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The maximum number of faces of the Minkowski sum of three convex polytopes

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Abstract

We derive tight expressions for the maximum number of k-faces, $0 \leq k \leq d-1$, of the Minkowski sum, $P_1 + P_2 + P_3$, of three d-dimensional convex polytopes P_1, P_2 and P_3 in $\mathbb{R}^d$, as a function of the number of vertices of the polytopes, for any $d \geq 2$. Expressing the Minkowski sum as a section of the Cayley polytope $\mathcal{C}$ of its summands, counting the k-faces of $P_1 + P_2 + P_3$ reduces to counting the $(k+2)$-faces of $\mathcal{C}$ which meet the vertex sets of the three polytopes. In two dimensions our expressions reduce to known results, while in three dimensions, the tightness of our bounds follows by exploiting known tight bounds for the number of faces of r d-polytopes in $\mathbb{R}^d$, where $r \geq d$. For $d \geq 4$, the maximum values are attained when P_1, P_2 and P_3 are d-polytopes, whose vertex sets are chosen appropriately from three distinct d-dimensional moment-like curves.

Key words: discrete & combinatorial geometry, combinatorial complexity, Cayley trick, tight bounds, Minkowski sum, convex polytopes

2010 MSC: 52B05, 52B11, 52C45, 68U05

1 Introduction

We study the Minkowski sum of three d-dimensional convex polytopes, or simply d-polytopes, in $\mathbb{R}^d$, and derive tight upper bounds for the number of its k-faces, for $0 \leq k \leq d-1$, with respect to the number of vertices of the summands. Given two convex polytopes P_1 and P_2, their *Minkowski sum* $P_1 + P_2$ is the set $\{p_1 + p_2 \mid p_1 \in P_1, p_2 \in P_2\}$. This definition extends to any number of summands and also, to non-convex sets of points. The Minkowski sum of convex polytopes is itself a convex polytope, namely, the convex hull of the Minkowski sum of the vertices of its summands.

Minkowski sums are widespread operations in Computational Geometry and find applications in a wide range of areas such as robot motion planning [14], pattern recognition [20], collision detection [15], Computer-Aided Design, and, very recently, Game Theory. They reflect geometrically some algebraic operations, and capture important properties of algebraic objects, such as polynomial systems. This makes them especially useful in Computational Algebra, see eg., [9, 19, 1].

The geometry of the Minkowski sum can be derived from that of its summands: its *normal fan* is the *common refinement* of the normal fans of the summands (see [23] for definitions and details). However, its combinatorial structure is not fully understood, partially due to the fact that most algorithms for computing Minkowski sums have focused on low dimensions (see, e.g., [4] for algorithms in three dimensions). The recent development of algorithms that target high dimensions [6], has led to a more extensive study of their properties (see, e.g., [21]).

A natural and fundamental question regarding the combinatorial properties of Minkowski sums, concerns their complexity measured as a function of the vertices, or the facets of the summands. A complete answer, in terms of the number of vertices or facets of the summands, does not yet exist although for certain classes of polytopes the question has been resolved (see Section 1.1 below). Most of the known results offer tight bounds with respect to the number of vertices of the summands; deriving tight upper bounds with respect to the number of facets seems much harder. Knowing the complexity of Minkowski sums is crucial in developing algorithms for their computation, since it allows to quantify their efficiency.

1.1 Previous work

The complexity of Minkowski sums depends on the geometry of their summands. Worst-case tight upper bounds offer the best possible alternative when the geometric characteristics of a specific instance of the problem are not accounted for. Gritzman and Sturmfels [9] have been the first to derive tight upper bounds for the number of k-faces $f_k(P_1 + P_2 + \cdots + P_r)$ of $P_1 + P_2 + \cdots + P_r$, for all $0 \le k \le d-1$, and $d, r \ge 2$, namely:

$$f_k(P_1 + P_2 + \cdots + P_r) \le 2\tbinom{m}{k} \sum_{j=0}^{d-k-1} \tbinom{m-k-1}{j},$$

where m denotes the number of non-parallel edges of $P_1, P_2, \ldots, P_r$. Equality occurs when P_i are generic zonotopes, i.e., when each P_i is a Minkowski sum of edges, and the generating edges of all polytopes are in general position.

Our knowledge of tight upper bounds for $f_k(P_1 + \cdots + P_r)$ as a function of the number of vertices or facets of the summands is much more limited, while the problem of finding such tight bounds is far from being fully understood and resolved. Given two polygons P_1, P_2 in two dimensions, with n_1, n_2 vertices (or edges) respectively, their Minkowski sum can have at most $n_1 + n_2$ vertices; clearly, this bound holds also for the number of edges of $P_1 + P_2$, and generalizes in the obvious way for any number of summands (cf. [2]).

In three or more dimensions, Fukuda and Weibel [7] have shown what they call the *trivial upper bound*: given r d-polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{R}^d$, where $d \ge 3$ and $r \ge 2$, we have, for all $k \ge 0$:

$$f_k(P_1 + P_2 + \cdots + P_r) \le \sum_{\substack{1 \le s_i \le n_i \\ s_1 + \ldots + s_r = k+r}} \prod_{i=1}^{r} \tbinom{n_i}{s_i}, \; s_i \in \mathbb{N}, \tag{1}$$

where n_i is the number of vertices of P_i, $1 \le i \le r$. In the same paper, Fukuda and Weibel have shown that the trivial upper bound is tight for: (i) $d \ge 4$, $2 \le r \le \lfloor \frac{d}{2} \rfloor$ and for all $0 \le k \le \lfloor \frac{d}{2} \rfloor - r$, and (ii) for the number of vertices, $f_0(P_1 + P_2 + \cdots + P_r)$, of $P_1 + P_2 + \cdots + P_r$, when $d \ge 3$ and $2 \le r \le d-1$. Karavelas and Tzanaki [12] recently extended the range of d, r and k for which the trivial upper bound (1) is attained. More precisely, they showed that for any $d \ge 3$, $2 \le r \le d-1$ and for all $0 \le k \le \lfloor \frac{d+r-1}{2} \rfloor - r$, there exist r d-polytopes $P_1, P_2, \ldots, P_r$ in $\mathbb{R}^d$, for which the number of k-faces of their Minkowski sum attains the trivial upper bound. For $r \ge d$, Sanyal [18] has shown that the trivial bound for $f_0(P_1 + P_2 + \cdots + P_r)$ cannot be attained, whereas tight upper bounds for this case were very recently shown by Weibel [22].

Tight bounds for *all* face numbers, i.e., for all $0 \le k \le d-1$, expressed as a function of the number of vertices or facets of the summands, are only known for *two* d-polytopes when $d \ge 3$. Fukuda and Weibel [7] were the first to derive such bounds for two 3-polytopes in $\mathbb{R}^3$ in terms of the number of vertices of the polytopes, while tight bounds in terms of the number of facets of the two polytopes were proved by Weibel [21]. Weibel's bound for $f_2(P_1 + P_2)$ in [21] has been generalized to the number of facets, $f_2(P_1 + P_2 + \cdots + P_r)$, of the Minkowski sum of any number of 3-polytopes by Fogel, Halperin and Weibel [5]. For d-polytopes in $\mathbb{R}^d$, where $d \ge 4$, Karavelas and Tzanaki [13], have shown that

$$f_{k-1}(P_1 + P_2) \le f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \tbinom{d+1-i}{k+1-i} \left(\tbinom{n_1-d-2+i}{i} + \tbinom{n_2-d-2+i}{i} \right), \tag{2}$$

for all $1 \le k \le d$, where $n_i = f_0(P_i)$, $i = 1, 2$, and $C_d(n)$ stands for the cyclic d-polytope with n vertices. The bounds in (2) have been shown to be tight, and match the corresponding bounds for 2- and 3-polytopes.

1.2 Overview

In this paper we use various basic concepts from discrete geometry and, in particular, polytope theory; the interested reader may refer to [23] for definitions and details. In this work we continue the line of research in [13], extending the methods to the case of three d-polytopes in $\mathbb{R}^d$. This turns out to be far from trivial. Allowing just one more summand significantly raises the problem's intricacy. In particular, deriving Lemmas 4, 5 and 8, which are essential in proving our upper bounds, requires much more involved techniques compared to the case of two polytopes. This is also the case when establishing the tightness of the upper bounds in Section 3: in our constructions an additional difficulty had to be overcome, since we require that not only the face numbers of the sum of the three polytopes are maximal, but also those of the three pairwise sums of the three polytopes. Even more importantly, the case of three d-polytopes provides a valuable insight towards our ultimate goal, the general case of r d-polytopes in $\mathbb{R}^d$, for any $d, r \geq 2$. Using the tools and methodology applied in this paper, some of the results obtained here can be generalized to the case $d, r \geq 2$ (see Section 4), while others still remain elusive. We state our main result to be proved in the following two sections.

