumps information can be obtained on the larger jumps of the process.

3. SIMULATION

It is noted that subordination of a Riesz-Bessel motion by a stable subordinator is again a Riesz-Bessel motion with a change of its parameters. Precisely, if $RB\left(t\right)$ is a Riesz-Bessel motion with characteristic function (1.2) and S_t is a stable subordinator with Laplace transform $\exp\left(-tz^{\beta}\right)$ then $RB\left(S_t\right)$ is a Riesz-Bessel motion with $\alpha := \alpha\beta$ and $\gamma := \gamma\beta$. Thus, all Riesz-Bessel motions can be reduced to the subordination of one of the two following cases: If $\gamma < 0$ then it can be obtained by subordination of a Riesz-Bessel motion with $\beta := \alpha, \gamma := \gamma/\beta$ and $\alpha := 1$. If $\gamma > 0$ then it can be obtained by subordination of a Riesz-Bessel motion with $\beta := \alpha + \gamma, \alpha := \alpha/\beta$ and $\gamma := \gamma/\beta$.

Simulation of a general Riesz-Bessel motion can be carried out by simulating an appropriate stable subordinator and one of the special cases. Simulation of stable random variables is detailed in Weron (1996). Details on the simulation of stable processes can be found in Janicki and Weron (1994). Simulation of the two special cases of Riesz-Bessel motion is discussed below.

First consider the case of $\alpha + \gamma = 1$. From Proposition 2.2 it follows that for $t \leq$

$\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\pi/\sin(\alpha\pi)$ a Riesz-Bessel random variable with $\alpha+\gamma=1$ can be represented by

$$RB_t \overset{d}{=} \sigma_t Z, \quad \sigma_t^2 \overset{d}{=} t + \delta W/A \tag{3.1}$$

where $Z \sim N(0,2)$, δ is a Bernoulli random variable with $\Pr(\delta=0)=\exp(-t\gamma)$, W is a exponential random variable with unit mean and A is a random variable with density

$$\frac{1}{\pi}\exp\left[-t\left(u^{\alpha}(1-u)^{1-\alpha}\cos(\alpha\pi)+u\right)\right]\sin\left(t\sin(\alpha\pi)u^{\alpha}(1-u)^{1-\alpha}\right)u^{-1}, \tag{3.2}$$

$0<u<1$. If $t>\alpha^{-\alpha}(1-\alpha)^{\alpha-1}\pi/\sin(\alpha\pi)$ then we can use the property that the Riesz-Bessel distribution is closed under convolution, that is,

$$RB(t)=\sum_{k=1}^{M}RB_k(t_k), \quad t=\sum_{k=1}^{M}t_k$$

with $t_k \le \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\pi/\sin(\alpha\pi)$ for all k and $RB_k(t)$ are independent. Simulation from density (3.2) can be achieved by a rejection sampling algorithm. The algorithm for simulating $RB(t)$ with $\alpha+\gamma=1$ is given below:

Algorithm 1: Case of $\alpha+\gamma=1$.

1. Repeat
   - Generate two independent random variates $U_1,\ U_2$ from the uniform distribution on $[0,1]$.
   - Set $V \leftarrow U_1^{1/\alpha}$.
   - Set $G \leftarrow t\sin(\alpha\pi)V^{\alpha-1} \times \max_{u\in[0,1]} \exp\left[-t\left(u^{\alpha}(1-u)^{1-\alpha}\cos(\alpha\pi)+u\right)\right]$.
   - Set $g \leftarrow \sin\left(t\sin(\alpha\pi)V^{\alpha}(1-V)^{1-\alpha}\right)\exp\left[-t\left(V^{\alpha}(1-V)^{1-\alpha}\cos(\alpha\pi)+V\right)\right]/V$.
   - Until $U_2 < g/G$.
2. Generate an exponential random variable W with unit mean and a Bernoulli random variable δ such that $\Pr(\delta=0)=\exp(-t\gamma)$.

