Mental Computation: Refining the Cognitive Frameworks

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This paper reports on a study of Year 3 and 4 students' addition and subtraction mental computation. The purpose of the study was to investigate the cognitive aspects of the mental computation conceptual frameworks that the author formulated from a study of Year 3 students who were accurate in their mental computation. These frameworks had accounted for differing levels of flexibility in mental computation.

While international and national research (e.g., Maclellan, 2001; McIntosh, 1998; Reys, Reys, Nohda, &Emori, 1995) recognise the importance of including mental computation in mathematics curriculum that promotes number sense, in Queensland, mental computation features only as extended number facts (Department of Education, Queensland, 1987a). Although, in the *Year 4 Mathematics Sourcebook* (Department of Education, Queensland, 1987b), mental computation strategies are mentioned (e.g., 52+29=52+30-1; 82-29=82-30+1). Further, the document recognises that "different strategies can be used for one calculation" and "procedures for mental calculation do not necessarily follow the written algorithm" (p. 79). With the revision of a new (draft) syllabus, it is hoped that mental computation will be incorporated into the Number strand. At present, it is not clear whether mental computation refers to calculating *with* the head, rather than merely, *in* the head (Anghileri, 1999); that is, calculating using strategies dependant on the numbers involved, and calculating with understanding. Also, whether students should be encouraged to develop their own mental strategies, or be taught specific strategies (regardless of the numbers involved) is not clear. Morgan (2000) suggested teaching mental strategies in a sequential fashion over the primary school years, the reason being that permitting students to develop their own strategies might challenge teachers' sense of efficacy. Surely, objections on these grounds should be dismissed. Further, the bad old days of "mental arithmetic" (where children were presented with difficult calculations that must be calculated mentally and quickly) hopefully are not to return. Nor should mental computation be viewed merely as extended number facts. Rather, emphasis should be placed on strategic flexibility and students' exploring, discussing, and justifying their strategies and solutions (e.g., Blöte, Klein, & Beishuizen, 2000; Buzeika, 1999; Hedrén, 1999; Kamii & Dominick, 1998).

Some research has indicated that mental computation is situated in a richly connected web (e.g., Blöte, Klein, & Beishuizen, 2000; Heirdsfield, 2001a, b, c; Maclellan, 2001). Blöte, Klein, and Beishuizen (2000) suggested that teachers should build on students' existing knowledge to extend their conceptual structures for numbers, the relations between numbers, and the relations between numbers and procedures. Maclellan (2001) suggested that mental computation required the student to have "a knowledge of number relationships, a facility with basic facts, an understanding of arithmetical operations, the ability to make comparisons between numbers and possession of base-ten place value concepts" (p. 153). Heirdsfield (2001c) reported the conceptual structures which appear to be associated with proficient mental computation (i.e., flexible and accurate mental computation). Frameworks were originally developed to explain the differences among proficient, accurate (but not flexible), and inaccurate mental computers (Heirdsfield, 2001a). These have been reported elsewhere (e.g., Heirdsfield, 2002a, b; 2001b; Heirdsfield & Cooper, 2002). The findings of this study showed that Year 3 students who were proficient in mental computation (accurate and flexible) exhibited strategic flexibility, dependant on the number combinations of the problems. It was posited that an integrated understanding of mental strategies, number facts, numeration, and effect of operation on number supported strategic flexibility (and accuracy). Moreover, this cohort of students also exhibited some metacognitive strategies, possessed reasonable short-term memory and executive functioning, and held strong beliefs about their self developed strategies. Where students exhibited less knowledge and fewer connections between knowledge, Heirdsfield (2001b) found that students compensated in different ways, depending on their beliefs and what knowledge they possessed. For instance, students who had sufficient knowledge to support the ability to mentally compute accurately (although not necessarily efficiently) generally held strong beliefs about teacher taught strategies, and used these strategies to successfully obtain answers to mental computations. The focus of the study reported here was to refine the cognitive aspects of the conceptual frameworks that Heirdsfield (2001a) developed for accurate mental computation, both flexible and inflexible in Year 3 students (8 and 9 year olds), and map the frameworks for Year 4 students (9 and 10 year olds).

