ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ - ΤΜΗΜΑ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ

Archimedes Center for Modeling, Analysis & Computation

UNIVERSITY OF CRETE - DEPARTMENT OF APPLIED MATHEMATICS

Archimedes Center for Modeling, Analysis & Computation

ACMAC's PrePrint Repository

On periodic orbits in a slow-fast system with normally elliptic slow manifold

Christos Sourdis

ON PERIODIC ORBITS IN A SLOW-FAST SYSTEM WITH NORMALLY ELLIPTIC SLOW MANIFOLD

CHRISTOS SOURDIS

Abstract. In this note we consider the bifurcation of a singular homoclinic orbit to periodic ones in a 4-dimensional slow-fast system of ordinary differential equations, having a 2-dimensional normally elliptic slow manifold, originally studied by Fečkan and Rothos in [11] (see also [13, Ch. 6]). Assuming an extra degree of differentiability on the system, we can refine their perturbation scheme, in particular the choice of approximate solution, and obtain improved estimates.

1. Introduction and statement of the main result

We consider the system

$$\begin{cases} \ddot{x} + h(x) = f(x, \dot{x}, y, \varepsilon\dot{y}, \varepsilon), \\ \\ \varepsilon^2\ddot{y} + y = \varepsilon^2 g(x, \dot{x}, y, \varepsilon\dot{y}, \varepsilon), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small singular perturbation parameter and

(A1) $h,\ f,\ g \in C^r$, for some $r \geq 1$, h is independent of ε, and the first order partial derivatives of f and g are bounded over compact sets of $\mathbb{R}^5$; $f(x_1, x_2, 0, 0, 0) = 0$,

(A2) $f(x_1, x_2, y_1, y_2, \varepsilon)$, $g(x_1, x_2, y_1, y_2, \varepsilon)$ are even in the variables x_2 and y_2, i.e.

$$f(x_1, -x_2, y_1, -y_2, \varepsilon) = f(x_1, x_2, y_1, y_2, \varepsilon), \quad g(x_1, -x_2, y_1, -y_2, \varepsilon) = g(x_1, x_2, y_1, y_2, \varepsilon),$$

(A3) $h(0) = 0,\ h'(0) = -\alpha^2 < 0$ ($\alpha > 0$), and there exists a homoclinic solution ϕ of

$$\ddot{x} + h(x) = 0, \quad t \in \mathbb{R}, \tag{1.2}$$

such that $\phi(t) = \phi(-t)$ and $\phi(t) \to 0$ as $t \to \pm\infty$ (see [6] for necessary and sufficient conditions; a typical example is $h(x) = -\alpha^2 x + |x|^{p-1}x$, $p > 1$). (We note that there exist at most two such solutions, see also [4] and [20]).

Systems of the form (1.1) appear in a variety of interesting physical situations. Firstly, note that if g is identically zero then (1.1) includes the rapidly forced Duffing equation (see [3, 14]). Next, we have the following two examples that are taken from [11], where we refer for more details (see also [13]). The equation

$$\varepsilon^2 u^{(4)} + \ddot{u} - u + u^2 = 0, \tag{1.3}$$

Key words and phrases. Singular perturbations; Resonance phenomena; Periodic orbits of vector fields and flows.

which arises in the theory of water–waves in the presence of surface tension (see [2] and the references therein), can be written as a fourth-order system of the form (1.1). Indeed, by setting $x = u$ and $y = \ddot{u} - u + u^2$, problem (1.3) is equivalent to

$$\begin{cases} \ddot{x} - x + x^2 = y, \\ \varepsilon^2 \ddot{y} + y = \varepsilon^2 \left[2(\dot{x})^2 + (y + x - x^2)(2x - 1) \right]. \end{cases} \tag{1.4}$$

System (1.1) also describes the reduced problem, after a center manifold reduction, in the study of traveling pulses for the discrete nonlinear Klein-Gordon equation

$$\ddot{u}_n - \frac{1}{\varepsilon^2}(u_{n+1} - 2u_n + u_{n-1}) - h(u_n) = 0;$$

we also refer to [5] for a variety of physical situations that are modeled by the above equation. Moreover, systems of the form (1.1), with (1.2) having heteroclinic orbits, describe a pendulum of unit length with a fixed end and large elastic constant of order $1/\varepsilon^2$, allowing the pendulum to be stretched or contracted slightly in the radial direction, see [10] and the references therein. Lastly, let us mention that a nonlocal perturbation of a system of the form (1.1), with h linear, appears in the study of concentrated solutions of the nonlinear Schrödinger equation along closed weighted geodiscs in the plane, after an infinite dimensional Lyapunov-Schmidt reduction (see [8]).