3. Set $V = t + \delta W/V$.

4. Generate a Gaussian random variable Z with mean zero and variance 2.

5. Return $Z\sqrt{V}$.

The expected number of iterations required to generate a single random variable is given by

$$\frac{t\sin(\alpha\pi)}{\alpha\pi\left(1-\exp\left(-t\left(1-\alpha\right)\right)\right)}\max_{u\in[0,1]}\exp\left[-t\left(u^{\alpha}\left(1-u\right)^{1-\alpha}\cos\left(\alpha\pi\right)+u\right)\right].$$

It is seen that for moderate values of t the efficiency of the algorithm increases with α while for t small the efficiency is symmetric about $\alpha = 1/2$ and increases as α approaches 1 and 0. The sample paths of Figure 3 were generated using algorithm 1 and time increments of 0.1.

Figure 3: Sample paths of Riesz-Bessel motion with $\alpha + \gamma = 1$. Note that as $\alpha \to 1$ the size of the jumps in the process becomes smaller.

Now we consider the case of $\alpha = 1$. This was briefly considered in Anh and McVinish (2004) where it was noted that the Lévy measure is given by

$$\nu(dx) = \frac{1}{\Gamma(-\gamma)}\left(x^{(1+\gamma)}e^{-x} + (1+\gamma)\,x^{-(2+\gamma)}e^{-x}\right)dx$$

from which it can be seen that RBLs is the sum of a tempered Stable (TS) subordinator, also called a CGMY process (see Carr, Geman and Yor (2002) for details) and an independent compound Poisson process with Gamma distributed jumps. In this paper the TS component will be simulated using a rejection sampling algorithm. The algorithm for simulating $RB(t)$ with $\alpha = 1$ is given below.

Algorithm 2: Case of $\alpha = 1$.

1. Generate a Poisson random variable N with mean t.

2. Generate a Gamma random variable G with shape parameter $-\gamma N$ and scale parameter 1.

3. Repeat

   – Generate a positive stable random variable V, with index $1+\gamma$ and scale parameter $t\cos\left(\pi\left(1+\gamma\right)/2\right)$.

   – Generate a random variable U from the uniform distribution of $[0,1]$.

   – Until $U < \exp\left(-V\right)$.

4. Generate a Gaussian random variable Z with mean zero and variance 2.

5. Return $Z\sqrt{V+G}$.

In this algorithm most computation is required for the rejection step which generates the TS component. The efficiency of the rejection algorithm will decrease quickly as t increase, through for t small the efficiency is near one. The sample paths of Figure 4 were generated using this algorithm with time increments of 0.1.

Figure 4: Sample paths of Riesz-Bessel motion with $\alpha = 1$. Note that as $\gamma \to -1$ RBLm approaches a compound Poisson process.

How to simulate random variables from the Riesz-Bessel distribution for the case of $\{\alpha \neq 1\} \cap \{\alpha+\gamma \neq 1\}$ using the above special cases was described at the beginning of this section. The efficiency of these algorithms will only be reasonable when the parameter values are near these special cases, that is, only if α or $\alpha+\gamma$ are not too far from one. Further research will hopefully provide more efficient algorithms.

4. PARAMETER ESTIMATION

For distributions which lack a closed form for the density function, maximum likelihood estimation of parameters is generally not feasible and so one needs an alternative approaches. For the stable and Linnik distributions estimates based on the empirical characteristic function have proven useful. For a symmetric distribution, the empirical characteristic function is defined as

$$\hat{\phi}_n\left(\lambda\right) = \frac{1}{n}\sum_{j=1}^{n}\cos\left(\lambda X_j\right)$$

from which it is clear that

$$E\,\hat{\phi}_n(\lambda)=\phi(\lambda)\,,\quad \mathrm{cov}\left(\hat{\phi}_n(\lambda_1)\,,\hat{\phi}_n(\lambda_2)\right)=\frac{1}{2n}\left[\phi(\lambda_1-\lambda_2)+\phi(\lambda_1+\lambda_2)-2\phi(\lambda_1)\,\phi(\lambda_2)\right].$$

The strong law of large numbers implies that $\hat{\phi}_n(\lambda)\to\phi(\lambda)$, almost surely, and hence estimators based on the empirical characteristic function are usually strongly consistent.