Theorem 1. *Let P_1, P_2 and P_3 be three d-polytopes in $\mathbb{R}^d$, $d \geq 2$, with $n_i \geq d+1$ vertices, $1 \leq i \leq 3$. Then, for all $1 \leq k \leq d$, we have:*

$$f_{k-1}(P_1 + P_2 + P_3) \leq f_{k+1}(C_{d+2}(n_{[3]})) - \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \tbinom{d+2-i}{k+2-i} \sum_{\emptyset \subset S \subset [3]} (-1)^{|S|} \tbinom{n_S - d - 3 + i}{i} - \delta \tbinom{\lfloor \frac{d}{2} \rfloor + 1}{k - \lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{3} \tbinom{n_i - \lfloor \frac{d}{2} \rfloor - 2}{\lfloor \frac{d}{2} \rfloor + 1},$$

where $[3] = \{1, 2, 3\}$, $\delta = d - 2\lfloor \frac{d}{2} \rfloor$, and $n_S = \sum_{i \in S} n_i$, $\emptyset \subset S \subseteq [3]$. Moreover, for any $d \geq 2$, there exist three d-polytopes in $\mathbb{R}^d$ for which the bounds above are attained.

To establish the upper bounds (cf. Section 2) we first lift the three d-polytopes in $\mathbb{R}^{d+2}$ using an affine basis of $\mathbb{R}^2$, and form the convex hull $\mathcal{C}$ of the embedded polytopes in $\mathbb{R}^{d+2}$. The polytope $\mathcal{C}$ is known as the *Cayley polytope* of the P_i's. Exploiting the bijection between the set $\mathcal{F}_{[3]}$, consisting of the k-faces of $\mathcal{C}$ that contain vertices from each P_i, and the $(k-2)$-faces of $P_1 + P_2 + P_3$, we reduce the derivation of upper bounds for $f_{k-2}(P_1 + P_2 + P_3)$ to deriving upper bounds for $f_k(\mathcal{F}_{[3]})$, $2 \leq k \leq d+1$.

The rest of our upper bound proof follows the main steps of McMullen's proof of the Upper bound Theorem for polytopes [17]. We add auxiliary vertices to appropriate faces of the Cayley polytope $\mathcal{C}$, resulting in a simplicial polytope $\mathcal{Q}$ whose face set contains $\mathcal{F}_{[3]}$. We then consider the f-vector $\boldsymbol{f}(\partial\mathcal{Q})$ and the h-vector $\boldsymbol{h}(\partial\mathcal{Q})$ of *the boundary complex* $\partial\mathcal{Q}$ of $\mathcal{Q}$, and derive expressions for their entries via the corresponding vectors for $\mathcal{F}_{[3]}$. Using these expressions, we continue by deriving Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$. As an intermediate step we define the subcomplex $\mathcal{K}_{[3]}$ of $\mathcal{C}$ as the closure, under subface inclusion, of $\mathcal{F}_{[3]}$, and derive expressions for its f- and h-vectors (cf. relations (3) and (9) with $R = [3]$). This allows us to write the Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$ in the very concise form:

$$h_{d+2-k}(\mathcal{F}_{[3]}) = h_k(\mathcal{K}_{[3]}), \qquad 0 \leq k \leq d+2.$$

Using a well known relation by McMullen (cf. rel. (12)), along with the expressions that relate the h-vector of $\partial\mathcal{Q}$ with those of $\mathcal{F}_{[3]}$ and $\mathcal{K}_{[3]}$, we establish a recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$ (see Lemma 6). This recurrence relation is then used to prove upper bounds on the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$ and $\boldsymbol{h}(\mathcal{K}_{[3]})$. These upper bounds combined with the Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$, yield refined upper bounds for the values $h_k(\mathcal{F}_{[3]})$ when $k > \lfloor \frac{d+2}{2} \rfloor$. We end by establishing our upper bounds on the number of k-faces, $0 \leq k \leq d-1$, of $P_1 + P_2 + P_3$ by computing $\boldsymbol{f}(\mathcal{F}_{[3]})$ from $\boldsymbol{h}(\mathcal{F}_{[3]})$. At the same time we establish conditions on a subset of the elements of the vectors $\boldsymbol{f}(\mathcal{F}_R)$, $\emptyset \subset R \subseteq [3]$, that are sufficient and necessary in order for the upper bounds in the number of k-faces of $P_1 + P_2 + P_3$ to be tight for all k ($\mathcal{F}_R$ stands for the set of faces of $\mathcal{C}$ that have at least one vertex from each P_i for all $i \in R$, but no vertex from any P_j with $j \notin R$).

In Section 3 we describe the constructions that establish the tightness of our upper bounds. For $d = 2$ and $d = 3$ we rely on previous results. For $d \geq 4$ we define three convex d-polytopes, whose vertices lie on three distinct moment-like d-curves, and show that the sets $\mathcal{F}_R$, $\emptyset \subset R \subseteq [3]$, associated with them satisfy the sufficient and necessary conditions mentioned above. We conclude with Section 4, where we discuss the case of four or more summands and directions for future work.

A more detailed presentation of the results presented in this paper, including the complete proofs, may be found in [11].

2 Upper bounds

2.1 The Cayley trick, f-vectors, h-vectors and Dehn-Sommerville-like equations

Recall that $[3]$ stands for the set $\{1,2,3\}$, and denote by $\mathfrak{S}_j := \{R \subseteq [3] \mid |R| = j\}$, the set of all subsets of $[3]$ of cardinality j, for $1 \leq j \leq 3$. Consider three d-polytopes P_1, P_2 and P_3 in $\mathbb{R}^d$, and choose the basis $\boldsymbol{e}_{2,1} = (0,0)$, $\boldsymbol{e}_{2,2} = (1,0)$, $\boldsymbol{e}_{2,3} = (0,1)$, as the preferred affine basis of $\mathbb{R}^2$. The *Cayley embedding* of the P_i's is defined via the maps $\mu_i(\boldsymbol{x}) = (\boldsymbol{e}_{2,i}, \boldsymbol{x})$, and we denote by $\mathcal{C}$ the $(d+2)$-polytope we get by taking the convex hull of the sets $\mathcal{V}_i = \{\mu_i(\boldsymbol{v}) \mid v \in V_i\}$, where V_i is the vertex set of P_i. This is known as the *Cayley polytope* of the P_i. Similarly, by taking appropriate affine bases, we define the Cayley polytope $\mathcal{C}_R$ of all polytopes P_i, $i \in R$, where $R \in \mathfrak{S}_j$, $j = 1, 2$. These are the Cayley polytopes of all pairs of P_i's and, trivially, the P_i's themselves. Clearly, $\mathcal{C}_R \equiv P_i$, for $R \in \mathfrak{S}_1$. Moreover, $\mathcal{C} \equiv \mathcal{C}_{[3]}$.

For any $\emptyset \subset R \subseteq [3]$, let $\mathcal{V}_R$ denote the union of the sets $\mathcal{V}_i$, $i \in R$. In the sequel we shall identify $\mathcal{C}_R \subset \mathbb{R}^{d+|R|-1}$, for all $R \in \mathfrak{S}_j$, $j = 1, 2$, with the affinely isomorphic and combinatorially equivalent polytope $\operatorname{conv}(\mathcal{V}_R) \subset \mathcal{C} \subset \mathbb{R}^{d+2}$. This will allow us to study properties of these subsets of $\mathcal{C}$ by examining the corresponding Cayley polytopes which lie in lower dimensional spaces.

We shall denote by $\mathcal{F}_R$, $\emptyset \subset R \subseteq [3]$, the set of proper faces of $\mathcal{C}_R$, with the property that $F \in \mathcal{F}_R$ if $F \cap \mathcal{V}_i \neq \emptyset$, for all $i \in R$. In other words, $\mathcal{F}_R$ consists of all the proper faces of $\mathcal{C}_R$ that have at least one vertex from each $\mathcal{V}_i$, for all $i \in R$. Clearly, if $|R| \geq 2$, then $f_0(\mathcal{F}_R) = 0$. Moreover, if $R \in \mathfrak{S}_1$ then $\mathcal{F}_R \equiv \partial P_i$. The dimension of $\mathcal{F}_R$ is the maximum dimension of the faces in $\mathcal{F}_R$, i.e., $\dim(\mathcal{F}_R) = \max_{F \in \mathcal{F}_R} \dim(F) = d + |R| - 2$.

Let $\overline{W}$ be the d-flat of $\mathbb{R}^{d+2}$:

$$\overline{W} = \{\tfrac{1}{3}\boldsymbol{e}_{2,1} + \tfrac{1}{3}\boldsymbol{e}_{2,2} + \tfrac{1}{3}\boldsymbol{e}_{2,3}\} \times \mathbb{R}^d,$$

and consider the *weighted* Minkowski sum $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$. Note that this is nothing more than $P_1 + P_2 + P_3$, scaled down by $\frac{1}{3}$, hence these two sums are combinatorially equivalent. The *Cayley trick* [10] says that the intersection of $\overline{W}$ with $\mathcal{C}$ is combinatorially equivalent (isomorphic) to the weighted Minkowski sum $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$, hence, also to the unweighted Minkowski sum $P_1 + P_2 + P_3$ (see also Fig. 1). Moreover, every face of $P_1 + P_2 + P_3$ is the intersection of a face of $\mathcal{F}_{[3]}$ with $\overline{W}$. This implies that $f_{k-1}(P_1 + P_2 + P_3) = f_{k+1}(\mathcal{F}_{[3]})$, for all $1 \leq k \leq d$.