The Study

The research consisted of a series of interviews based on those used by Heirdsfield (2001a) to investigate strategic flexibility and proficiency in mental computation and associated factors. The current research extended to include Year 4 students, in the hope of mapping development across the two years.

Participants

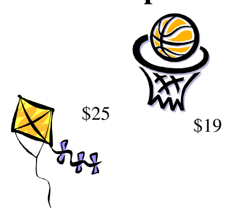
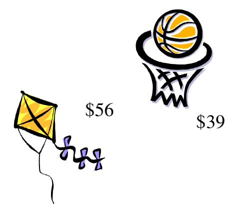
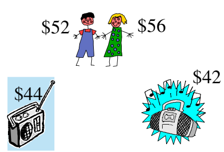
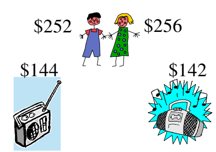
Initially, forty-one Year 3 students (from 4 classes) and thirty-three Year 4 students (from 3 classes) were selected by their teachers to form the initial cohort and participate in selection interviews (the teachers selected students they believed were reasonably proficient in mathematics). The students attended a Brisbane government school that served a middle socioeconomic area. All students participated in selection mental computation interviews, and from this cohort, eight Year 3 students and eight Year 4 students were selected on the basis of high accuracy (at least 80% of addition tasks were solved correctly), to participate in further indepth interviews. Further, of the eight accurate Year 4 students, four were identified as being flexible (and 4 inflexible), and of the eight accurate Year 3 students, six were flexible (and 2 were inflexible).

Instruments

The instruments were adapted from previously developed instruments (Heirdsfield, 2001a). The Year 3 instruments remained similar to those developed in Heirdsfield

(2001a); whereas, the instruments for Year 4 included tasks with more complex numbers. Examples of the instruments are presented in Table 1.

Table 1
Examples of tasks presented to Years 3 and 4 students

Year 3 tasks		Year 4 tasks	
Mental computation			
	What is the total cost of the basketball and the kite?		What is the total cost of the basketball and the kite?
Number facts			
8+2	7-2	8+2	7-2
6+7	9-3	6+7	9-3
9+7	16-7	9+7	16-7
Computational estimation			
	Your friend had $52 and spent $44 on the radio. You had $56 and spent $42 on the radio. Who had more money left?		Your friend had $252 and spent $144 on the radio. You had $256 and spent $142 on the radio. Who had more money left?
Numeration			
Writing and reading two-, three-, and four-digit numbers and canonical and non-canonical understanding (Sierink & Watson, 1991)			
"Write: 15, 54, 103, 690". "Read: 19, 83, 209, 560". What can you tell me about these numbers (underlined) – in as many ways as possible?		"Write: 103, 690, 1634". "Read: 209, 560, 1026". What can you tell me about these numbers (underlined) – in as many ways as possible?	
Effect of operation on number			
Given 43+26=69, solve without calculating		Given: 143+126=269, solve without calculating	
43+27	430+260	143+127	1430+1260
42+27	260+430	142+127	1260+1430
69-26	69-43	269-126	269-143
70-26	70-43	270-126	270-143
70-25	70-44	270-125	270-144
70-27	690-260	270-127	2690-1260

Procedure

For all interviews, the students were withdrawn, individually, from class and interviewed in a quiet room in the school. The indepth interviews consisted of three sessions: (1) number facts and mental computation, (2) computational estimation and numeration, and (3) effect of operation on number. Students were directed to solve the tasks and explain their solution strategies. In the mental computation interviews, if the students did not select what was considered an efficient mental strategy, they were also asked if they could think of alternative strategies.

Analysis

Mental computation strategies were analysed for strategy choice, flexibility (in terms of variety across all the mental computation tasks, and access to alternative strategies), and understanding of aspects of number sense (understanding of the effect of operation on number, numeration, computational estimation, and number facts). Analysis of the interviews investigating these aspects of number sense was undertaken with the intention of exploring connections with mental computation. Each student's results for cognitive factors were summarised and compared with the appropriate conceptual framework (accurate and flexible – see Figure 1; or accurate and inflexible). The inflexible framework was deplete (c.f., proficient framework) of estimation, effect of operation on number, number facts strategies, and aspects of numeration.