On the other side, the main mathematical interest for studying (1.1) is the following: Writing (1.1) as a slow-fast system of ordinary differential equations, by setting $x_1 = x$, $x_2 = \dot{x}$, $y_1 = y$, $y_2 = \varepsilon \dot{y}$, it is easy to see, letting $\varepsilon = 0$, that the corresponding system has $(y_1, y_2) = (0, 0)$ as a two-dimensional slow manifold (recall (A1)), see [15]. By linearizing, we find that the slow eigenvalues are imaginary ($\pm i$ to be exact), see also a related discussion in the introduction of [19]. This implies that the slow manifold is normally elliptic, see [10], and nearby orbits oscillate rapidly in the normal direction. Thus, in contrast to the normally hyperbolic case (for example when $\varepsilon^2\ddot{y} + y$ is replaced by $\varepsilon^2\ddot{y} - y$ in (1.1)), the smooth persistence of the slow-manifold, for small $\varepsilon > 0$, is not guaranteed. Consequently, it is an interesting problem to rigorously relate (1.1) with the limit problem (1.2) for small $\varepsilon > 0$. In geometric singular perturbation theory, the latter problem is frequently referred to as the limit slow system, and the image of its trajectories on the slow manifold $(y_1, y_2) = (0, 0)$, namely $(x, \dot{x}, 0, 0)$, are called singular orbits (see [15]). In these terms, the homoclinic solution ϕ of (1.2) gives rise to a singular homoclinic orbit of (1.1).

The singular homoclinic orbit of (1.1) is not expected to survive the singular perturbation, and persist as a homoclinic orbit, for small $\varepsilon > 0$. In fact, it may persist as a homoclinic orbit to an exponentially small amplitude periodic orbit (as opposed to a homoclinic to an equilibrium), see [2, 16] and the references therein. Nevertheless, it has been shown that, at least for a sequence $\varepsilon_i \to 0$, there are periodic solutions $(x_{\varepsilon_i}, y_{\varepsilon_i})$ of (1.1), whose periods diverge as $i \to \infty$, that are close (in some sense) to $(\phi, 0)$ (keep in mind that, in the phase plane of (1.2), the interior of the homoclinic loop is exhausted by periodic orbits, see for instance Chapter 12 in [4]). To be more precise, if $h,\ f,\ g \in C^1$, Fečkan and Rothos in [11] (see also Chapter 6 in [13]) proved by a perturbation argument, using the pair $(x_0, y_0) = (\phi, 0)$ as an approximate solution to (1.1), for small $\varepsilon > 0$, the following:

Theorem 1.1. [11, 13] For any $k_0 \in \mathbb{N}$, there is an ε_0 such that for any $0 < \varepsilon < \varepsilon_0$ and $T = \varepsilon\left(2k[\varepsilon^{-3/2}]\pi + \tau\right)$ with $k \in \mathbb{N}$, $k \le k_0$, $\tau \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$, system (1.1) has an even $2T$-periodic solution $(x_\varepsilon(t), y_\varepsilon(t))$ near $(\phi(t), 0)$ on $[0, T]$, where $[a]$ denotes the integer part of $a > 0$.

Remark 1.1. If in addition f, g and h are analytic, the Melnikov function (see [14]) of (1.1) is known to be exponentially small (see [11, 16] and the references therein; see also [3]).