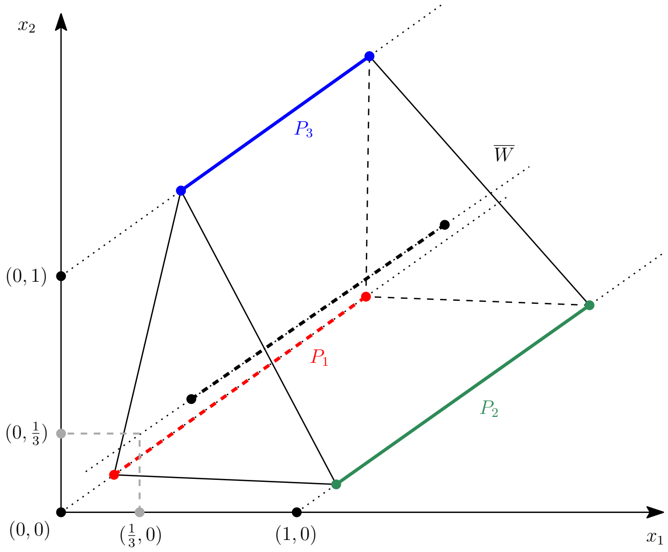

Figure 1: Schematic of the Cayley trick for three polytopes. The three polytopes P_1, P_2 and P_3 are shown in red, green and blue, respectively. The polytope $\frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3$ is shown in black.

To compute the upper bounds for the number of k-faces of $P_1 + P_2 + P_3$, in the rest of the paper we assume that $\mathcal{C}$ is *"as simplicial as possible"*, i.e., all faces of $\mathcal{C}$ are simplicial except for the trivial faces of $\mathcal{C}_R$, for all $\emptyset \subset R \subseteq [3]$. Otherwise, we can employ the so called *bottom-vertex triangulation* [16], where we triangulate every face of $\mathcal{C}$ except the trivial faces of $\mathcal{C}_R$ (i.e., $\mathcal{C}_R$ themselves and not their proper

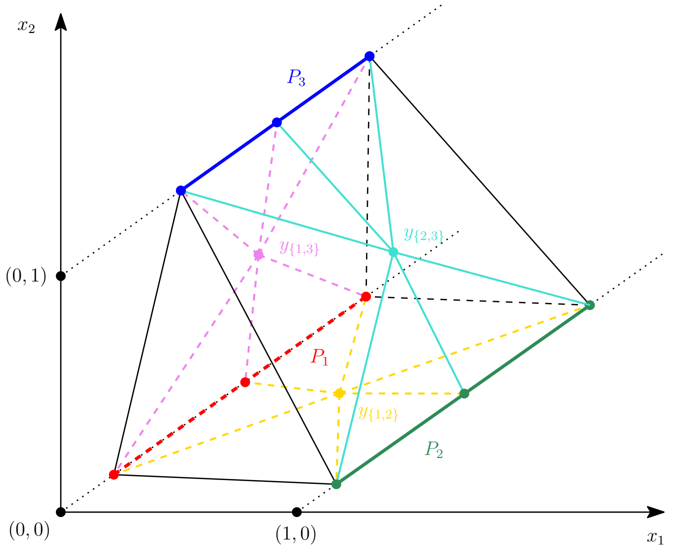

Figure 2: The $(d+2)$-polytope $\mathcal{Q}$.

faces) for all $\emptyset \subset R \subseteq [3]$. The resulting complex is polytopal (cf. [3]) and all of its faces are simplicial, except for the seven trivial faces above. Moreover, it has the same number of vertices as $\mathcal{C}$, while the number of its k-faces is never less than the number of k-faces of $\mathcal{C}$.

Under the *"as simplicial as possible"* assumption above, the faces in $\mathcal{F}_R$ are simplicial. We shall denote by $\mathcal{K}_R$ the *closure*, under subface inclusion, of $\mathcal{F}_R$, i.e., $\mathcal{K}_R$ contains all the faces in $\mathcal{F}_R$ and all the faces that are subfaces of faces in $\mathcal{F}_R$. It is easy to see that $\mathcal{K}_R$ does not contain any of the trivial faces of $\mathcal{C}_S$, $S \subseteq R$, and, thus, $\mathcal{K}_R$ is a pure simplicial $(d+|R|-2)$-complex, whose facets are precisely the facets in $\mathcal{F}_R$. It is also clear that $\mathcal{F}_R \equiv \mathcal{K}_R \equiv \partial P_R$, for $R \in \mathfrak{S}_1$. Moreover, $\mathcal{K}_{[3]}$ is the boundary complex $\partial\mathcal{C}$ of the Cayley polytope $\mathcal{C}$, except for its three facets (i.e., $(d+1)$-faces) $\mathcal{C}_R$, $R \in \mathfrak{S}_2$, and its three ridges (i.e., d-faces) P_i, $1 \leq i \leq 3$.

Consider a k-face F of $\mathcal{K}_R$, $\emptyset \subset R \subseteq [3]$. By the definition of $\mathcal{K}_R$, F is either a k-face of $\mathcal{F}_R$, or a k-face of $\mathcal{F}_S$ for some nonempty subset S of R. Hence:

$$f_k(\mathcal{K}_R) = \sum_{\emptyset \subset S \subseteq R} f_k(\mathcal{F}_S), \qquad -1 \leq k \leq d+|R|-2, \tag{3}$$

where, in order for the above equation to hold for $k=-1$, we set $f_{-1}(\mathcal{F}_R) = (-1)^{|R|-1}$. In what follows we use the convention that $f_k(\mathcal{F}_R) = 0$, for any $k < -1$ or $k > d+|R|-2$.

We are going to define auxiliary vertices in $\mathbb{R}^{d+2}$ not contained in $\mathcal{V}_i, i = 1,2,3$. For every $\emptyset \subset R \subset [3]$ we add a vertex y_R in the relative interior of $\mathcal{C}_R$ and, following [3], we consider the complex arising by taking successive stellar subdivisions of $\partial\mathcal{C}$ as follows:

(i) we form the complex arising from $\partial\mathcal{C}$ by taking the stellar subdivisions $\mathrm{st}(y_{\{i\}}, \mathcal{C}_{\{i\}})$ for all $1 \leq i \leq 3$, then

(ii) we form the complex arising from the one constructed in the previous step by taking the stellar subdivisions $\mathrm{st}(y_R, \mathcal{C}'_R)$ for every $R \in \mathfrak{S}_2$. $\mathcal{C}'_R$ is the complex obtained by taking, for every $S \subset R$, the stellar subdivision of y_S over the boundary complex of $\mathcal{C}_S$.

This complex is polytopal and isomorphic to the boundary complex of a $(d+2)$-polytope, which we shall denote as $\mathcal{Q}$ (see also Fig. 2). The boundary complex $\partial\mathcal{Q}$ is a simplicial $(d+1)$-sphere. The simpliciality of $\partial\mathcal{Q}$ will allow us to utilize its Denh-Sommerville equations in order to prove Dehn-Sommerville-like equations for $\mathcal{F}_{[3]}$ in the upcoming Lemma 2. We shall denote by $\mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3 \cup \{y_R \mid \emptyset \subset R \subset [3]\}$ the vertex set of $\mathcal{Q}$.

By distinguishing cases with respect to the auxiliary vertices y_R that a k-face F of $\partial\mathcal{Q}$ contains, we can count the number of all k-faces of $\partial\mathcal{Q}$, for all $0 \leq k \leq d+1$:

$$f_k(\partial\mathcal{Q}) = f_k(\mathcal{F}_{[3]}) + \sum_{R \in \mathfrak{S}_2} [f_k(\mathcal{F}_R) + f_{k-1}(\mathcal{F}_R)] \quad + \sum_{R \in \mathfrak{S}_1} [f_k(\mathcal{F}_R) + 3f_{k-1}(\mathcal{F}_R) + 2f_{k-2}(\mathcal{F}_R)]. \tag{4}$$

Relation (4) also holds for $k \in \{-1, 0\}$, since, by convention, we have set $f_l(\mathcal{F}_S) = 0$ for all $l < -1$ and $\emptyset \subset S \subseteq [3]$.

Denote by $\mathcal{Y}$ a generic subset of faces of $\mathcal{C}$. $\mathcal{Y}$ will either be a subcomplex of the boundary complex $\partial\mathcal{C}$ of $\mathcal{C}$, or one of the $\mathcal{F}_R$'s. Let δ be the dimension of $\mathcal{Y}$. Then we can define the h-vector of $\mathcal{Y}$ as

$$h_k(\mathcal{Y}) = \sum_{i=0}^{\delta+1} (-1)^{k-i} \tbinom{\delta+1-i}{\delta+1-k} f_{i-1}(\mathcal{Y}). \tag{5}$$

Another quantity that will be heavily used in the rest of the paper is that we call the m-order g-vector of $\mathcal{Y}$, the k-th element of which is given by the following recursive formula:

$$g_k^{(m)}(\mathcal{Y}) = \begin{cases} h_k(\mathcal{Y}), & m = 0, \\ g_k^{(m-1)}(\mathcal{Y}) - g_{k-1}^{(m-1)}(\mathcal{Y}), & m > 0. \end{cases} \tag{6}$$

Observe that for $m = 0$ we get the h-vector of $\mathcal{Y}$, for $m = 1$ we get the g-vector of $\mathcal{Y}$, while, in general, $\boldsymbol{g}^{(m)}(\mathcal{Y})$ is nothing but the m-order backward finite difference of $\boldsymbol{h}(\mathcal{Y})$.

