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Abstract

We consider elliptic PDEs (partial differential equations) in the framework of isogeometric analysis, i.e., we treat the physical domain by means of a B-spline or NURBS mapping which we assume to be regular. The numerical solution of the PDE is computed by means of tensor product B-splines mapped onto the physical domain. We construct additive multilevel preconditioners and show that they are asymptotically optimal, i.e., the spectral condition number of the resulting preconditioned stiffness matrix is independent of h. Together with a nested iteration scheme, this enables an iterative solution scheme of optimal linear complexity. The theoretical results are substantiated by numerical examples in two and three space dimensions.

Keywords: Isogeometric analysis, elliptic PDE, B-splines, multilevel preconditioning, BPX-preconditioner, uniformly bounded condition number.

1. Introduction

Isogeometric analysis as introduced in [15] employs modern techniques from computer aided geometric design for the solution of PDEs on general domains which can be represented as unions of parametric mappings of squares or cubes. The physical domain is represented in terms of splines or NURBS and the same description is adopted for unknown fields.

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Typically, isogeometric methods employ B-splines of degree higher than one in order to generate highly accurate solutions. In this paper, our focus is on the construction of optimal preconditioners for isogeometric discretizations of elliptic PDEs. Our model problem will be the second order Laplacian with homogeneous Dirichlet boundary conditions,

$$-\Delta u = f \quad \text{on } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

where $\partial\Omega$ denotes the boundary of $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and f is any square integrable function. More generally, we will seek approximate solution of a problem in variational form:

$$a(u, v) = \int_\Omega f\, v \tag{2}$$

where $H_0^1(\Omega)$ denotes the subset of $H^1(\Omega)$ with homogenous Dirichlet boundary conditions, $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is continuous and coercive, and $f \in l^2(\Omega)$.

In fact, the theory we propose covers any elliptic operator of even order $2r$ (for r positive integer) and, for the sake of generality, we adopt this general setting in the presentation. In this general setting, the model problems is of the form (2) with $a(\cdot,\cdot) : H_0^r(\Omega) \times H_0^r(\Omega) \to \mathbb{R}$, $r \geq 1$ verifying the assumptions declared in Section 2. Here, $H_0^r(\Omega)$ denotes the subset of $H^r(\Omega)$ with homogenous essential boundary conditions.

Essential, for a favorable performance of the discretization method for (1), is a fast solution scheme for the final, large, linear system which has to be solved. In view of its size and the sparsity and structure of the system matrix, one typically employs an iterative solver for the resulting linear system of equations whose convergence speed depends on the spectral condition number $\kappa_2(\mathbf{A})$ of the system matrix $\mathbf{A}$. For any discretization on a grid of grid spacing $0 < h < 1$, this condition number grows like $C(p)h^{-2}$, where $C(p)$ is a constant growing with p which is the degree of the isogeometric approximation. The dependence on h induces a dramatic increase of the number of iterations to reach discretization error accuracy as the grid size h decreases. A remedy to overcome this problem is to employ a *preconditioner* $\mathbf{C}$ for $\mathbf{A}$ whose set-up, storage and application is of linear complexity of the number of unknowns N but for which $\kappa_2(\mathbf{CA}) \ll \kappa_2(\mathbf{A})$. The ideal case when $\kappa_2(\mathbf{CA})$ is proportional to a constant independent of h can be achieved by preconditioners of multilevel form. We call this an *(asymptotically) optimal preconditioner*. The type of schemes for which this can be shown are the so-called additive preconditioners like the wavelet preconditioner and the BPX-preconditioner [9] whose optimality was proved independently in [10, 20], multiplicative versions like multigrid [7, 13], and algebraic multilevel iteration (AMLI) methods [1, 2]. Multigrid preconditioners for isogeometric analysis have been analysed in [12], whereas domain decomposition type preconditioners have been proposed in [4, 5, 16].

Note that the hierarchical basis (HB-)preconditioner proposed in [22] does not have this optimality property: for problems of order $r = 1$ on two-dimensional domains, $\kappa_2(\mathbf{CA})$ still grows like $|\log(h)|$. To illustrate this effect, in [19], BPX-type preconditioners were presented together with proofs of optimality for second and fourth order problems on the two-sphere $S \subset \mathbb{R}^3$, involving the Laplace-Beltrami operator Δ_S and Δ_S^2. There also numerical computations were displayed to illustrate the effect of the BPX- versus the HB-preconditioning for $\mathcal{C}^0$ and $\mathcal{C}^1$ finite elements.

Our construction of optimal multilevel preconditioners will rely on tensor products so that principally any space dimension $d \in \mathbb{N}$ is permissible as long as storage permits; we will, however, mostly consider the cases $d = 2, 3$. As discretization space, we choose in each spatial direction B-splines of (the same) degree p on quasi-uniform grids and with maximal smoothness; all just for notational convenience.

The remainder of this paper is organized as follows. In the next section, we introduce the mathematical framework in terms of tensor product B-splines and some necessary tools from approximation theory like direct and inverse inequalities. We propose additive multilevel preconditioners including BPX-type versions in Section 3 and prove their optimality. Section 4 contains some numerical results confirming the theory. We conclude in Section 5 with a short summary and some outlook.