It follows from the analysis in [11], see Remark 3.2 therein, that the following estimates hold:

$$|x_\varepsilon(t) - \phi(t)| + |\dot{x}_\varepsilon(t) - \dot{\phi}(t)| \le C\varepsilon^{1/4}, \quad |y_\varepsilon(t)| + |\varepsilon \dot{y}_\varepsilon(t)| \le C\varepsilon^{1/2}, \quad t \in [0, T].$$

Remark 1.2. Throughout this article, we will denote by C a large generic constant that is independent of small ε and large T.

If $h \in C^2$ (h'' being bounded around $[0, \phi(0)]$ is enough), pushing their arguments further, it is easy to see that one can choose

$$T = \varepsilon\left(2k[\varepsilon^{\delta-2}]\pi + \tau\right) \tag{1.5}$$

with $k \in \mathbb{N}$, $k \le k_0$, $\tau \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$, $0 < \delta < 1$, and obtain the estimates

$$|x_\varepsilon(t) - \phi(t)| + |\dot{x}_\varepsilon(t) - \dot{\phi}(t)| + |y_\varepsilon(t)| + |\varepsilon \dot{y}_\varepsilon(t)| \le C\varepsilon^{\delta}, \quad t \in [0, T]. \tag{1.6}$$

In this note, we will revisit the proof of [11] (see also Chapter 6 in [13]), applying their perturbation scheme, but starting with a more refined initial approximate solution to (1.1), for small $\varepsilon > 0$, than $(\phi, 0)$. This will eventually lead to an improvement of estimates (1.6). A minor drawback of our argument is that we require an extra derivative on g and h, namely $g, h \in C^2$. More precisely, instead of $(\phi, 0)$, we will choose as approximate solution the pair (ϕ, y_1), where y_1 is obtained by applying one iteration in the recursive relation

$$y_{n+1} = \varepsilon^2 g(\phi, \dot{\phi}, y_n, \varepsilon \dot{y}_n, \varepsilon) - \varepsilon^2 \ddot{y}_n, \quad y_0 = 0.$$

Note that y_n is defined as long as g, h are sufficiently smooth, and one can show that

$$|E_n| = |\varepsilon^2 \ddot{y}_n + y_n - \varepsilon^2 g(\phi, \dot{\phi}, y_n, \varepsilon \dot{y}_n, \varepsilon)| \le C_n \varepsilon^{2+2n}, \quad |y_n| + |\dot{y}_n| + |\ddot{y}_n| \le C_n \varepsilon^2.$$

(The important thing is that T does not appear in the above estimates). After studying the corresponding linearized operators around (ϕ, y_1), along the lines of [11, 13] (but offering some simplified proofs), we obtain the existence of a genuine solution by applying Schauder's fixed point theorem on a carefully chosen closed neighborhood of (ϕ, y_1) which is smaller than the corresponding one in [11][13] (this point requires that $h \in C^2$).

Our main result is

Theorem 1.2. Assume that $f \in C^1$ and $g, h \in C^2$, satisfy conditions (A1)–(A3). For any $k_0 \in \mathbb{N}$, and $0 < \delta < 1$, there is an ε_0 such that for any $0 < \varepsilon < \varepsilon_0$ and $T = \varepsilon\left(2k[1/\varepsilon^{2-\delta}]\pi + \tau\right)$ with $k \in \mathbb{N}$, $k \le k_0$, $\tau \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$, system (1.1) has an even $2T$-periodic solution $(x_\varepsilon(t), y_\varepsilon(t))$ near $(\phi(t), 0)$ on $[0, T]$. In fact,

$$|x_\varepsilon(t) - \phi(t)| + |\dot{x}_\varepsilon(t) - \dot{\phi}(t)| \le C\varepsilon, \quad |y_\varepsilon(t)| + |\varepsilon \dot{y}_\varepsilon(t)| \le C\varepsilon^{1+\delta}, \quad t \in [0, T]. \tag{1.7}$$

Remark 1.3. If $f(x_1, x_2, 0, 0, \varepsilon) = 0$, as is the case in (1.4), it follows from the proof that the estimates in (1.7) are both of order ε^2.

Remark 1.4. Estimate (1.7) implies the convergence $\|y_\varepsilon\|_{C^1[0,T]} \to 0$, as $\varepsilon \to 0$, which does not follow from (1.6).