We next define the summation operator $\mathcal{S}_k(\cdot; D, \nu)$ whose action on $\mathcal{Y}$ is as follows:

$$\mathcal{S}_k(\mathcal{Y}; D, \nu) = \sum_{i=0}^{D+1} (-1)^{k-i} \tbinom{D+1-i}{D+1-k} f_{i-1-\nu}(\mathcal{Y}). \tag{7}$$

Assuming that the dimension of $\mathcal{Y}$ is δ, $\nu \geq 0$, $\delta \leq D$, and $D - \delta - \nu \geq 0$, it is easy to verify that for any $k \geq 0$ we have:

$$\mathcal{S}_k(\mathcal{Y}; D, \nu) = g_{k-\nu}^{(D-\delta-\nu)}(\mathcal{Y}). \tag{8}$$

Applying the summation operator in (7) to relations (3) and (4), while using (8), we can prove the following lemma which relates the h-vectors of $\mathcal{F}_R$ and $\mathcal{K}_R$ with each other, and with the h-vector of $\partial\mathcal{Q}$. The last among the relations proved in the following lemma can be thought of as the analogue of the Dehn-Sommerville equations for $\mathcal{F}_{[3]}$ and $\mathcal{K}_{[3]}$.

Lemma 2. *The following relations hold:*

$$h_k(\mathcal{K}_R) = \sum_{\emptyset \subset S \subseteq R} g_k^{(|R|-|S|)}(\mathcal{F}_S),\ 0 \leq k \leq d + |R| - 1,\ \emptyset \subset R \subseteq [3], \tag{9}$$

$$h_k(\partial\mathcal{Q}) = h_k(\mathcal{F}_{[3]}) + \sum_{R \in \mathfrak{S}_2} h_k(\mathcal{F}_R) + \sum_{R \in \mathfrak{S}_1} [h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R)], \quad 0 \leq k \leq d + 2, \tag{10}$$

$$h_{d+2-k}(\mathcal{F}_{[3]}) = h_k(\mathcal{K}_{[3]}), \qquad 0 \leq k \leq d + 2. \tag{11}$$

2.2 Recurrence relation for $\boldsymbol{h}(\mathcal{F}_{[3]})$

Recall that we denote by $\mathcal{V}$ the vertex set of $\partial\mathcal{Q}$ and by $\mathcal{V}_i$ the (Cayley embedding of the) vertex set of ∂P_i, $1 \leq i \leq 3$. Let $\mathcal{Y}/v$ denote the link of vertex v of $\mathcal{Y}$ in the simplicial complex $\mathcal{Y}$. McMullen [17] showed that for any δ-dimensional polytope P the following relation holds for all $0 \leq k \leq \delta - 1$:

$$(k + 1)h_{k+1}(\partial P) + (d - k)h_k(\partial P) = \sum_{v \in \mathrm{vert}(\partial P)} h_k(\partial P/v). \tag{12}$$

Applying relation (12) to the $(d + 2)$-dimensional polytope $\mathcal{Q}$, we have, for all $0 \leq k \leq d + 1$:

$$(k + 1)h_{k+1}(\partial\mathcal{Q}) + (d + 2 - k)h_k(\partial\mathcal{Q}) = \sum_{v \in \mathcal{V}_{[3]}} h_k(\partial\mathcal{Q}/v) + \sum_{\emptyset \subset R \subset [3]} h_k(\partial\mathcal{Q}/y_R), \tag{13}$$

where we used the fact that $\mathcal{V}$ is the disjoint union of the vertex sets $\mathcal{V}_{[3]} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ and $\{y_R \mid \emptyset \subset R \subset [3]\}$. The following lemma offers convenient expressions for the elements in the sums of the right-hand side of (13) in terms of the h-vectors of the $\mathcal{F}_R$'s and $\mathcal{K}_R$'s.

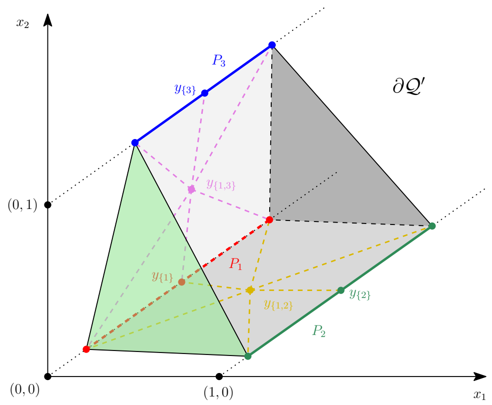

Figure 3: The $(d+1)$-complex $\partial\mathcal{Q}'$ that we get from $\partial\mathcal{Q}$ be removing all faces incident to $y_{\{2,3\}}$.

Lemma 3. *The h-vectors of the complexes $\partial\mathcal{Q}/v$, $v \in \mathcal{V}_i$, $i = 1, 2, 3$, $\partial\mathcal{Q}/y_R$, $R \in \mathfrak{S}_1$, and $\partial\mathcal{Q}/y_R$, $R \in \mathfrak{S}_2$ are given by the following relations:*

$$h_k(\partial Q/v) = h_k(\mathcal{K}_{[3]}/v) + \sum_{\{i\}\subseteq R\subset[3]} h_{k-1}(\mathcal{K}_R/v) + h_{k-2}(\mathcal{K}_{\{i\}}/v),\ v \in \mathcal{V}_i, \quad i \in [3], \tag{14}$$

$$h_k(\partial\mathcal{Q}/y_R) = h_k(\mathcal{F}_R) + h_{k-1}(\mathcal{F}_R), \qquad R \in \mathfrak{S}_1, \tag{15}$$

$$h_k(\partial\mathcal{Q}/y_R) = \sum_{\emptyset\subset S\subseteq R} h_k(\mathcal{F}_S), \qquad R \in \mathfrak{S}_2. \tag{16}$$

Using Lemmas 2 and 3, we can manipulate relation (13), to arrive at the generalization of relation (12) for $\mathcal{F}_{[3]}$.

Lemma 4. *The following relation holds, for all $0 \le k \le d+1$:*

$$(k+1)h_{k+1}(\mathcal{F}_{[3]}) + (d+2-k)h_k(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset R\subseteq[3]} (-1)^{3-|R|} \sum_{v\in\mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R/v). \tag{17}$$

The last intermediate step that we need in order to derive the recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$ is to bound the right-hand side of (17) by an expression that does not involve the links $\mathcal{K}_R/v$. This is the subject of the following lemma.

Lemma 5. *The following relation holds, for all $0 \le k \le d+1$:*

$$\sum_{\emptyset\subset R\subseteq[3]} (-1)^{3-|R|} \sum_{v\in\mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R/v) \le \sum_{\emptyset\subset R\subseteq[3]} (-1)^{3-|R|} \sum_{v\in\mathcal{V}_R} g_k^{(3-|R|)}(\mathcal{K}_R). \tag{18}$$

Sketch of proof. First observe that, by rearranging terms, we can rewrite relation (18) as follows:

$$\sum_{i=1}^{3}\sum_{v\in\mathcal{V}_i}\sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|}\, g_k^{(3-|R|)}(\mathcal{K}_R/v) \le \sum_{i=1}^{3}\sum_{v\in\mathcal{V}_i}\sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|}\, g_k^{(3-|R|)}(\mathcal{K}_R). \tag{19}$$

Clearly, to show that relation (19) holds, it suffices to prove that:

$$\sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|}\, g_k^{(3-|R|)}(\mathcal{K}_R/v) \le \sum_{\{i\}\subseteq R\subseteq[3]} (-1)^{3-|R|}\, g_k^{(3-|R|)}(\mathcal{K}_R), \quad v \in \mathcal{V}_i, \quad i \in [3]. \tag{20}$$

We shall sketch a proof of relation (20) for $i = 1$ and for any $v \in \mathcal{V}_1$; the remaining two cases are entirely similar. We shall define a subset $\mathcal{G}$ of $\partial\mathcal{Q}$ such that it contains all faces in $\mathcal{F}_{[3]}$ and it satisfies a relation from which we can easily obtain relation (20). The first step towards this construction is to consider the polytopal $(d+1)$-complex $\partial\mathcal{Q}'$ we get by removing from $\partial\mathcal{Q}$ the faces that are incident to $y_{\{2,3\}}$ (see Fig. 3). Let $\mathcal{X}$ denote the set of faces of $\partial\mathcal{Q}'$ that are either faces in the star $\mathcal{S}_{\{1,2\}}$ of $y_{\{1,2\}}$ in $\partial\mathcal{Q}'$ or faces in the star $\mathcal{S}_{\{1,3\}}$ of $y_{\{1,3\}}$ in $\partial\mathcal{Q}'$. Now define $\mathcal{G}$ as the set of faces of $\partial\mathcal{Q}'$ that are either faces in $\mathcal{F}_{[3]}$ or faces in $\mathcal{F}_{\{2,3\}}$.