2. Construction of the discrete problem

Throughout this paper, we assume that the bilinear form $a(\cdot,\cdot) : H_0^r(\Omega) \times H_0^r(\Omega) \to \mathbb{R}$ appearing in (2) is symmetric, continuous and coercive, i.e., there exist constants $0 < c_A \leq C_A < \infty$ such that the induced self-adjoint operator $\langle Av, w\rangle := a(v, w)$ satisfies the isomorphism relation

$$c_A \|v\|_{H^r(\Omega)} \leq \|Av\|_{H^{-r}(\Omega)} \leq C_A \|v\|_{H^r(\Omega)}, \quad v \in H_0^r(\Omega). \tag{3}$$

Here, $H^{-r}(\Omega)$ stands for the dual of $H_0^r(\Omega)$ with respect to the pivot space $L_2(\Omega)$ and $\langle \cdot, \cdot \rangle$ for the respective dual form. We assume that r is a strictly positive integer even though the more general case of $r > 0$ and real can be covered with minor technical changes. The parameter $2r$ denotes the order of the PDE operator. If the precise format of the constants in (3) does not matter, we abbreviate this relation as $\|v\|_{H^r(\Omega)} \lesssim \|Av\|_{H^{-r}(\Omega)} \lesssim \|v\|_{H^r(\Omega)}$, or shortly as $\|Av\|_{H^{-r}(\Omega)} \sim \|v\|_{H^r(\Omega)}$. Under these conditions, Lax-Milgram's theorem guarantees that, for any given $f \in H^{-r}(\Omega)$, the operator equation

$$Au = f \quad \text{ in } H^{-r}(\Omega) \tag{4}$$

has a unique solution $u \in H_0^r(\Omega)$.

In order to approximate the solution of (2) or (4), we choose a finite-dimensional subspace $V_h \subset H_0^r(\Omega)$. We will construct these approximation spaces by using tensor product splines. We first revise the main definitions and fix the notation.

2.1. B-splines, geometry and push forward

Given two positive integers p and n, we say that $\Xi := \{\xi_1, \ldots, \xi_{n+p+1}\}$ is a p-open knot vector if

$$0 = \xi_1 = \ldots = \xi_{p+1} < \xi_{p+2} \leq \ldots \leq \xi_n < \xi_{n+1} = \ldots = \xi_{n+p+1} = 1,$$

where repeated knots are allowed, but all internal knots have multiplicity less than or equal to $p - r + 1$. From the knot vector Ξ, B-spline functions of degree p are defined following the well-known Cox-DeBoor recursive formula. We start with piecewise constants ($p = 0$):

$$N_{i,0}(\zeta) = \begin{cases} 1, & \text{if } \xi_i \le \zeta < \xi_{i+1}, \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

and for $p \ge 1$ the *B-spline* functions are defined as

$$N_{i,p}(\zeta) = \frac{\zeta - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\zeta) + \frac{\xi_{i+p+1} - \zeta}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\zeta). \tag{6}$$

This gives a set of n B-splines that form a basis of the space of *splines*, that is, piecewise polynomials of degree p with $p - m_j$ continuous derivatives at the internal knots ξ_j, for $j = p + 2, \ldots, n$, where m_j is the multiplicity of the knot ξ_j. This means the chosen B-splines functions are at least C^0 when $r = 1$ (i.e., the case of our numerical tests in Section 4) and C^{r-1} in general. In what follows, we attach the index r to objects and spaces to remind that, depending on r, the considered B-splines functions have different minimal regularity.

Notice moreover, that the B-spline function $N_{i,p}$ is supported in the interval $[\xi_i, \xi_{i+p+1}]$, and in fact its definition only depends on the knots within that interval.

In dimension d, $d = 2, 3$, the space of B-splines is obtained by tensor product construction. To fix ideas, let us consider $d = 3$. Indeed, let Ξ_ℓ, $\ell = 1, \ldots, 3$, be three open knot vectors of length $n_\ell + p + 1$, and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$, we define:

$$N_{i_1,i_2,i_3;p}(\boldsymbol{\zeta}) = N_{i_1,p}(\zeta_1) N_{i_2,p}(\zeta_2) N_{i_3,p}(\zeta_3), \quad 0 \le \zeta_\ell \le 1, \quad 1 \le i_\ell \le n_\ell.$$

The trivariate B-splines functions above, for the sake of convenience, are also denoted as

$$B_i(\boldsymbol{\zeta}) = N_{i_1,i_2,i_3;p}(\boldsymbol{\zeta}), \quad i = i_1 + n_1(i_2 - 1) + n_1 n_2 (i_3 - 1) \in \mathbb{I} = \{1, 2, \ldots, n_1 n_2 n_3\} \tag{7}$$

and are defined on the cube $\hat{\Omega} = (0,1)^3$. We define:

$$S_h(\hat{\Omega}) = \operatorname{span}\{B_i(\boldsymbol{\zeta}),\, i \in \mathbb{I}\}. \tag{8}$$

We denote by $\mathcal{Q}_h$ the tensor product mesh composed of all non-empty elements of the so-called knot mesh, i.e., the tensor product mesh having as vertices the points $\Xi_1 \times \Xi_2 \times \Xi_3$ and h stands for the maximum diameter of the elements Q of the tensor product mesh $\mathcal{Q}_h$. From now on we suppose that $\mathcal{Q}_h$ is a shape regular and quasi uniform mesh in the following sense: there exists a constant C such that

$$Ch \le \operatorname{diam}(Q), \text{ and } Ch \le |Q|^{1/d}, \quad \forall Q \in \mathcal{Q}_h. \tag{9}$$

In the spirit of isogeometric analysis, we suppose that also the computational domain is described in terms of B-spline functions. We suppose then that the computational domain

Ω is the image of a mapping $\mathbf{F} : \hat{\Omega} \to \Omega$ where each component F_i of $\mathbf{F}$ belongs to $S_{\bar{h}}(\hat{\Omega})$ for some given $\bar{h}$. In most of applications, the geometry can be described in terms of a very coarse mesh, namely $\bar{h} \gg h$. Then, a natural assumption in this context is that $\mathbf{F}$ is invertible and verifies:

$$\|D^\alpha \mathbf{F}\|_{L^\infty(\hat{\Omega})} \le C_{\mathbf{F}}, \quad \|D^\alpha \mathbf{F}^{-1}\|_{L^\infty(\Omega)} \le c_{\mathbf{F}}^{-1}, \qquad |\alpha| \le r \tag{10}$$

where $c_{\mathbf{F}}$ and $C_{\mathbf{F}}$ are independent constants bounded away from 0.