A number of estimates based on the empirical characteristic function are described in the literature. In the context of the stable distribution, Koutrouvelis (1980) proposed the least squares regression of $\log(-\log|\hat{\phi}_n(\lambda)|^2)$ on $\log|t|$, Press (1972) suggested a method of moments type estimator, Paulson et al. (1975) gave a minimal distance method of estimation, where the estimate is given by

$$\min_{\theta}\int_{-\infty}^{\infty}\left|\hat{\phi}_n(\lambda)-\phi(\lambda)\right|^2 e^{-\lambda^2}d\lambda$$

and the integral is approximated by Gauss-Hermite quadrature, and Feuerverger and McDunnogh (1981) proposed the k–L procedure. The method of moments type estimate, minimal distance estimate and k–L procedure can be written as the solution to an estimating equation

$$\sum_i a(\lambda_i;\theta)\left(\hat{\phi}_n(\lambda_i)-\phi(\lambda_i;\theta)\right)=0 \tag{4.1}$$

for particular choices of $a(\lambda;\theta)$. The theory of quasi-likelihood provides a framework in which an optimal choice for $a(\lambda;\theta)$ can be made within a given class of estimating functions such as (4.1) (see Heyde (1997)). Taking the sequence $\{\lambda_i\}$ as fixed, the optimal estimating equation within the class (4.1) is

$$Z(\theta)^T V^{-1}(\theta)\left(\hat{\phi}_n-\phi(\theta)\right)=0 \tag{4.2}$$

where

$$Z(\theta)_{ij}=\frac{\partial\phi(\lambda_i;\theta)}{\partial\theta_j},\quad V(\theta)_{ij}=\mathrm{cov}\left(\hat{\phi}_n(\lambda_i)\,,\hat{\phi}_n(\lambda_j)\right)$$

(from page 15 of Heyde (1997)), which is precisely the estimator obtained from the k–L procedure. It was proved by Feuerverger and McDunnogh (1981) that this estimate can be made to have arbitrarily high asymptotically efficiency. Using the quasi-likelihood framework it is possible to see that for any finite sample size the estimate is an approximate maximum likelihood estimate. Note that equation (4.2) can be written in the form

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{i} a(\lambda_i;\theta)\left(\cos(\lambda_i X_j) - \phi(\lambda_i;\theta)\right) = 0.$$

The quasi-score function minimises the distance to the true score function, that is, it minimises

$$E\left[\left(\frac{\partial \log f(X;\theta)}{\partial\theta} - \sum_{i} a(\lambda_i;\theta)\left(\cos(\lambda_i X_j) - \phi(\lambda_i;\theta)\right)\right)^2\right], \tag{4.3}$$

see Heyde (1997) page 13 for details. Now consider the space $L_2(f)$ of functions which a square integrable with respect to $f(x;\theta)$. This Hilbert space can be decomposed into the subspace of functions that are constant f–$a.e.$, denoted by H, and the subspace of functions orthogonal to it, that is

$$L_2(f) = H \oplus H^{\perp}.$$

Clearly, the score function belongs to the subspace $H^{\perp}$. It is known that the space of trigonometric functions is dense in $L_2(f)$. The function $\cos(\lambda x) - \phi(\lambda;\theta)$ is in $H^{\perp}$ as it is the result of $\cos(\lambda x)$ being made orthogonal to H. It follows that the functions $\cos(\lambda x) - \phi(\lambda;\theta)$ are dense in $H^{\perp}$ and hence (4.3) can be made arbitrarily small. In summary, the estimating equation (4.2) can be made arbitrarily close to the true score function by taking a sufficiently fine sequence of $\{\lambda_i\}$.