The sets $\mathcal{X}$ and $\mathcal{G}$ form a disjoint union of the faces in $\partial\mathcal{Q}'$, which implies that $f_k(\partial\mathcal{Q}') = f_k(\mathcal{X}) + f_k(\mathcal{G})$, for all $-1 \le k \le d+1$. By applying the summation operator $\mathcal{S}_k(\cdot; d+1, 0)$ we immediately get the corresponding h-vector relation: $h_k(\partial\mathcal{Q}') = h_k(\mathcal{X}) + h_k(\mathcal{G})$, for all $0 \le k \le d+2$. We next show that there exists a shelling $\mathsf{S}(\partial\mathcal{Q}')$ of $\partial\mathcal{Q}'$ such that the facets in $\mathcal{X}$ appear before the facets in $\mathcal{G}$, and for this particular shelling, $h_k(\mathcal{G})$ counts the number of restrictions of size k that correspond to facets of $\partial\mathcal{Q}'$ that are also facets of $\mathcal{G}$. Notice that $\mathsf{S}(\partial\mathcal{Q}')$ is an initial segment of a shelling of $\partial\mathcal{Q}$ that shells the star of $y_{\{2,3\}}$ last.

The same argument also shows that $\partial\mathcal{Q}'/v$ can be seen as the disjoint union of the sets $\mathcal{X}/v$ and $\mathcal{G}/v$, and that $h_k(\partial\mathcal{Q}'/v) = h_k(\mathcal{X}/v) + h_k(\mathcal{G}/v)$, for all $0 \le k \le d+1$. As for $h_k(\mathcal{G})$, we can argue that $h_k(\mathcal{G}/v)$ counts the number of restrictions of size k that correspond to facets of $\partial\mathcal{Q}'/v$ that are also facets of $\mathcal{G}/v$.

To prove that $h_k(\mathcal{G}/v) \le h_k(\mathcal{G})$, for all $0 \le k \le d+2$, we consider the dual graph $G^{\Delta}(\partial\mathcal{Q})$ of $\partial\mathcal{Q}$, oriented according to the shelling $\mathsf{S}(\partial\mathcal{Q})$, as well as the dual graph $G^{\Delta}(\partial\mathcal{Q}/v)$ of $\partial\mathcal{Q}/v$, also oriented according to the shelling $\mathsf{S}(\partial\mathcal{Q}/v)$. We will denote by $\mathcal{V}^{\Delta}(\mathcal{Y})$ the subset of vertices of $G^{\Delta}(\partial\mathcal{Q})$ that are the duals of the facets in $\partial\mathcal{Q}$ that belong to $\mathcal{Y}$, where $\mathcal{Y}$ stands for a subset of the set of faces of $\partial\mathcal{Q}$. Since $\mathsf{S}(\partial\mathcal{Q}/v)$ is induced from $\mathsf{S}(\partial\mathcal{Q})$, $G^{\Delta}(\partial\mathcal{Q}/v)$ is isomorphic to the subgraph of $G^{\Delta}(\partial\mathcal{Q})$ defined over $\mathcal{V}^{\Delta}(\mathrm{star}(v, \partial\mathcal{Q}))$. Moreover, $h_k(\partial\mathcal{Q})$ counts the number of vertices of $\mathcal{V}^{\Delta}(\partial\mathcal{Q})$ with in-degree equal to k, while $h_k(\mathcal{G})$ counts the number of vertices of $\mathcal{V}^{\Delta}(\mathcal{G})$ of in-degree k in $G^{\Delta}(\partial\mathcal{Q})$ (for the particular shelling $\mathsf{S}(\partial\mathcal{Q})$ of $\partial\mathcal{Q}$ that we have chosen). Consequently, $h_k(\mathcal{G})$ counts the number of vertices of $\mathcal{V}^{\Delta}(\mathcal{G})$ of in-degree k in $G^{\Delta}(\partial\mathcal{Q})$; in an analogous manner, we can conclude that $h_k(\mathcal{G}/v)$ counts the number of vertices of $\mathcal{V}^{\Delta}(\mathrm{star}(v, \mathcal{G}))$ with in-degree k in $G^{\Delta}(\partial\mathcal{Q}/v)$. Since, however, $G^{\Delta}(\partial\mathcal{Q}/v)$ is the subgraph of $G^{\Delta}(\partial\mathcal{Q})$ that corresponds to the face v^{Δ} of $G^{\Delta}(\partial\mathcal{Q})$, the number of vertices of $\mathcal{V}^{\Delta}(\mathrm{star}(v, \mathcal{G}))$ with in-degree k cannot exceed the number of vertices of $\mathcal{V}^{\Delta}(\mathcal{G})$ with in-degree k. Hence, $h_k(\mathcal{G}/v) \le h_k(\mathcal{G})$, for all $0 \le k \le d+2$. Inequality (20) in now established by showing that its left- and right-hand side are equal to $h_k(\mathcal{G}/v)$ and $h_k(\mathcal{G})$, respectively. □

Using inequality (18) in Lemma 5, we finally arrive at the following recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F}_{[3]})$.

Lemma 6. *For all $0 \le k \le d+1$, we have:*

$$h_{k+1}(\mathcal{F}_{[3]}) \le \frac{n_{[3]} - d - 2 + k}{k+1} h_k(\mathcal{F}_{[3]}) + \sum_{i=1}^{3} \frac{n_i}{k+1} g_k(\mathcal{F}_{[3]\setminus\{i\}}). \tag{21}$$

Sketch. Using Lemma 5, we can bound the left hand side of relation (17) by the right hand side of relation (18), which involves g-vectors, of various orders, of the complexes $\mathcal{K}_R$, where $\emptyset \subset R \subseteq [3]$. These can be substituted by their equal values from relation (9) with $R = [3]$ and for all $R \in \mathfrak{S}_2$. This gives an inequality involving h-vectors and g-vectors of $\mathcal{F}_{[3]}$ and $\mathcal{F}_R$, $R \in \mathfrak{S}_2$, which simplifies to relation (21). □

2.3 Establishing the upper bounds

In this paragraph we establish upper bounds for the number of $(k+2)$-faces of $\mathcal{F}_{[3]}$, $0 \le k \le d-1$, which immediately yield upper bounds for the number of k-faces of $P_1 + P_2 + P_3$. Our starting point is the recurrence relation (21). Using this recurrence relation, along with the corresponding relation for $\boldsymbol{h}(\mathcal{F}_R)$, $R \in \mathfrak{S}_2$ (cf. [13, Lemma 3.2]), we can derive the upper bounds for $\boldsymbol{h}(\mathcal{F}_{[3]})$ and $\boldsymbol{h}(\mathcal{K}_{[3]})$ stated in the following two lemmas, as well as necessary and sufficient conditions for these bounds to be tight. These conditions will be exploited in Section 3 in order to prove the tightness of our upper bounds.

Lemma 7. *For all $0 \le k \le d+2$, we have:*

$$h_k(\mathcal{F}_{[3]}) \le \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \tbinom{n_S - d - 3 + k}{k}, \quad n_S = \sum_{i \in S} n_i. \tag{22}$$

Equality holds for some $0 \le k \le \lfloor \frac{d+2}{2} \rfloor$, if and only if $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \binom{n_S}{l}$, for all $0 \le l \le k$.

Lemma 8. *For all $0 \le k \le d+2$, we have:*

$$h_k(\mathcal{K}_{[3]}) \le \tbinom{n_{[3]} - d - 3 + k}{k}. \tag{23}$$

Furthermore, for $d \geq 3$ and d odd, we have:

$$h_{\lfloor \frac{d}{2} \rfloor+1}(\mathcal{K}_{[3]}) \leq \binom{n_{[3]}-\lfloor \frac{d}{2} \rfloor-3}{\lfloor \frac{d}{2} \rfloor+1} - \sum_{i=1}^{3} \binom{n_i-\lfloor \frac{d}{2} \rfloor-2}{\lfloor \frac{d}{2} \rfloor+1}. \tag{24}$$

Equality holds for some k, where $0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$, if and only if, for all $\emptyset \subset R \subseteq [3]$, $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R}(-1)^{|R|-|S|}\binom{n_S}{l}$, for all $0 \leq l \leq \min\{k, \lfloor \frac{d+|R|-1}{2} \rfloor\}$.

Utilizing the bounds from Lemmas 7 and 8, along with the Dehn-Sommerville-like equations (11), we arrive at the following theorem concerning upper bounds on the number of k-faces of the Minkowski sum of three convex d-polytopes in $\mathbb{R}^d$.

Theorem 9. *Let P_1, P_2 and P_3 be three d-polytopes in $\mathbb{R}^d$, $d \geq 2$, with $n_i \geq d+1$ vertices, $1 \leq i \leq 3$. Then, for all $1 \leq k \leq d$, we have:*

$$f_{k-1}(P_1+P_2+P_3) \leq f_{k+1}(C_{d+2}(n_{[3]})) - \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k+2-i} \sum_{\emptyset \subset S \subset [3]} (-1)^{|S|} \binom{n_S-d-3+i}{i} - \delta \binom{\lfloor \frac{d}{2} \rfloor+1}{k-\lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{3} \binom{n_i-\lfloor \frac{d}{2} \rfloor-2}{\lfloor \frac{d}{2} \rfloor+1}, \tag{25}$$

where $\delta = d-2\lfloor \frac{d}{2} \rfloor$, and $n_S = \sum_{i \in S} n_i$. Equality holds for all $1 \leq k \leq d$, if and only if

$$f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S}{l}, \tag{26}$$

for all $\quad 0 \leq l \leq \lfloor \frac{d+|R|-1}{2} \rfloor, \quad \emptyset \subset R \subseteq [3].$