Indeed, this assumption on the geometry could be weakened a lot: (i) the mapping $\mathbf{F}$ can be a piecewise C^∞ function on the mesh $\mathcal{Q}_{\bar{h}}$, independent of the h, with the same inter-element regularity as the splines in $S_{\bar{h}}(\hat{\Omega})$ and, (ii) the domain Ω can have a *multi-patch* representation, that is, it can be the union of Ω_k, each one parametrized by a spline mapping of the unit cube. The theory presented here would apply also in this more general setting, as long as the spline discretization space on the physical domain Ω remains conforming in $H_0^r(\Omega)$.

2.2. Properties of the ansatz spaces

With the above definitions at hand, we are ready to define appropriate ansatz spaces. We define the discrete space

$$V_h^r := \{v_h : \ v_h \circ \mathbf{F} \in S_h(\hat{\Omega})\} \cap H_0^r(\Omega), \tag{11}$$

where the intersection with $H_0^r(\Omega)$ is used to incorporate boundary conditions. We have three important properties which will play a crucial role later on for the construction of the preconditioners. For this purpose, we suppose from now on that the B-spline basis is L^2-*normalized*, i.e., that it holds

$$\|B_i\|_{L^2(\hat{\Omega})} \sim 1, \text{ and thus also } \|B_i \circ \mathbf{F}^{-1}\|_{L^2(\Omega)} \sim 1 \text{ for all } i \in \mathbb{I}, \tag{12}$$

(see Section 4.1 for the concrete choice of the normalization factor in the context of nested spaces.)

Theorem 1. *Let* $\{B_i\}_{i\in\mathbb{I}}$ *be the B-spline basis defined in* (7)*, normalized as in (12),* $N = \#\mathbb{I}$ *and* V_h^r *defined in* (11)*. Then:*

(S) (Uniform stability with respect to $L_2(\Omega)$*) for any* $\mathbf{c} \in \ell_2$*,*

$$\left\| \sum_{i=1}^N c_i \, B_i \circ \mathbf{F}^{-1} \right\|_{L_2(\Omega)}^2 \sim \sum_{i=1}^N |c_i|^2 =: \|\mathbf{c}\|_{\ell_2}^2 \tag{13}$$

with constants independent of h *and* $\mathbf{c} := (c_i)_{i=1,\dots,N}$ *but depending on* $\mathbf{F}$ *(that is,* Ω*),* p *and the spatial dimension* d*;*

(J) (direct or Jackson estimates)

$$\inf_{v_h \in V_h^r} \|v - v_h\|_{L_2(\Omega)} \lesssim h^s \, |v|_{H^s(\Omega)}, \qquad \textit{for any } v \in H^s(\Omega), \quad 0 \leq s \leq r, \tag{14}$$

where $|\cdot|_{H^s(\Omega)}$ *denotes the Sobolev seminorm of highest weak derivatives* s*; the constant depends on* $\mathbf{F}$ *(that is* Ω*),* p *and* d*;*

(B) (inverse or Bernstein estimates)

$$\|v_h\|_{H^s(\Omega)} \lesssim h^{-s} \|v_h\|_{L_2(\Omega)} \qquad \textit{for any } v_h \in V_h^r \textit{ and } 0 \leq s \leq r; \tag{15}$$

the constant depends on $\mathbf{F}$ *(that is* Ω*),* p *and* d*.*

Proof. The proof for (13) with respect to $\hat{\Omega}$ is classical and can be found, e.g., in [11]; then by construction and by (10) we have, for all $v \in L_2(\Omega)$,

$$\|v\|^2_{L_2(\Omega)} = \int_{\hat{\Omega}} (v(\mathbf{F}(\mathbf{x})))^2 |\det(D\mathbf{F}(\mathbf{x}))| d\mathbf{x} \sim \|v \circ \mathbf{F}\|^2_{L_2(\hat{\Omega})}$$

The proof of estimates (14) and (15) are a special cases of the ones in, e.g., [3, 6, 21].

We consider now the following discretization of the abstract problem (4):

$$\text{Find } u \in V_h^r \; : \; a(u,v) = \langle f, v \rangle \quad \forall v \in V_h^r; \tag{16}$$

as a particular case, the discretization of (1) reads:

$$\text{Find } u \in V_h^1 \; : \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x} = \int_\Omega f \, v \, d\mathbf{x} \quad \forall v \in V_h^1. \tag{17}$$

In the next section, we construct the classical BPX-preconditioner for these problems and show their optimality.

3. Additive multilevel preconditioners

The construction of optimal preconditioners are based on a *multiresolution analysis* of the underlying energy function space $H_0^r(\Omega)$. As before, $2r$ stands for the order of the PDEs we are solving, and is always kept fixed.

3.1. Abstract framework

It will be convenient to first describe the necessary ingredients within an abstract basis-free framework, see, e.g., [11]; we specify in Subsection 3.2 the realization for the parametrized tensor product spaces in (11).

Let $\mathcal{V}$ be a sequence of strictly nested spaces V_j, starting with some fixed coarsest index $j_0 > 0$ (determined by the polynomial degree p which determines the support of the basis functions) and terminating with a highest index J,

$$V_{j_0} \subset V_{j_0+1} \subset \cdots \subset V_j \subset \cdots \subset V_J \subset H_0^r(\Omega). \tag{18}$$

The index j will later be identified as the level of resolution defining approximations on a grid with dyadic grid spacing $h = 2^{-j}$, i.e., we replace V_h by V_j, and V_J will be the space relative to the finest grid 2^{-J}. We associate with $\mathcal{V}$ a sequence of linear projectors $\mathcal{P}$ with the following properties.

Properties 2. *We assume that:*

(P1) P_j *maps* $H_0^r(\Omega)$ *onto* V_j,

(P2) $P_j P_\ell = P_j$ *for* $j \leq \ell$,

(P3) $\mathcal{P}$ *is uniformly bounded on* $L_2(\Omega)$, *i.e.,* $\|P_j\|_{L_2(\Omega)} \lesssim 1$ *for any* $j \geq j_0$ *with a constant independent of* j.