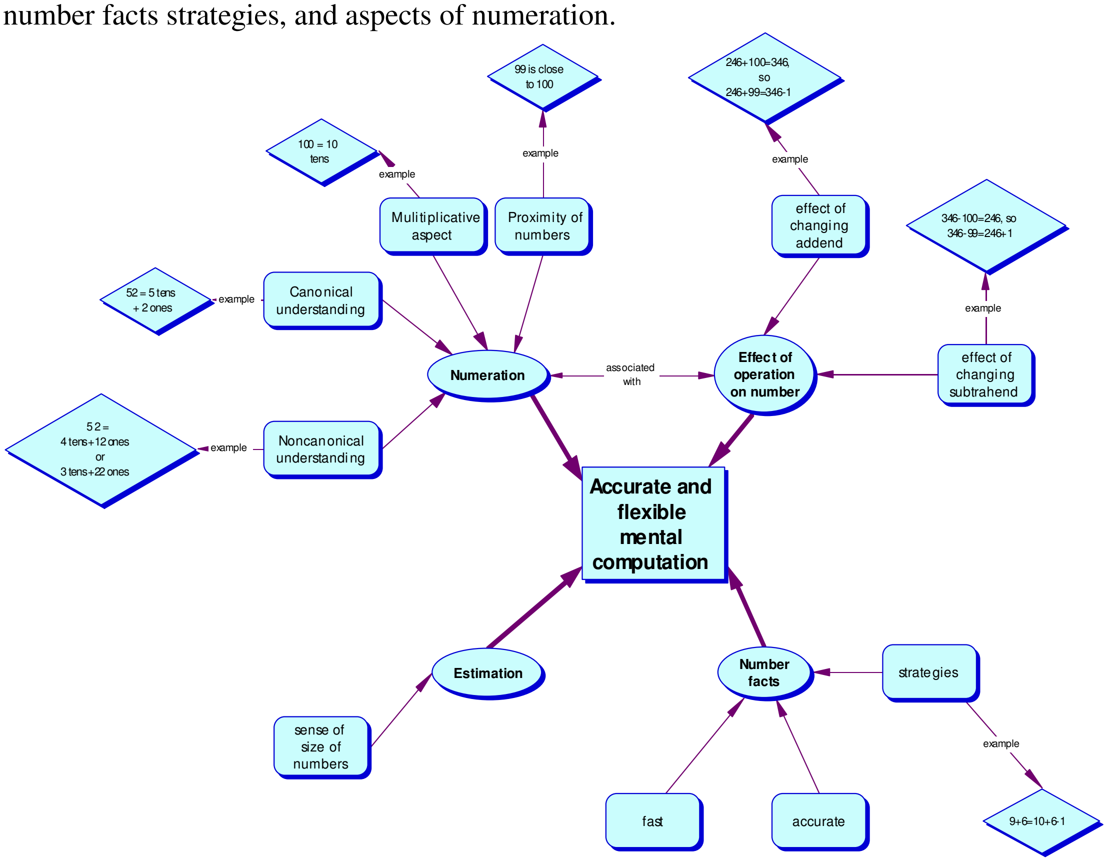


Figure 1. Cognitive aspects of conceptual framework for accurate and flexible mental computers (adapted from Heirdsfield, 2001a).

Findings

The results for the proficient (accurate and flexible) mental computers are described first, then the results for the accurate and inflexible mental computers are described, and finally, a comparison is made between the two. Year 3 and 4 students are reported together, as the findings for the two year levels were similar to each other.

Flexible mental computers

A variety of strategies was documented for the flexible computers, with each student employing at least three different strategies over the interview. These strategies included *separation* (e.g., 38+17: 30+10=40, 8+7 = 15 = 10+5, 40+10+5 = 55; 30+10=40, 40+8=48, 48+7=55), *aggregation* (e.g., 38+17: 38+10=48, 48+7 = 55), and *wholistic* (e.g., 38+17 = 40+17-2 = 57-2 = 55 or 35+20=55). All flexible students spontaneously employed *wholistic* at least once.

Computational estimation was poorly understood. Only one Year 4 student successfully completed all the estimation tasks. Others just guessed wildly, not being able to give any reasonable response, or unsuccessfully attempted to employ rounding or truncation. In fact, one student stated that estimation was not important.