Sketch. Our upper bounds will follow from the fact that $f_{k-1}(P_1+P_2+P_3) = f_{k+1}(\mathcal{F}_{[3]})$, $1 \leq k \leq d$. It suffices to establish upper bounds for $f_k(\mathcal{F}_{[3]})$ for all $0 \leq k \leq d+1$. Indeed, writing the f-vector of $\mathcal{F}_{[3]}$ in terms of its h-vector, and using relation (11), we get:

$$f_{k-1}(\mathcal{F}_{[3]}) = \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k-i} h_i(\mathcal{F}_{[3]}) + \sum_{j=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{j}{k-d-2+j} h_j(\mathcal{K}_{[3]}). \tag{27}$$

From Lemmas 7 and 8, the two sums above are bounded, respectively, by the following quantities:

$$\sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k-i} \sum_{\emptyset \subset S \subseteq [3]} (-1)^{3-|S|} \binom{n_S-d-3+i}{i},$$

$$\sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{n_{[3]}-d-3+j}{j} - \delta \binom{\lfloor \frac{d}{2} \rfloor+1}{k-\lfloor \frac{d}{2} \rfloor-2} \sum_{i=1}^{3} \binom{n_i-\lfloor \frac{d}{2} \rfloor-2}{\lfloor \frac{d}{2} \rfloor+1},$$

where $\delta = d-2\lfloor \frac{d}{2} \rfloor$. Combining relation (27) with the bounds above, and after a few calculations we arrive at the following:

$$f_{k-1}(\mathcal{F}_{[3]}) \leq f_{k-1}(C_{d+2}(n_{[3]})) - \sum_{i=0}^{\lfloor \frac{d+2}{2} \rfloor} \binom{d+2-i}{k-i} \sum_{\emptyset \subset S \subset [3]} (-1)^{|S|} \binom{n_S-d-3+i}{i} - \delta \binom{\lfloor \frac{d}{2} \rfloor+1}{k-\lfloor \frac{d}{2} \rfloor-2} \sum_{i=1}^{3} \binom{n_i-\lfloor \frac{d}{2} \rfloor-2}{\lfloor \frac{d}{2} \rfloor+1}.$$

From the derivation of the upper bounds above (see also relation (27)), it is clear that the bounds are tight if and only if: (1) $h_k(\mathcal{F}_{[3]})$ is maximal, for all $0 \leq k \leq \lfloor \frac{d+2}{2} \rfloor$, and (2) $h_k(\mathcal{K}_{[3]})$ is maximal, for all $0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$. According to Lemma 7 and Lemma 8, these conditions are, respectively, equivalent to requiring that:

(i) $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset \subset S \subseteq [3]}(-1)^{3-|S|}\binom{n_S}{l}$, for all $0 \leq l \leq \lfloor \frac{d+2}{2} \rfloor$, and

(ii) $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R}(-1)^{|R|-|S|}\binom{n_S}{l}$, for all $0 \leq l \leq \min\{\lfloor \frac{d+1}{2} \rfloor, \lfloor \frac{d+|R|-1}{2} \rfloor\}$, and for all $\emptyset \subset R \subseteq [3]$.

For $R \equiv [3]$, condition (i) implies condition (ii), while for $R \subset [3]$, $\min\{\lfloor \frac{d+1}{2} \rfloor, \lfloor \frac{d+|R|-1}{2} \rfloor\} = \lfloor \frac{d+|R|-1}{2} \rfloor$. We, therefore, conclude that the bounds in (25) are attained if and only if, conditions (26) hold true for all $0 \leq k \leq \lfloor \frac{d-|R|+1}{2} \rfloor$ and for all $\emptyset \subset R \subseteq [3]$. $\square$

3 Tightness of upper bounds

In this section we show that the bounds in Theorem 9 are tight. We distinguish between the cases $d = 2$, $d = 3$ and $d \geq 4$. For $d = 2$, it is easy to verify that for $k = 0, 1$, the right-hand side of inequality (25) evaluates to $n_1 + n_2 + n_3$, which is known to be tight.

3.1 Three dimensions

In order to prove that our upper bounds are tight, we exploit two results: one by Fukuda and Weibel [7] and one by Weibel [22]. Theorem 3 in [22] relates the number of k-faces of the Minkowski sum of r d-polytopes $P_1, \ldots, P_r$ in $\mathbb{R}^d$, where $r \geq d$, to the number of k-faces of the Minkowski sum of subsets of these polytopes of size at most $d-1$. In the same paper, Weibel also presented a construction of r simplicial d-polytopes, such that any subset S of these polytopes of size at most $d-1$ has the maximum possible number of vertices, namely, $f_0(P_S) = \prod_{i \in S} n_i$. Specializing this construction for $r = d = 3$, we deduce that it is possible to construct three simplicial 3-polytopes P_1, P_2, P_3 in $\mathbb{R}^3$, such that $f_0(P_i) = n_i$, $1 \leq i \leq 3$, and $f_0(P_i + P_j) = n_i n_j$, $1 \leq i < j \leq 3$. Then, from [22, Theorem 3] we get: $f_0(P_1 + P_2 + P_3) = n_1 n_2 + n_2 n_3 + n_1 n_3 - n_1 - n_2 - n_3 + 2$, which matches the expression for the upper bound in Theorem 9 for $k = 0$. Since all P_i's are simplicial, we have that $f_1(P_i) = 3n_i - 6$ and $f_2(P_i) = 2n_i - 4$, for all $1 \leq i \leq 3$. On the other hand, since $f_0(P_i + P_j)$ is maximal, for all $1 \leq i < j \leq 3$, we get, by [7, Corollary 4], that $f_k(P_i + P_j)$ is also maximal for $k = 1, 2$, and for all $1 \leq i < j \leq 3$. Hence: $f_1(P_i + P_j) = 2n_i n_j + n_i + n_j - 8$, $f_2(P_i + P_j) = n_i n_j + n_i + n_j - 6$. Combining the above with [22, Theorem 3] we obtain: $f_1(P_1 + P_2 + P_3) = 2n_1 n_2 + 2n_2 n_3 + 2n_1 n_3 - n_1 - n_2 - n_3 - 6$, and $f_2(P_1 + P_2 + P_3) = n_1 n_2 + n_2 n_3 + n_1 n_3 - 6$, which again match the expressions for the upper bounds in Theorem 9 for $k = 1, 2$.

3.2 Four or more dimensions

We now focus on the case $d \geq 4$. We shall construct three d-polytopes P_1, P_2 and P_3 in $\mathbb{R}^d$, such that they satisfy the conditions in relation (26). Consequently, as Theorem 9 asserts, these polytopes attain the upper bounds in (25).

Consider the following d-dimensional moment-like curves in $\mathbb{R}^d$:

$$\begin{aligned} \boldsymbol{\gamma}_1(t) &= (t, \zeta t^2, \zeta t^3, t^4, t^5, \ldots, t^d), \\ \boldsymbol{\gamma}_2(t) &= (\zeta t, t^2, \zeta t^3, t^4, t^5, \ldots, t^d), \\ \boldsymbol{\gamma}_3(t) &= (\zeta t, \zeta t^2, t^3, t^4, t^5, \ldots, t^d), \end{aligned} \tag{28}$$

where $t > 0$, and $\zeta \geq 0$. Let $\boldsymbol{e}_{1,1} = (0), \boldsymbol{e}_{1,2} = (1)$ be the standard affine basis of $\mathbb{R}$ and recall that $\boldsymbol{e}_{2,1} = (0,0), \boldsymbol{e}_{2,2} = (1,0), \boldsymbol{e}_{2,3} = (0,1)$ is the standard affine basis of $\mathbb{R}^2$. We shall define three polytopes as the convex hulls of points, chosen appropriately on each of these d-curves. Let $x_{i,j}$, $1 \leq j \leq n_i$, $1 \leq i \leq 3$, be $n_{[3]}$ positive real numbers, such that $x_{i,j} < x_{i,j+1}$, $1 \leq j \leq n_i - 1$, and let τ be a positive real parameter. Also let $\nu_i = 3 - i$, $1 \leq i \leq 3$, and set $\zeta = \tau^M$, where $M \geq d(d+1)$. We are going to define three vertex sets V_i as follows:

$$V_i = \{\boldsymbol{\gamma}(x_{i,1}\tau^{\nu_i}), \boldsymbol{\gamma}(x_{i,2}\tau^{\nu_i}), \ldots \boldsymbol{\gamma}(x_{i,n_i}\tau^{\nu_i})\}, \quad 1 \leq i \leq 3. \tag{29}$$

Call P_i the d-polytope we get as the convex hull of the vertices in V_i, and call $\mathcal{V}_i$ the image of V_i via the Cayley embedding. As in Section 2, let $\mathcal{C}$ be the Cayley polytope of the P_i's in $\mathbb{R}^{d+2}$, and let $\mathcal{F}_R$, $\emptyset \subset R \subseteq [3]$, be the set of faces of $\mathcal{C}$ with at least one vertex from each $\mathcal{V}_i$, $i \in R$. Note that, by construction, P_i is a $\lfloor \frac{d}{2} \rfloor$-neighborly polytope in $\mathbb{R}^d$ with n_i vertices, which immediately implies that conditions (26) hold for $R \in \mathfrak{S}_1$ and for all $0 \leq l \leq \lfloor \frac{d}{2} \rfloor$. Hence, it suffices to show that for all $0 \leq l \leq \lfloor \frac{d+|R|-1}{2} \rfloor$:

$$f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|R|-|S|} \binom{n_S}{l}, \quad 2 \leq |R| \leq 3, \tag{30}$$

which we succeed by choosing a sufficiently small value for τ.