These conditions are satisfied, for example, for $L_2(\Omega)$-orthogonal projectors, or, in the case of splines, for the quasi-interpolant proposed and analysed in [21, Chapter 4]. The second condition (P2) ensures that the differences $P_j - P_{j-1}$ are also projectors for any $j > j_0$. Next we define a sequence $\mathcal{W} := \{W_j\}_{j \geq j_0}$ of complement spaces

$$W_j := (P_{j+1} - P_j) V_{j+1} \tag{19}$$

which then yields the decomposition

$$V_{j+1} = V_j \oplus W_j. \tag{20}$$

Thus, for the finest level J, we can express V_J in its *multilevel decomposition*

$$V_J = \bigoplus_{j=j_0-1}^{J-1} W_j \tag{21}$$

upon setting $W_{j_0-1} := V_{j_0}$. Setting $P_{j_0-1} := 0$, the corresponding representation of any $v \in V_J$ is then

$$v = \sum_{j=j_0}^{J} (P_j - P_{j-1}) v. \tag{22}$$

We now have the following result which will be used later for the proof of the optimality of the multilevel preconditioners.

Theorem 3. *Let* $\mathcal{P}, \mathcal{V}$ *be as above where, in addition, we require that for each* V_j, $j_0 \leq j \leq J$, *a Jackson and Bernstein estimate as in Theorem 1 (J) and (B) hold with* $h = 2^{-j}$. *Then one has the function space characterization*

$$\|v\|_{H^r(\Omega)} \sim \left(\sum_{j=j_0}^{J} 2^{2rj} \|(P_j - P_{j-1}) v\|_{L_2(\Omega)}^2 \right)^{1/2} \qquad \textit{for any } v \in V_J. \tag{23}$$

Such a result holds, in fact, for a much larger class of function spaces (so-called Besov spaces which are subsets of $L_q(\Omega)$ for general q different from 2) and for any function $v \in H^r(\Omega)$ (then with an infinite sum on the right hand side), see, e.g. [11].

We demonstrate next how to exploit the norm equivalence (23) in the construction of an optimal multilevel preconditioner. Define for any $v, w \in V_J$ the linear self-adjoint positive-definite operator $C_J : V_J \to V_J$ given by

$$(C_J^{-1} v, w)_{L_2(\Omega)} := \sum_{j=j_0}^{J} 2^{2rj} \left((P_j - P_{j-1})v, (P_j - P_{j-1})w\right)_{L_2(\Omega)}, \tag{24}$$

which we denote as multilevel *BPX-type preconditioner* and let $A_j : V_j \to V_j$ be the finite-dimensional operator defined by $(A_j v, w)_{L_2(\Omega)} := a(v, w)$ for all $v, w \in V_j$.

Theorem 4. *With the same prerequisites as in Theorem 3, C_J is an asymptotically optimal symmetric preconditioner for A_J, i.e., $\kappa_2(C_J^{1/2} A_J C_J^{1/2}) \sim 1$ with constants independent of J.*

Proof. For the parametric domain $\hat{\Omega}$, the result was proved independently in [10, 20] and is based on the combination of (23) with the well-posedness of the continuous problem. The result on the physical domain follows then together with (10). □

Concrete realizations of this preconditioner based on B-splines lead to representations of the complement spaces W_j whose bases are called *wavelets*. For these, efficient implementations of optimal linear complexity involving the Fast Wavelet Transform can be derived, see, e.g., [10, 18].

However, since the order of the PDE operator r is positive, we can use here the argumentation from [9] which ultimately will allow to work with the same basis functions as for the spaces V_j. The first part of the argument relies on the assumption that the P_j are L^2-*orthogonal* projectors. For a clear distinction, we shall use the notation O_j for L^2-orthogonal projectors and reserve the notation P_j for linear operators. Then, the BPX-type preconditioner (24) reads as

$$C_J^{-1} := \sum_{j=j_0}^{J} 2^{2jr}(O_j - O_{j-1}), \tag{25}$$

which is by Theorem 4 a BPX-type preconditioner for the self-adjoint positive definite operator A_J. By the orthogonality of the projectors O_j, we can immediately derive from (25) that

$$C_J = \sum_{j=j_0}^{J} 2^{-2jr}(O_j - O_{j-1}). \tag{26}$$

Since $r > 0$, by rearranging the sum, the exponentially decaying scaling factors allow to replace C_J by the spectrally equivalent operator

$$C_J = \sum_{j=j_0}^{J} 2^{-2jr} O_j. \tag{27}$$

Recall that in this setting two linear operators $\mathcal{A} : V_J \to V_J$ and $\mathcal{B} : V_J \to V_J$ are called *spectrally equivalent* if they satisfy, uniformly in the number of levels J,

$$\frac{(\mathcal{A}v, v)_{L_2(\Omega)}}{(v, v)_{L_2(\Omega)}} \sim \frac{(\mathcal{B}v, v)_{L_2(\Omega)}}{(v, v)_{L_2(\Omega)}} , \quad v \in V_J. \tag{28}$$

Thus, the realization of the preconditioner is reduced to a computation in terms of the bases of the spaces V_j instead of W_j. The orthogonal projector O_j can, in turn, be replaced by a simpler local operator which is spectrally equivalent to O_j, see [17].

3.2. BPX for isogeometric analysis

Up to this point, the discussion of multilevel preconditioners has been basis-free. We now show how this framework can be used to construct a BPX-preconditioner for the linear systems deriving from the problem (16). To this aim, we need to construct a sequence of spaces satisfying (18), and such that $V_J = V_h^r$. Suppose that for each space dimension, we are given with a sequence of knot vectors $\Xi_{j_0,\ell}, \ldots, \Xi_{J,\ell} = \Xi_\ell$, $\ell = 1, 2, 3$, such that:

- $\Xi_{j_0,\ell}$, $\ell = 1, 2, 3$, provide (up to repetitions) a quasi-uniform partition of the segment $(0, 1)$;
- $\Xi_{j,\ell} \subset \Xi_{j+1,\ell}$, $j = j_0, j_0 + 1, \ldots, J$;
- all knot vectors $\Xi_{j,\ell}$ are open;
- the knot vectors $\Xi_{j+1,\ell}$ are obtained by $\Xi_{j,\ell}$ by diadic refinement.