No student in either Year 3 or Year 4 knew all their number facts by recall. All students solved the number facts test using *derived facts strategies* and occasionally used *count* for some subtraction examples. *Derived facts strategies* and *fact recall* were generally used for interim calculations in the mental computations. All students, except one Year 3 student, achieved one hundred percent accuracy in the number facts test. This Year 3 student was weaker in subtraction in both the number facts test and the mental computation items.

For the numeration tasks, all students were able to regroup/rename numbers, but they also required MAB material to give complete descriptions. Some appeared to treat different representations of number as if using a numeral expander.

Few students exhibited understanding of the effect of operation on number in the tasks specifically designed to investigate this. Although most students attempted to use previous examples to solve others, they were rarely successful, and often calculated the answers. However, all the students were able to use the concept of changing the addend and changing the subtrahend to solve mental computation tasks. Therefore, they appeared to have a working knowledge of these concepts.

Inflexible mental computers

The inflexible students generally employed a single mental strategy throughout their mental computation interview. This strategy was the pen and paper algorithm, performed mentally. One student was so comfortable with this strategy that she tried to solve 300-298 using the algorithm. Having trouble remembering the interim calculations, she decided that it was impossible to solve. When prompted to do so, the students could sometimes access *wholistic*, but not always successfully.

Without exception, performance in computational estimation was poor. There was a great deal of guessing and attempted calculations.

All students used *derived facts strategies* to solve the number facts tasks. Some *count* strategies were also used. One student also employed *count* for interim calculations in the mental computation tasks.

None of these students appeared confident with the numeration tasks. They were slow at representing numbers in a variety of ways, and needed to use MAB, sometimes having to count the blocks because they did not see relationships between the numbers.

No student was able to solve the tasks which addressed the effect of operation on number. They did not see connections among the expressions. In fact, one student refused to use previous examples to help solve another one, as he said it was cheating!

Comparison between flexible and inflexible mental computers

Although both groups of students were accurate, the flexible students used a variety of (mostly) efficient mental strategies, whereas the inflexible students employed a mental image of the pen and paper algorithm, predominantly. Computational estimation was poor in both groups; however, one flexible student exhibited good understanding. Students in both groups employed *derived facts strategies* in the number facts test. However, inflexible students also used *count* in the test and again in the mental computation interviews (for interim calculations). The inflexible students were also slower and less accurate than the flexible students in the number facts test. Numeration was not well understood by any student; however, the flexible students exhibited better understanding than the inflexible students. Finally, the effect of operation on number was not well understood by any of the students. However, some flexible students attempted to use previous examples to help them solve other examples. The flexible students exhibited a working knowledge of the principles of changing the addend and subtrahend in the mental computation tasks.

Discussion

In general, the flexible students possessed a more extensive and connected knowledge base than the inflexible students. However, overall, the scope of even the flexible students' knowledge base was not as extensive as those students who were investigated in Heirdsfield (2001a, b, c). In general, even though four of the flexible students were in Year 4, they were not as flexible with number as the Year 3 students who were investigated by Heirdsfield (2001a, b, c), nor were their numbers facts as proficient, nor their explicit understanding of the effect of operation on number. The refined cognitive frameworks are presented.

One can only guess at why this was so. There was never any intention to investigate classroom practice. However, from some of the statements made by students in the interviews, some of the teaching might not have been promoting student reasoning. For instance:

> You can't write a number like that. (Referring to 038 as the smallest number that can be made with the three digits).
>
> You can't do that. (Referring to 300-298).
>
> You can't take this from ….. (Referring to interim calculations in a subtraction examples).
>
> We don't discuss the strategies. (Referring to mental arithmetic done in class).
>
> I do them this way in class, but the teacher doesn't know. (Responding to a question about when he uses these strategies or has learnt the strategies)

> The teacher stopped me, because she realised I was too good. (Referring to being prevented from calculating mentally in class).

Although the flexible students did not exhibit a very extensive knowledge base to support mental computation (c.f., Heirdsfield, 2001a, b, c), they had nevertheless developed quite efficient mental strategies, without being taught. Therefore, it is posited that students do not need to be taught mental strategies, merely encouraged to use them. The results of this study also beg the question, why are students taught computational strategies if they can already solve computational tasks in arguably more efficient ways?