Let us briefly comment on the structure of the set in which the parameter ε can be chosen. As will be apparent in the proof, our construction does not hold for all values of the parameter ε close to 0. There is a resonance phenomenon which prevents the construction to hold for any small value of ε and which forces ε to be taken away from certain intervals. In this context, such a phenomenon was first discovered in [17] and [19].

In [12], the authors considered the case where the corresponding equation to (1.2) has heteroclinic solutions. Our observations seem to apply in this setting without any problems.

Even, small amplitude, high frequency, periodic solutions of (1.3) have been constructed in [2] for small $\varepsilon > 0$. These solutions bifurcate from the trivial solution of (1.3). Based on these, similarly to the proof of Theorem 1.2, we can construct a plethora of approximate, large period, periodic solutions to (1.4), "near" $(\phi, 0)$, for small $\varepsilon > 0$ (one may also try to use the periodic solutions of (1.2) that are accumulating, in the phase plane, to the homoclinic loop). It is natural to wonder whether the presence of the extra parameters in these approximate solutions (which are free to adjust conveniently) offers any advantage in dealing with the aforementioned issue of resonance. In the context of homoclinic orbits to (1.3), this strategy has been successfully employed in [2]. In the context of partial differential equations, an approach in the same spirit has been applied recently in [9].

We will devote the rest of this note in proving Theorem 1.2.

2. Proof of Theorem 1.2

The following lemma is motivated from [18] (see also [10]).

Lemma 2.1. Let

$$y_1 = \varepsilon^2 g(\phi, \dot{\phi}, 0, 0, \varepsilon). \tag{2.1}$$

Then,

$$|y_1| + |\dot{y}_1| + |\ddot{y}_1| \leq C\varepsilon^2, \quad t \in [0, T], \tag{2.2}$$

$$|E_1| = \left| \varepsilon^2 \ddot{y}_1 + y_1 - \varepsilon^2 g(\phi, \dot{\phi}, y_1, \varepsilon \dot{y}_1, \varepsilon) \right| \leq C\varepsilon^4, \quad t \in [0, T], \tag{2.3}$$

and

$$\dot{y}_1(0) = 0, \quad |\dot{y}_1(T)| \leq C\varepsilon^2 e^{-\alpha T}. \tag{2.4}$$

Proof Relation (2.2) follows at once from the fact that $g \in C^2$, $h \in C^1$, observing that $\phi,\ \dot{\phi},\ \ddot{\phi},\ \dddot{\phi} \in L^\infty(\mathbb{R})$.

We have

$$\begin{aligned} |E_1(t)| &= \left| \varepsilon^2 \ddot{y}_1 + y_1 - \varepsilon^2 g(\phi, \dot{\phi}, y_1, \varepsilon \dot{y}_1, \varepsilon) \right| \\ &\leq \varepsilon^2 |\ddot{y}_1| + \varepsilon^2 \left| g(\phi, \dot{\phi}, 0, 0, \varepsilon) - g(\phi, \dot{\phi}, y_1, \varepsilon \dot{y}_1, \varepsilon) \right| \\ &\overset{(A1)}{\leq} \varepsilon^2 |\ddot{y}_1| + C\varepsilon^2 \left(|y_1| + \varepsilon |\dot{y}_1| \right), \end{aligned}$$

for $t \in [0, T]$, and estimate (2.3) follows via (2.2).

By differentiating (2.1), we get

$$\dot{y}_1(t) = \varepsilon^2 g_{x_1}(\phi, \dot{\phi}, 0, 0, \varepsilon)\dot{\phi} + \varepsilon^2 g_{x_2}(\phi, \dot{\phi}, 0, 0, \varepsilon)\ddot{\phi}.$$

Now, keeping in mind that $\dot{\phi}(0) = 0$, the exponential decay estimates

$$|\phi(t)| + |\dot{\phi}(t)| + |\ddot{\phi}(t)| \leq Ce^{-\alpha|t|}, \quad t \in \mathbb{R} \ \ (\text{from (A3)}), \tag{2.5}$$

and $g_{x_2}(x_1, 0, y_1, 0, \varepsilon) = 0$, $x_1, y_1 \in \mathbb{R}$ (from (A2)), we infer that relation (2.4) holds.
The proof of the lemma is complete.