An immediate consequence of the above is that the corresponding meshes $\mathcal{Q}_{h_j}$ (composed of all non empty elements of knot meshes) are quasi uniform and shape regular (see (9)) with mesh size $h \sim h^{-j}$. Moreover, if we consider the sequence of splines spaces $S_j(\hat{\Omega})$ defined on the knot vectors $\Xi_{j,\ell}$, we have that $\mathbf{F} \in S_{j_0}(\hat{\Omega})$.

Notice that it holds:

$$S_{j_0}(\hat{\Omega}) \subset S_{j_0+1}(\hat{\Omega}) \subset \ldots \subset S_J(\hat{\Omega})$$

and, setting $V_j^r(\Omega) = \{v \in H_0^r(\Omega) : \ v \circ \mathbf{F} \in S_j(\hat{\Omega})\}$, it also holds:

$$V_{j_0}^r(\Omega) \subset V_{j_0+1}^r(\Omega) \subset \ldots \subset V_J^r(\Omega).$$

Setting $\mathbb{I}_j := \{1, \ldots, \dim(S_j(\hat{\Omega})\}$, we denote by B_i^j, $i \in \mathbb{I}_j$ the set of L_2-normalized B-spline basis functions for the space $S_j(\hat{\Omega})$. Define now the positive definite operator $P_j : L^2(\Omega) \to V_j^r$

$$P_j = \sum_{i \in \mathbb{I}_j} (\,\cdot\,, B_i^j \circ \mathbf{F}^{-1})_{L_2(\Omega)}\, B_i^j \circ \mathbf{F}^{-1}. \tag{29}$$

Corollary 5. *For the basis $\{B_i^j \circ \mathbf{F}^{-1},\ i \in \mathbb{I}_j\}$, the operators P_j defined above and the L^2-projectors O_j are spectally equivalent for any j.*

Proof. The assertion follows by combining (10), (13), with Remark 3.7.1 from [17], see [9] for the main ingredients. □

Finally, we obtain an explicit representation of the preconditioner C_J in terms of the mapped spline bases of V_j, $j = j_0, \ldots, J$,

$$C_J = \sum_{j=j_0}^{J} 2^{-2jr} \sum_{i \in \mathbb{I}_j} (\,\cdot\,, B_i^j \circ \mathbf{F}^{-1})_{L_2(\Omega)}\, B_i^j \circ \mathbf{F}^{-1}. \tag{30}$$

Remark 6. *The* hierarchical basis (HB) preconditioner *introduced in two spatial dimensions in [22] for piecewise linear B-splines fits into this framework by choosing* Lagrangian interpolants *in place of the projectors P_j in (24). However, since they do not satisfy (P3) in Properties 2, they are not asymptotically optimal for $d \geq 2$. Specifically, for $d = 3$, this preconditioner does not have an effect at all.*

Remark 7. *So far we have not explicitly thematized the dependence of the preconditioned system on p. Since all estimates in Theorem 1 which enter the proof of optimality depend on p, it is to be expected that the absolute values of the condition numbers, i.e., the values of the constants, depend on p and increases with p. Indeed, in the next section, we will propose a series of numerical tests which also aim at studying this dependance.*

4. Numerical Results

In this section, we describe the implementation of the BPX-preconditioner and we test it for the model problem (1). From now on $r = 1$.

4.1. Prolongations and restrictions

As the main ingredient of the BPX-preconditioner, we need to define prolongation and restriction operators. Since $V_j \subset V_{j+1}$, each B-spline B_i^j on the level j can be represented by a linear combination of B-splines B_k^{j+1} on the level $j+1$. Arranging the B-splines into a vector $\mathbf{B}^j := (B_1^j, \ldots, B_{N_j}^j)^T$, this relation denoted as *refinement relation* can be written as

$$\mathbf{B}^{j+1} = \mathbf{I}_j^{j+1} \mathbf{B}^j \tag{31}$$

with *prolongation operator* $\mathbf{I}_j^{j+1}$ from the trial space V_j to the trial space V_{j+1}. The restriction $\mathbf{I}_{j+1}^j$ is then simply defined as the transposed operator, i.e., $\mathbf{I}_{j+1}^j = (\mathbf{I}_j^{j+1})^T$.

In case of piecewise linear B-splines, our definition coincides with the well known prolongation and restriction operators from finite element textbooks. We thus shall exemplify the construction in case of C^1 quadratic and C^2 cubic B-splines on the interval, see, e.g., [8]. To this end, we equidistantly subdivide the interval $[0, 1]$ into 2^j subintervals. We obtain 2^j and $2^j + 1$ B-splines in case of the quadratic and cubic spline space V_j which is given on this partition, respectively, see Figure 1 for an illustration. Note that the two boundary functions which do not vanish at the boundary are removed, in order to guarantee that V_j

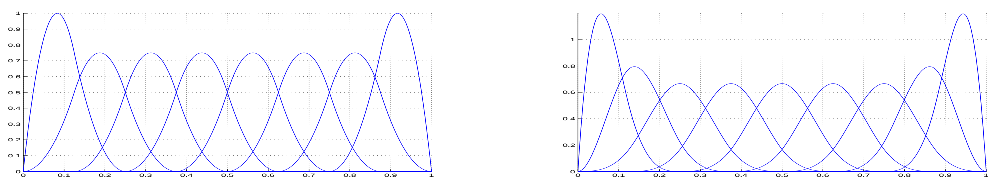

Figure 1: Quadratic (left) and cubic (right) L^2-normalized (12) B-splines on level $j = 3$ on the interval $[0, 1]$, yielding the basis functions for $V_j \subset H_0^1(\Omega)$.

is a subspace of $H_0^1(\Omega)$. Moreover, recall that the B-splines are L^2 normalized (12) which means that B_i^j is of the form $B_i^j(\zeta) = 2^{j/2} B(2^j \zeta - i)$ if B_i^j is an interior function, and correspondingly for the boundary functions.