In more detail, to prove that conditions (30) hold for $R \in \mathfrak{S}_2 \cup \mathfrak{S}_3$ and for all $|R| \leq l \leq \lfloor \frac{d+|R|-1}{2} \rfloor$, we adopt the key idea used in the proofs of [23, Theorem 0.7 & Corollary 0.8] on basic properties of cyclic d-polytopes, and adapt this idea to our setting. Let us choose some $R \in \mathfrak{S}_2 \cup \mathfrak{S}_3$. We essentially show that the parameter τ can be chosen so that for any $0 \leq l \leq \lfloor \frac{d+|R|-1}{2} \rfloor$, any subset $\mathcal{U}$ of $\mathcal{V}_R = \cup_{i \in R} \mathcal{V}_i$ of size l, such that $\mathcal{U}$ contains at least one vertex from each P_i, $i \in R$, defines a $(l-1)$-face of $\mathcal{F}_R$. At a more technical level, for each $\mathcal{U} \subseteq \mathcal{V}_R$, such that $\mathcal{U} \cap \mathcal{V}_i \neq \emptyset$, for all $i \in R$, we define a hyperplane $H_{\mathcal{U}}(\boldsymbol{x})$ in $\mathbb{R}^{d+|R|-1}$ that passes through the vertices in $\mathcal{U}$. We then show that for τ small enough $H_{\mathcal{U}}(\boldsymbol{x})$ is, in fact, a supporting hyperplane for $\mathcal{C}_R$, where recall that $\mathcal{C}_R$ stands for the Cayley polytope of the polytopes P_i with $i \in R$. Let us call $\tau^\star$ the value of τ for which relation (30) holds true for all $R \in \mathfrak{S}_2 \cup \mathfrak{S}_3$ and for all $|R| \leq l \leq \lfloor \frac{d+|R|-1}{2} \rfloor$. Since $f_{-1}(\mathcal{F}_R) = (-1)^{|R|-1}$, for all $\emptyset \subset R \subseteq [3]$, while $f_{l-1}(\mathcal{F}_R) = 0$, for all $1 \leq l \leq |R|$, we conclude that for $\tau \equiv \tau^\star$, conditions (30) actually hold for all $0 \leq l \leq \lfloor \frac{d+|R|-1}{2} \rfloor$.

Combining the analysis above with that for three 3-polytopes in $\mathbb{R}^3$ at the beginning of this section, we conclude that the upper bounds in Theorem 9 are actually tight for any $d \geq 2$, as already stated in Theorem 1 in the introductory section of the paper.

4 Open problems

Our ultimate goal is to extend our results for the Minkowski sum of r d-polytopes in $\mathbb{R}^d$, for $r \geq 4$ and $d \geq 3$. Towards this direction, we can extend our methodology and tools so as to prove relations for r polytopes that generalize certain relations that hold true for two or three polytopes. For example, the Dehn-Sommerville-like equations in the Lemma 2 (cf. rel. (11)) generalize to:

$$h_{d+r-1-k}(\mathcal{F}_{[r]}) = h_k(\mathcal{K}_{[r]}), \qquad 0 \leq k \leq d+r-1, \tag{31}$$

where $[r] = \{1, 2, \ldots, r\}$, while $\mathcal{F}_R$ and $\mathcal{K}_R$, $\emptyset \subset R \subseteq [r]$, are defined as in Section 2. Notice that, since for $r = 1$ we have $\mathcal{F}_{[1]} \equiv \mathcal{K}_{[1]} \equiv \partial P_1$, the equations in (31) reduce to the well-known Dehn-Sommerville equations for a simplicial d-polytope.

On the other hand, a recurrence relation similar to (21) in Lemma 6 is not as straightforward to obtain. However, we conjecture that the following recurrence relation holds for all $0 \leq k \leq d+r-2$:

$$h_{k+1}(\mathcal{F}_{[r]}) \leq \frac{n_{[r]} - d - r + 1 + k}{k+1} h_k(\mathcal{F}_{[r]}) \quad + \sum_{i=1}^{r} \frac{n_i}{k+1} g_k(\mathcal{F}_{[r]\setminus\{i\}}),$$

where $n_{[r]} = \sum_{i=1}^{r} n_i$. The bounds presented in this paper refer to polytopes of the same dimension. We would like to derive similar bounds for two or more polytopes when the dimensions of these polytopes differ, as well as in the special case of simple polytopes. Finally, a similar problem is to express the number of k-faces of the Minkowski sum of r d-polytopes in terms of the number of facets of these polytopes. Results in this direction are known for $d = 2$ and $d = 3$ only. We would like to derive such expressions for any $d \geq 4$ and any number, r, of summands.

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Appendix

Proof of Lemma 7

We start by proving a lemma that establishes bounds for the g-vector of $\mathcal{F}_R$, $R \in \mathfrak{S}_2$.

Lemma 10. *Let R be a non-empty subset of $[3]$ of cardinality 2. Then, for all $0 \le k \le d+2$, we have:*

$$g_k(\mathcal{F}_R) \le \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \binom{n_S - d - 3 + k}{k}. \tag{32}$$

Equality holds for some k, where $0 \le k \le \lfloor \frac{d+1}{2} \rfloor$, if and only if $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \binom{n_S}{l}$, for all $0 \le l \le k$.

Proof. The bound clearly holds, as equality, for $k = 0$. For $k \ge 1$, from [13, Lemma 3.2] we have:

$$h_k(\mathcal{F}_R) \le \tfrac{n_R - d - 2 + k}{k} h_{k-1}(\mathcal{F}_R) + \sum_{\emptyset \subset S \subset R} \tfrac{n_{R \setminus S}}{k} g_{k-1}(\mathcal{F}_S). \tag{33}$$

Subtracting $h_{k-1}(\mathcal{F}_R)$ from both sides of (33) we get:

$$g_k(\mathcal{F}_R) \le \tfrac{n_R - d - 2}{k} h_{k-1}(\mathcal{F}_R) + \sum_{\emptyset \subset S \subset R} \tfrac{n_{R \setminus S}}{k} g_{k-1}(\mathcal{F}_S). \tag{34}$$

Using now the upper bounds for $h_{k-1}(\mathcal{F}_R)$, $g_{k-1}(\mathcal{F}_S)$, $\emptyset \subset S \subset R$, and noting that $n_R - d - 2 \ge 2(d+1) - d - 2 = d > 0$, we deduce, for any $k \ge 1$:

$$\begin{aligned}
g_k(\mathcal{F}_R) &\le \tfrac{n_R - d - 2}{k} \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \tbinom{n_S - d - 3 + k}{k-1} + \sum_{\emptyset \subset S \subset R} \tfrac{n_{R \setminus S}}{k} \tbinom{n_S - d - 3 + k}{k-1} \\
&= \tfrac{n_R - d - 2}{k} \tbinom{n_R - d - 3 + k}{k-1} - \sum_{\emptyset \subset S \subset R} \tfrac{n_R - d - 2}{k} \tbinom{n_S - d - 3 + k}{k-1} + \sum_{\emptyset \subset S \subset R} \tfrac{n_{R \setminus S}}{k} \tbinom{n_S - d - 3 + k}{k-1} \\
&= \tfrac{n_R - d - 2 + k}{k} \tbinom{n_R - d - 3 + k}{k-1} - \tbinom{n_R - d - 3 + k}{k-1} - \sum_{\emptyset \subset S \subset R} \tfrac{n_R - d - 2 - n_{R \setminus S}}{k} \tbinom{n_S - d - 3 + k}{k-1} \\
&= \tbinom{n_R - d - 2 + k}{k} - \tbinom{n_R - d - 3 + k}{k-1} - \sum_{\emptyset \subset S \subset R} \tfrac{n_S - d - 2}{k} \tbinom{n_S - d - 3 + k}{k-1} \\
&= \tbinom{n_R - d - 3 + k}{k} - \sum_{\emptyset \subset S \subset R} \left[\tfrac{n_S - d - 2 + k}{k} \tbinom{n_S - d - 3 + k}{k-1} - \tbinom{n_S - d - 3 + k}{k-1} \right] \\
&= \tbinom{n_R - d - 3 + k}{k} - \sum_{\emptyset \subset S \subset R} \left[\tbinom{n_S - d - 2 + k}{k} - \tbinom{n_S - d - 3 + k}{k-1} \right] \\
&= \tbinom{n_R - d - 3 + k}{k} - \sum_{\emptyset \subset S \subset R} \tbinom{n_S - d - 3 + k}{k} \\
&= \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \tbinom{n_S - d - 3 + k}{k}.
\end{aligned}$$

We focus now on the equality claim. Suppose first that $f_{l-1}(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \binom{n_S}{l}$, for all $0 \le l \le k$. Then, by [13, Lemma 3.3], $h_\lambda(\mathcal{F}_R) = \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \binom{n_S - d - 2 + \lambda}{\lambda}$, for $\lambda = k-1, k$, which gives:

$$\begin{aligned}
g_k(\mathcal{F}_R) &= h_k(\mathcal{F}_R) - h_{k-1}(\mathcal{F}_R) \\
&= \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \tbinom{n_S - d - 2 + k}{k} - \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \tbinom{n_S - d - 2 + k - 1}{k-1} \\
&= \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \left[\tbinom{n_S - d - 2 + k}{k} - \tbinom{n_S - d - 2 + k - 1}{k-1} \right] \\
&= \sum_{\emptyset \subset S \subseteq R} (-1)^{|S|} \tbinom{n_S - d - 3 + k}{k}.
\end{aligned}$$