The estimating equation (4.2) can be solved iteratively given a good initial estimate θ_0 as follows:

$$Z(\theta_m)^T V(\theta_m)^{-1} Z(\theta_m)\,\delta_m = Z(\theta_m)^T V(\theta_m)^{-1}\left(\hat{\phi}_n - \phi(\theta_m)\right) \tag{4.4}$$

$$\theta_{m+1} = \theta_m + \delta_m$$

The resulting estimate is consistent, asymptotically normal with covariance matrix given by

$$E\ (\hat{\theta}-\theta)(\hat{\theta}-\theta)^T = \left[Z(\theta)^T V^{-1}(\theta) Z(\theta)\right]^{-1}.$$

Equation (4.4) is just generalised least squares and so the method can be easily implemented is most statistical packages. As the parameters of the Riesz-Bessel distribution need to satisfy certain constraints it is advisable to transform the parameters to remove these constraints, for example, set

$$\alpha^* = \log\left(\frac{\alpha}{1-\alpha}\right), \quad \gamma^* = \log\left(\frac{\alpha+\gamma}{1-\alpha-\gamma}\right), \quad t^* = \log(t).$$

and the appropriate changes made to the matrix Z.

The only choice to be made is the sequence $\{\lambda_i\}$ which is common to the other methods previously mentioned. A large number of ordinates at which the empirical characteristic function is computed will lead to more efficient estimates, however there is also an increase in computational cost and stability problems may arise with a near singular matrices. A simulation study was performed to assess the method in small samples for the Reisz-Bessel distribution. For each value of the parameter the estimation scheme was applied 50 times to a sample of size 250. The $\{\lambda_i\}$ was taken to be a sequence from 0.1 to 5 with spacing of 0.1 The results of the simulation study are reported in Table I.

Table I: Performance of estimator in the simulation study.

(α, γ, t)	Average	Standard Error
(0.9,-0.8,1)	(0.9050, -0.7936, 0.9930)	(0.0766, 0.1210, 0.1764)
(0.9,-0.6,1)	(0.9229, -0.6045, 0.9933)	(0.0673, 0.1406, 0.1712)
(0.9,-0.4,1)	(0.9524, -0.4247, 1.0001)	(0.0653, 0.1855, 0.1587)
(0.9,-0.2,1)	(0.9637, -0.2497, 1.0008)	(0.0526, 0.1879, 0.1431)
(0.7,-0.6,1)	(0.7276, -0.6253, 1.0083)	(0.0884, 0.1369, 0.1709)
(0.7,-0.4,1)	(0.7579, -0.4476, 1.0176)	(0.0797, 0.1513, 0.1553)
(0.7,-0.2,1)	(0.8269, -0.3072, 1.0080)	(0.0611, 0.1698, 0.1717)
(0.7,0.2,1)	(0.7098, 0.1376, 1.0482)	(0.0661, 0.1958, 0.1617)
(0.5,0.2,1)	(0.4826, 0.2672, 0.9589)	(0.0664, 0.2142, 0.1808)
(0.5,0.4,1)	(0.5181, 0.3335, 1.0633)	(0.0739, 0.2074, 0.1889)

We note that the bias appears to be considerable for small negative values of γ. The bias does not appear to be as great for small positive values of γ.

Finally, we illustrate the Riesz-Bessel distribution and the fitting procedure on the exchange rate of the Japanese Yen against the US dollar. The data were taken daily close during the period 12 December 1983 to 8 October 2001. The approach used here assumes that the Japanese Yen exchange rate follows an exponential transformation of a Riesz-Bessel-Lévy motion whose distribution is parameterised

$$\hat{p}(t,\lambda) = \exp\left[-\kappa t\,|\lambda|^{2\alpha}\left(c^2+\lambda^2\right)^{\gamma}\right], \quad \lambda \in R.$$

Admittedly, this ignores certain dependencies that financial data are known to display, however, the aim of this application is to demonstrate the fit of the distribution to some real data. Applying the estimation method described above the following parameter estimates were obtained with $t = 1$,

Table II: Parameter estimates for Japanese Yen exchange rate data.

κ	α	γ	c
(0.0057 (0.0015)	0.9999 (0.0015)	-0.4869 (0.0703)	181.1382 (24.1096)

The value of α being very close to one indicates that the distribution appears to have finite variance but not Gaussian as γ is significantly different from 0. A plot of the fitted Riesz-Bessel density with a non-parametric density estimate of the data is given below.

Figure 5: Sample density (solid curve) of the returns of Japanese Yen series and the fitted Riesz-Bessel density function (dashed curve).

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