In case of quadratic B-Splines, the restriction $\mathbf{I}_{j+1}^j$ reads as

$$\mathbf{I}_{j+1}^j = 2^{-1/2} \begin{bmatrix} \frac{1}{2} & \frac{9}{8} & \frac{3}{8} & & & & & & & \\ & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & & & & & \\ & & & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & \\ & & & & & & & \frac{3}{8} & \frac{9}{8} & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{2^j \times 2^{j+1}},$$

and, in case of cubic B-Splines, as

$$\mathbf{I}_{j+1}^j = 2^{-1/2} \begin{bmatrix} \frac{1}{2} & \frac{9}{8} & \frac{3}{8} & & & & & & & & & \\ & \frac{1}{4} & \frac{11}{12} & \frac{2}{3} & \frac{1}{6} & & & & & & & \\ & & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & & & & & \\ & & & & \ddots & \ddots & \ddots & & & & & \\ & & & & & \frac{1}{8} & \frac{1}{2} & \frac{3}{4} & \frac{1}{2} & \frac{1}{8} & & \\ & & & & & & & \frac{1}{6} & \frac{2}{3} & \frac{11}{12} & \frac{1}{4} & \\ & & & & & & & & & \frac{3}{8} & \frac{9}{8} & \frac{1}{2} \end{bmatrix} \in \mathbb{R}^{(2^j+1)\times(2^{j+1}+1)}.$$

Note that the normalization factor $2^{-1/2}$ stems from the L^2-normalization (12) of the B-splines. The matrix entries are scaled in the usual fashion such that their rows sum to two.

From these one-dimensional restriction operators, we obtain the related restriction operators on arbitrary unit cubes $[0,1]^d$ via tensor products. Finally, we set $\mathbf{I}_j^J := \mathbf{I}_{J-1}^J \mathbf{I}_{J-2}^{J-1} \cdots \mathbf{I}_j^{j+1}$ and $\mathbf{I}_J^j := \mathbf{I}_{j+1}^j \mathbf{I}_{j+2}^{j+1} \cdots \mathbf{I}_J^{J-1}$ to define the prolongations and restrictions between arbitrary levels.

4.2. Discretized BPX-preconditioner

For given functions $u_J = \sum_{k\in\mathbb{I}_J} u_{J,k}\, B_k^J \circ \mathbf{F}^{-1} \in V_J$ and $v_J = \sum_{\ell\in\mathbb{I}_J} v_{J,\ell}\, B_\ell^J \circ \mathbf{F}^{-1} \in V_J$, we conclude from (30) that

$$\begin{aligned}(C_J u_J, v_J)_{L^2(\Omega)} &= \sum_{k,\ell\in\mathbb{I}_J} u_{J,k} v_{J,\ell} (C_J(B_k^J \circ \mathbf{F}^{-1}), B_\ell^J \circ \mathbf{F}^{-1})_{L^2(\Omega)} \\ &= \sum_{k,\ell\in\mathbb{I}_J} u_{J,k} v_{J,\ell} \sum_{j=j_0}^{J} 2^{-2j} \sum_{i\in\mathbb{I}_j} (B_k^J \circ \mathbf{F}^{-1}, B_i^j \circ \mathbf{F}^{-1})_{L_2(\Omega)} \\ &\qquad\qquad (B_i^j \circ \mathbf{F}^{-1}, B_\ell^J \circ \mathbf{F}^{-1})_{L^2(\Omega)}.\end{aligned}$$

We introduce the mass matrix $\mathbf{M}_J = [(B_k^J \circ \mathbf{F}^{-1}, B_\ell^J \circ \mathbf{F}^{-1})_{L_2(\Omega)}]_{k,\ell}$ and obtain by the use of restrictions and prolongations

$$\begin{aligned}(C_J u_J, v_J)_{L^2(\Omega)} &= \sum_{j=j_0}^{J} 2^{-2j} \sum_{i\in\mathbb{I}_j} [\mathbf{I}_J^j \mathbf{M}_J \mathbf{u}_J]_i [\mathbf{I}_J^j \mathbf{M}_J \mathbf{v}_J]_i \\ &= \sum_{j=j_0}^{J} 2^{-2j} \mathbf{u}_J^T \mathbf{M}_J \mathbf{I}_j^J \mathbf{I}_J^j \mathbf{M}_J \mathbf{v}_J.\end{aligned}$$

The mass matrices which pop up in this expression disappear in practice since dual basis functions should in fact be used to discretize the preconditioner, and $\mathbf{M}_J$ is spectrally equivalent to the identity matrix.

Finally, the *discretized BPX-preconditioner* to be implemented is

$$\mathbf{C}_J = \sum_{j=j_0}^{J} 2^{-2j} \mathbf{I}_j^J \mathbf{I}_J^j. \tag{32}$$

A simple improvement is obtained by replacing the scaling factor 2^{-2j} by $\operatorname{diag}(\mathbf{A}_j)^{-1}$, where $\operatorname{diag}(\mathbf{A}_j)$ denotes the diagonal matrix built from the diagonal entries of the stiffness matrix $\mathbf{A}_j$. This diagonal scaling has the same effect as the levelwise scaling by 2^{-2j} but improves the condition numbers considerably, particularly if mappings are involved. We arrive thus at the (discretized) BPX-preconditioner

$$\mathbf{C}_J = \sum_{j=j_0}^{J} \mathbf{I}_j^J \operatorname{diag}(\mathbf{A}_j)^{-1} \mathbf{I}_J^j \tag{33}$$

which we will use in the subsequent computations.