Next, we recall two useful lemmas from [11] (see also Chapter 6 in [13]). We point out that they hold for $h \in C^1$.

Lemma 2.2. There exist $T_0, C > 0$ such that, if $T \geq T_0$ and $z \in C[0,T]$, there exists a unique solution $u = \tilde{L}_T[z]$ of

$$\begin{cases} \ddot{u} + h'(\phi(t))u = z(t), & 0 \leq t \leq T, \\ \\ \dot{u}(0) = \dot{u}(T) = 0. \end{cases} \tag{2.6}$$

Moreover, we have that

$$\|u\| + \|\dot{u}\| \leq C\|z\|,$$

where

$$\|x\| = \max_{[0,T]} |x(t)|.$$

Proof We note that, by standard theory of ODEs (see for example Chapter VI in [20]), existence and uniqueness of a solution will follow from the a-priori estimate for $z \equiv 0$. The validity of the a-priori estimate essentially relies on the property that there are no nontrivial bounded and even solutions to

$$\ddot{u} + h'\left(\phi(t)\right)u = 0, \quad t \in \mathbb{R}, \tag{2.7}$$

see for example Lemma 3.2 in [5] (note that $\dot{\phi}$ is an odd solution). The proof in [11] is in the spirit of Lyapunov and Perron, making use of exponential dichotomies and the variation of constants formula. Another way is to argue by contradiction, similarly to Lemma 1.3.6 in [7], as follows. Firstly, we will establish the weaker a-priori estimate

$$\|u\| \leq C\|z\|, \tag{2.8}$$

for some constant C, for all $u \in C^2[0,T]$ and $z \in C[0,T]$ that satisfy (2.6), provided that T is sufficiently large. Suppose to the contrary that there are sequences $T_i \to \infty$, $u_i \in C^2[0, T_i]$, and $z_i \in C[0, T_i]$, satisfying (2.6) on $[0, T_i]$, such that $\|u_i\| = 1$ and $\|z_i\| \to 0$. Without loss of generality, we may assume that there are $t_i \in [0, T_i]$ such that $u_i(t_i) = 1$, $\dot{u}_i(t_i) = 0$, $\ddot{u}_i(t_i) \leq 0$. It follows that $h'\left(\phi(t_i)\right) \geq z_i(t_i)$. Therefore, via (A3), we infer that the t_i's are uniformly bounded. Using elliptic estimates to find that the u_i's are uniformly bounded in $C^2(\mathbb{R})$ (with respect to i, see also (2.9) below), Arzcela-Ascoli's theorem, and a standard Cantor-type diagonal argument, we can extract a subsequence such that $u_i \to u_*$ in $C^1_{loc}[0,\infty)$, and we may further assume that $t_i \to t_* \geq 0$, where

$$\ddot{u}_* + h'\left(\phi(t)\right)u_* = 0, \quad t > 0, \quad u_*'(0) = 0, \ u_*(t_*) = 1, \ \max_{[0,\infty)} |u_*| = 1.$$

Extending u_* evenly, we get a nontrivial bounded and even solution to (2.7). On the other hand, as we have already discussed, this is not possible. Consequently, estimate (2.8) holds. Then, we deduce the validity of the full estimate of the lemma by (2.6) and the elementary interpolation inequality

$$\|\dot{u}\| \leq 2\|u\| + \|\ddot{u}\|, \tag{2.9}$$

(keep in mind that $T \geq 1$).

The proof of the lemma is complete.

Remark 2.1. Related estimates to those of Lemma 2.2 can be found in [9].