Suppose now that $g_k(\mathcal{F}_R)=\sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S-d-3+k}{k}$. By relation (34), we conclude that $h_{k-1}(\mathcal{F}_R)$ must be equal to its upper bound (cf. [13, Lemma 3.3]), since, otherwise, $g_k(\mathcal{F}_R)$ would not be maximal, which contradicts our assumption on the value of $g_k(\mathcal{F}_R)$. This gives:

$$\begin{aligned}
h_k(\mathcal{F}_R) &= g_k(\mathcal{F}_R)+h_{k-1}(\mathcal{F}_R)\\
&= \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S-d-3+k}{k}+\sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S-d-2+k-1}{k-1}\\
&= \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\left[\binom{n_S-d-2+k-1}{k}+\binom{n_S-d-2+k-1}{k-1}\right]\\
&= \sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S-d-2+k}{k}.
\end{aligned}$$

Now the fact that $h_k(\mathcal{F}_R)$ is maximal, implies that $h_l(\mathcal{F}_R)$ must be equal to its maximal value for all $0\le l<k$. To see this suppose that $h_l(\mathcal{F}_R)$ is not maximal for some l, with $0\le l<k$, and among all such l choose the largest one. Then, Lemmas 3.2 and 3.3 in [13] imply that $h_{l+1}(\mathcal{F}_R)$ cannot be maximal, which contradicts the maximality of l. Summarizing, we deduce that if $g_k(\mathcal{F}_R)$ is equal to its upper bound in (32), so is $h_l(\mathcal{F}_R)$ for all $0\le l\le k$. By Lemma 3.3 in [13], this implies that $f_{l-1}(\mathcal{F}_R)=\sum_{\emptyset\subset S\subseteq R}(-1)^{|S|}\binom{n_S}{l}$, for all $0\le l\le k$. □

We are now ready to prove Lemma 7; we do so by induction on k.

The result clearly holds for $k=0$, since

$$h_0(\mathcal{F}_{[3]})=1=1-3+3=\binom{n_{[3]}-d-3}{0}-\sum_{i=1}^{3}\binom{n_{[3]\setminus\{i\}}-d-3}{0}+\sum_{i=1}^{3}\binom{n_i-d-3}{0}.$$

Suppose the bound holds for some $k\ge 0$. We will show that it holds for $k+1$. Using relation (21), Lemma 10, and the fact that, for any $k\ge 0$, $n_{[3]}-d-2+k\ge 3(d+1)-d-2=2d+1>0$, we have:

$$\begin{aligned}
h_{k+1}(\mathcal{F}_{[3]}) &\le \tfrac{n_{[3]}-d-2+k}{k+1}h_k(\mathcal{F}_{[3]})+\sum_{i=1}^{3}\tfrac{n_i}{k+1}g_k(\mathcal{F}_{[3]\setminus\{i\}})\\
&\le \tfrac{n_{[3]}-d-2+k}{k+1}\sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S-d-3+k}{k}+\sum_{i=1}^{3}\tfrac{n_i}{k+1}\sum_{\emptyset\subset S\subseteq[3]\setminus\{i\}}(-1)^{|S|}\binom{n_S-d-3+k}{k}\\
&= \tfrac{n_{[3]}-d-2+k}{k+1}\binom{n_{[3]}-d-3+k}{k}-\sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k}{k+1}\binom{n_{[3]\setminus\{i\}}-d-3+k}{k}+\sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k}{k+1}\binom{n_i-d-3+k}{k}\\
&\qquad+\sum_{i=1}^{3}\tfrac{n_i}{k+1}\binom{n_{[3]\setminus\{i\}}-d-3+k}{k}-\sum_{i=1}^{3}\tfrac{n_i}{k+1}\sum_{j\in[3]\setminus\{i\}}\binom{n_j-d-3+k}{k}\\
&= \binom{n_{[3]}-d-2+k}{k+1}-\sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k-n_i}{k+1}\binom{n_{[3]\setminus\{i\}}-d-3+k}{k}+\sum_{i=1}^{3}\tfrac{n_{[3]}-d-2+k-n_{[3]\setminus\{i\}}}{k+1}\binom{n_i-d-3+k}{k}\\
&= \binom{n_{[3]}-d-2+k}{k+1}-\sum_{i=1}^{3}\tfrac{n_{[3]\setminus\{i\}}-d-2+k}{k+1}\binom{n_{[3]\setminus\{i\}}-d-3+k}{k}+\sum_{i=1}^{3}\tfrac{n_i-d-2+k}{k+1}\binom{n_i-d-3+k}{k}\\
&= \binom{n_{[3]}-d-2+k}{k+1}-\sum_{i=1}^{3}\binom{n_{[3]\setminus\{i\}}-d-2+k}{k+1}+\sum_{i=1}^{3}\binom{n_i-d-2+k}{k+1}\\
&= \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S-d-2+k}{k+1},
\end{aligned}$$

where we used the fact that:

$$\begin{aligned}
\sum_{i=1}^{3}\tfrac{n_{[3]\setminus\{i\}}}{k+1}\binom{n_i-d-3+k}{k} &= \sum_{i=1}^{3}\left(\sum_{j\in[3]\setminus\{i\}}\tfrac{n_j}{k+1}\right)\binom{n_i-d-3+k}{k}=\sum_{i=1}^{3}\sum_{j\in[3]\setminus\{i\}}\tfrac{n_j}{k+1}\binom{n_i-d-3+k}{k}\\
&= \sum_{i=1}^{3}\sum_{j\in[3]\setminus\{i\}}\tfrac{n_i}{k+1}\binom{n_j-d-3+k}{k}=\sum_{i=1}^{3}\tfrac{n_i}{k+1}\sum_{j\in[3]\setminus\{i\}}\binom{n_j-d-3+k}{k}.
\end{aligned}$$

The rest of the proof is concerned with the equality claim. Assume first that $f_{l-1}(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S}{l}$, for all $0 \le l \le k$. Then we have:

$$\begin{aligned}
h_k(\mathcal{F}_{[3]}) &= \sum_{i=0}^{d+2}(-1)^{k-i}\binom{d+2-i}{d+2-k} f_{i-1}(\mathcal{F}_{[3]}) \\
&= (-1)^k \sum_{i=0}^{d+2}(-1)^i \binom{d+2-i}{d+2-k} \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|}\binom{n_S}{i} \\
&= (-1)^k \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|} \sum_{i=0}^{d+2}(-1)^i \binom{d+2-i}{d+2-k}\binom{n_S}{i} \\
&= \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|}\binom{n_S-d-3+k}{k}.
\end{aligned}$$

In the above relation we used the combinatorial identity (cf. [8, eq. (5.25)]):

$$\sum_{0\le k\le l} \binom{l-k}{m}\binom{s}{k-n}(-1)^k = (-1)^{l+m}\binom{s-m-1}{l-m-n},$$

where $k \leftarrow i$, $l \leftarrow d+2$, $m \leftarrow d+2-k$, $n \leftarrow 0$, and $s \leftarrow n_S$.

Suppose now that $h_k(\mathcal{F}_{[3]}) = \sum_{\emptyset\subset S\subseteq[3]}(-1)^{3-|S|}\binom{n_S-d-3+k}{k}$. Since relation (21) holds for all $k \ge 0$, we conclude that $h_l(\mathcal{F}_{[3]})$ must be equal to its upper bound in (22), for all $0 \le l < k$. To see this suppose that (22) is not tight for some l, with $0 \le l < k$, and among all such l choose the largest one. Then, relation (21) implies that $h_{l+1}(\mathcal{F}_{[3]})$ cannot be equal to its upper bound from (22), which contradicts the maximality of l. Hence, if $h_k(\mathcal{F}_{[3]})$ is equal to its upper bound in (22), so is $h_l(\mathcal{F}_{[3]})$ for all $0 \le l < k$, which gives, for all l with $0 \le l \le k$:

$$\begin{aligned}
f_{l-1}(\mathcal{F}_{[3]}) &= \sum_{i=0}^{d+2}\binom{d+2-i}{l-i} h_i(\mathcal{F}_{[3]}) \\
&= \sum_{i=0}^{d+2}\binom{d+2-i}{l-i} \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|}\binom{n_S-d-3+i}{i} \\
&= \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|} \sum_{i=0}^{d+2}\binom{d+2-i}{l-i}\binom{n_S-d-3+i}{i} \\
&= \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|} \sum_{i=0}^{d+2}\binom{d+2-i}{d+2-l}\binom{n_S-d-3+i}{n_S-d-3} \qquad (35) \\
&= \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|}\binom{n_S}{n_S-l} \qquad (36) \\
&= \sum_{\emptyset\subset S\subseteq[3]} (-1)^{3-|S|}\binom{n_S}{l},
\end{aligned}$$

where, in order to get from (35) to (36), we used the combinatorial identity (cf. [8, eq. (5.26)]):

$$\sum_{0\le k\le l} \binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1},$$

with $k \leftarrow i$, $l \leftarrow d+2$, $m \leftarrow d+2-l$, $q \leftarrow n_S-d-3$, and $n \leftarrow n_S-d-3$. □