level	interval				square				cube			
	$p=1$	$p=2$	$p=3$	$p=4$	$p=1$	$p=2$	$p=3$	$p=4$	$p=1$	$p=2$	$p=3$	$p=4$
3	7.43	3.81	7.03	5.93	5.93	7.31	22.8	133	3.49	39.5	356	5957
4	8.87	4.40	9.47	7.81	5.00	9.03	40.2	225	4.85	50.8	624	9478
5	10.2	4.67	11.0	9.36	5.70	9.72	51.8	293	5.75	56.6	795	11887
6	11.3	4.87	12.1	10.7	6.27	10.1	58.7	340	6.40	59.7	895	13185
7	12.2	5.00	12.7	11.5	6.74	10.4	63.1	371	6.91	61.3	961	13211
8	13.0	5.10	13.0	11.9	7.14	10.5	66.0	391	7.34	62.2	990	13234
9	13.7	5.17	13.2	12.1	7.48	10.6	68.0	403	7.70	62.6	1016	13255
10	14.2	5.22	13.4	12.2	7.77	10.6	69.3	411	7.99	62.9	1040	—

Table 1: Condition numbers of the BPX-preconditioned Laplacian on the unit interval / square /cube. The value for the cube with $p = 4$ and level 10 is missing since its computation would require about 8 months.

We want to mention one further improvement. Let $\mathbf{A}_{j_0}$ denote the operator on the coarsest level j_0. If the condition number $\kappa(\mathbf{A}_{j_0})$ is already high on the coarsest level j_0, it is worth to use its exact inverse on the coarse grid, i.e., to apply

$$\mathbf{C}_J = \mathbf{I}_{j_0}^J \mathbf{A}_{j_0}^{-1} \mathbf{I}_J^{j_0} + \sum_{j=j_0+1}^{J} \mathbf{I}_j^J \operatorname{diag}(\mathbf{A}_j)^{-1} \mathbf{I}_J^j.$$

Further improvement of the BPX-preconditioner can be achieved by replacing the diagonal scaling on each level by, e.g., a SSOR preconditioning (see Subsection 4.5).

4.3. Dependence on the spatial dimension d and the spline degree p

We shall provide numerical results in order to demonstrate the preconditioning and to specify the dependence on the spatial dimension d and the spline degree p. We consider the discretization of the homogeneous Dirichlet problem for the Poisson equation on the d-dimensional unit cube $\hat{\Omega} = [0,1]^d$ $(d = 1,2,3)$. To get the mesh on level j, we subdivide the cube j-times dyadically into 2^d subcubes of mesh size $h_j = 2^{-j}$. On this subdivision, we consider smoothest B-splines of degree $p = 1,2,3,4$. The ℓ^2-condition numbers of the related stiffness matrices, preconditioned by the BPX-preconditioner (33), are tabulated in Table 1. Indeed, the condition numbers seem to be independent of the level j, but they depend on the spline degree p and the space dimension d. Observe though that for $d = 1$ the condition number does not depend on p. Nevertheless, the condition numbers of the preconditioned stiffness matrices using cubic B-splines in three dimensions are about 1000 which is acceptable.

4.4. Dependence on the parametric mapping **F**

In our second test, we demonstrate the influence of the parametric mapping $\mathbf{F}$. To this end, we restrict ourselves to two spatial dimensions and consider again the Laplacian operator with homogeneous Dirichlet boundary conditions. We first consider the ℓ^2-condition

level	$p=1$	$p=2$	$p=3$	$p=4$
3	5.04 (21.8)	12.4 (8.64)	31.8 (31.8)	184 (184)
4	11.1 (90.2)	16.3 (34.3)	54.7 (32.9)	291 (173)
5	25.3 (368)	19.0 (139)	70.1 (98.9)	376 (171)
6	31.9 (1492)	21.4 (560)	79.2 (401)	436 (322)
7	37.4 (6015)	23.1 (2255)	84.4 (1620)	471 (1297)
8	42.1 (241721)	24.3 (9062)	87.3 (6506)	490 (5217)
9	45.7 (969301)	25.2 (36353)	89.0 (26121)	500 (20945)
10	48.8 (388690)	25.9 (145774)	90.1 (104745)	505 (83975)

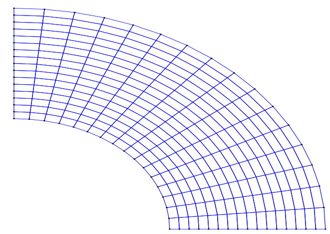

Table 2: Condition numbers of the BPX-preconditioned Laplacian on the analytic arc seen on the right hand side. The bracketed numbers are the related condition numbers without preconditioning.

numbers of the BPX-preconditioned system matrix in case of a smooth mapping (see the plot on the right hand side of Table 2 for an illustration of the mapping). As one can see from Table 2, the condition numbers are at most about a factor of five higher than the related values in Table 1. Nearly the same observation holds if we replace the parametric mapping by a $\mathcal{C}^0$-parametrization which maps the unit square onto an L-shaped domain (see the plot on the right hand side of Table 3 for an illustration of the mapping). The condition numbers are now at most 10 times higher than on the unit square. Nevertheless, it is remarkable that, for both mappings, the condition numbers in case of the cubic B-splines are nearly the same as on the unit square.

If we consider a singular map $\mathbf{F}$, that is the bound (10) fails, the condition numbers grow considerably. As an example, we consider a $\mathcal{C}^1$-parametrization of the L-shape (see the plot on the right hand side of Table 4 for an illustration of the mapping). As seen from Table 4, the condition numbers of the preconditioned system matrix more or less doubles now from level to level. Note that we do observe only a slight dependence on the polynomial degree p.