In contrast, the inflexible students had a very basic understanding of numeration and number facts (although they used *derived facts strategies* in the number facts test, they generally used *count* to solve interim calculations in the mental computation interviews). It is posited that some associated understandings are required for students to access/develop efficient mental strategies.

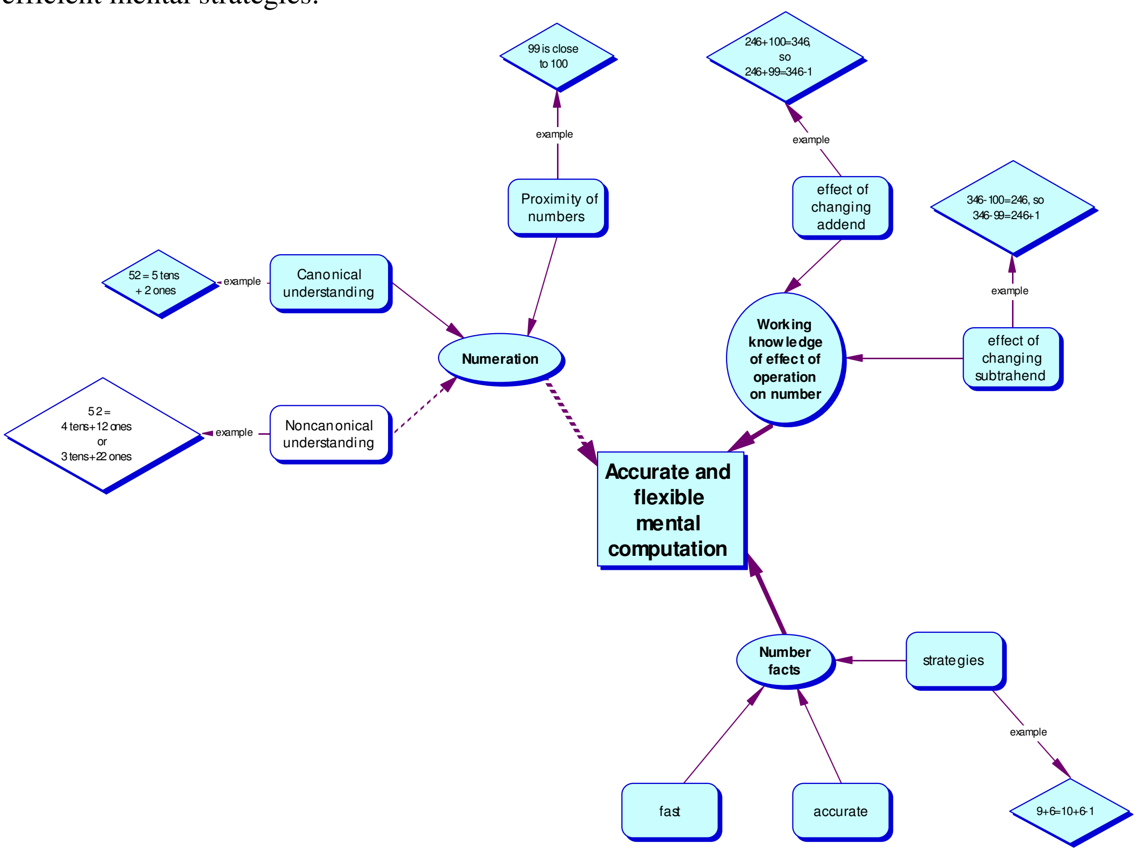


Figure 2. Revised cognitive conceptual framework for accurate and flexible mental computers (clear cells indicate some understanding; dashed lines indicate weak connections).

Finally, the refined framework for accurate and flexible computers is presented (Figure 2). It is diminished, compared with the original conceptual framework presented in Figure 1. While Heirdsfield (2001a, b, c) posited that students need a well integrated and extensive knowledge base to be proficient in mental computation, the findings of the present study do not support this. However, the accurate and flexible students exhibited more understandings than the inflexible students. It might be fair to say that some connected understandings are required. If students can exhibit proficiency in mental computation without supporting knowledge, then it is feasible that students can develop proficient mental computation strategies with less connected knowledge than was previously hypothesised. However, if the benefit of inclusion of mental computation in the curriculum is to promote the development of number sense, then it would still be advisable to provide opportunities for the development of other aspects of number sense, that were included in the previous conceptual framework (Figure 1).

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