The bracketed values in Tables 2–4 are the condition numbers of the unpreconditioned (but diagonally scaled) system matrix. From level to level, they obviously grow by a factor 4, also in the case of the singular map. But for a fixed level, the condition numbers become slightly better when p increases which is in contrast to the behaviour of the BPX-preconditioner. Nevertheless, the BPX-preconditioner impressively reduces the condition numbers of the system matrix in all examples.

level	$p=1$	$p=2$	$p=3$	$p=4$
3	14.0 (25.8)	13.4 (10.2)	33.5 (33.5)	194 (194)
4	25.2 (108)	20.6 (41.1)	56.7 (34.7)	301 (182)
5	36.9 (452)	26.8 (168)	72.1 (123)	383 (180)
6	47.9 (1845)	31.8 (689)	80.5 (500)	442 (400)
7	57.4 (7465)	35.4 (2790)	85.5 (2025)	477 (1620)
8	65.3 (30047)	38.0 (11244)	88.3 (8157)	496 (6533)
9	71.8 (120603)	40.0 (45172)	90.0 (32773)	505 (26264)
10	77.0 (483618)	41.2 (181140)	91.0 (131418)	511 (105381)

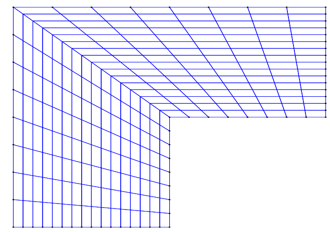

Table 3: Condition numbers of the BPX-preconditioned Laplacian relative to a $\mathcal{C}^0$-parametrization of the L-shape seen on the right hand side. The bracketed numbers are the related condition numbers without preconditioning.

level	$p=1$	$p=2$	$p=3$	$p=4$
3	15.0 (28.8)	14.7 (13.2)	32.8 (32.8)	185 (185)
4	44.1 (133)	36.2 (53.7)	56.7 (38.5)	303 (189)
5	91.1 (568)	70.9 (225)	95.7 (158)	388 (196)
6	167 (2341)	147 (931)	155 (639)	463 (557)
7	443 (9502)	385 (3804)	385 (2587)	887 (2350)
8	1136 (38544)	960 (15353)	1021 (10491)	2417 (9604)
9	2797 (155276)	2301 (61619)	2588 (42355)	6251 (38695)
10	6664 (622565)	5362 (247091)	6318 (169844)	15505 (155143)

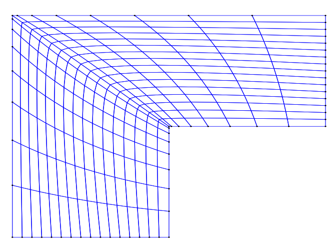

Table 4: Condition numbers of the BPX-preconditioned Laplacian relative to a singular $\mathcal{C}^1$-parametrization of the L-shape seen on the right hand side. The bracketed numbers are the related condition numbers without preconditioning.

level	square	analytic arc	$\mathcal{C}^0$-map of the L-shape	singular $\mathcal{C}^1$-map of the L-shape
3	3.61	3.65	3.67	3.80
4	6.58	6.97	7.01	7.05
5	8.47	10.2	10.2	14.8
6	9.73	13.1	13.2	32.2
7	10.5	14.9	15.2	77.7
8	11.0	15.9	16.3	180
9	11.2	16.5	17.0	411
10	11.4	16.9	17.7	933

Table 5: Condition numbers of the BPX-preconditioned Laplacian for cubic B-splines on different geometries in case of using a SSOR preconditioning on each level.

4.5. Improvement of the BPX-preconditioner

We shall use the standard decomposition $\mathbf{A}_j = \mathbf{L}_j + \mathbf{D}_j + \mathbf{L}_j^T$ of the system matrix $\mathbf{A}_j$ with the diagonal matrix $\mathbf{D}_j$, the lower triangular part $\mathbf{L}_j$, and the upper triangular part $\mathbf{L}_j^T$. Then, by replacing the diagonal scaling on each level of the BPX-preconditioner (33) by the SSOR preconditioner, i.e., instead of (33) applying the preconditioner

$$\mathbf{C}_J = \sum_{j=j_0}^{J} \mathbf{I}_j^J (\mathbf{D}_j + \mathbf{L}_j)^{-T} \mathbf{D}_j (\mathbf{D}_j + \mathbf{L}_j)^{-1} \mathbf{I}_J^j, \tag{34}$$

the condition numbers can be improved impressively. In Table 5, we list the ℓ^2-condition numbers for the BPX-preconditioned Laplacian in case of cubic B-splines in two spatial dimensions. By comparing the numbers with those found in Tables 1–4 one infers that the related condition numbers are all reduced by a factor about five.

5. Conclusions and outlook

We presented in this paper an optimal multilevel preconditioner for isogeometric analysis for which we have shown both by theoretical analysis as well as numerical experiments that the spectral condition number of the corresponding stiffness matrix does not depend on the grid spacing, for different degrees p, spatial dimensions d and different mappings. A further drastic improvement of the absolute values of the constants was provided by employing at the heart of the scheme an SSOR decomposition of the stiffness matrix.

The numerical experiments have focused on the important case of the second order model problem (1), but the theory presented adresses the general arbitrary (even) order case. The fourth-order case is also important in some applications and, to this end, we refer to [19]. There, the focus is on the bi-Laplacian operator on the sphere; the context is different (C^1 quadratic discrete ansatz functions on a Powell-Sabin triangulation are adopted) but we expect similar asymptotic behaviour of the condition number (with and without BPX preconditioning) for the isogeometric C^1 discretization of the bi-Laplacian, since both approaches share the same theoretical background.

We also expect that very similar results can be achieved by multiplicative multilevel preconditioners like multigrid methods.

For problems with variable coefficients, we expect, moreover, that our BPX preconditioner with SSOR acceleration will perform very well since, like for the parametric mapping, the properties of the variable coefficients will implicitly been included, see (33) and (34).

A future reduction of the absolute complexity of the solution scheme can be achieved by employing adaptive schemes for B-splines of higher order when the solution of (2) is not smooth. This, however, requires appropriate a-posteriori error estimation and, ideally, results on the convergence and optimal computational complexity of such an adaptive scheme.

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