In the remainder of the article, given $k_0 \in \mathbb{N}$ and $\delta \in (0,1)$, we will always assume that small $\varepsilon > 0$ and large T satisfy relation (1.5) for $k \leq k_0$ with $k \in \mathbb{N}$, $\tau \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$. So,

$$\left|\frac{T}{\varepsilon} - 2m\pi - \frac{\pi}{2}\right| \leq \frac{\pi}{4} \quad \text{and} \quad \left|\frac{T}{\varepsilon} - 2m\pi - \frac{3\pi}{2}\right| \leq \frac{\pi}{4} \quad \text{for } m = k[\varepsilon^{\delta-2}] \in \mathbb{N}. \tag{2.10}$$

Lemma 2.3. If ε and T satisfy (1.5), given $z \in C[0,T]$, there exists a unique solution $v = \tilde{M}_{\varepsilon,T}[z]$ of

$$\begin{cases} \varepsilon^2 \ddot{v} + v = \varepsilon z(t), & 0 \leq t \leq T, \\ \dot{v}(0) = 0, \quad \dot{v}(T) = 0. \end{cases}$$

Moreover, we have that

$$\|v\| + \|\varepsilon \dot{v}\| \leq 2(\sqrt{2}+1)T\|z\|.$$

Proof The validity of the lemma follows directly from the explicit representation of the solution

$$v(t) = \tilde{M}_{\varepsilon,T}[z] = \frac{1}{\sin(T/\varepsilon)}\left(\int_0^T \cos\left(\frac{T-s}{\varepsilon}\right) z(s)ds\right)\cos(t/\varepsilon) + \int_0^t \sin\left(\frac{t-s}{\varepsilon}\right) z(s)ds, \tag{2.11}$$

and the fact that (2.10) implies that $|\sin(T/\varepsilon)| \geq \sqrt{2}/2$.

The proof of the lemma is complete.

Remark 2.2. A qualitatively analogous relation to (2.11) has been shown in [1] for the nonhomogeneous boundary value problem $\varepsilon^2 \ddot{v} + q^2(t)v = \varepsilon z(t)$, $t \in (0,T)$; $\dot{v}(0) = \dot{v}(T) = 0$, with $q \in C^1[0,T]$ and positive, via the classical Prüfer transform (see for instance [20]).

Remark 2.3. In order to avoid confusion, we point out that we have changed the original notation of [11], [13].

Since both $\dot{\phi}(T)$ and $\dot{y}_1(T)$ are of order $e^{-\alpha T}$ (recall (2.4), (2.5)), and T is of order $\varepsilon^{\delta-1}$ with $0 < \delta < 1$ (recall (1.5)), we can easily obtain the following corollaries:

Corrolarry 2.1. There exist $\varepsilon_1, C > 0$ such that, if $0 < \varepsilon < \varepsilon_1$ and $T \gg 1$ satisfy (1.5), given $z \in C[0,T]$, there exists a unique solution $u = L_T[z]$ of

$$\begin{cases} \ddot{u} + h'(\phi(t))u = z(t), & 0 \leq t \leq T, \\ \dot{u}(0) = 0, \quad \dot{u}(T) = -\dot{\phi}(T). \end{cases}$$

Moreover,

$$\|u\| + \|\dot{u}\| \leq C\|z\| + \varepsilon^{10}.$$

Proof Let

$$\tilde{u}(t) = u(t) + \frac{\dot{\phi}(T)}{2T}t^2.$$

In terms of $\tilde{u}$, we have the equivalent problem

$$\ddot{\tilde{u}} + h'(\phi(t))\,\tilde{u} = z(t) + \frac{\dot{\phi}(T)}{T} + h'(\phi(t))\,\frac{\dot{\phi}(T)}{2T}t^2, \quad t \in [0,T], \quad \dot{\tilde{u}}(0) = 0, \ \dot{\tilde{u}}(T) = 0.$$

Now, we can apply Lemma 2.2 to obtain existence, uniqueness, and the estimates

$$\|\tilde{u}\| + \|\dot{\tilde{u}}\| \le C\|z\| + CT|\dot{\phi}(T)|.$$

Since $T|\dot{\phi}(T)| \le C\varepsilon^{\delta-1}e^{-C^{-1}\varepsilon^{\delta-1}}$, via the definition of $\tilde{u}$, we deduce that the assertion of the lemma holds.

The proof of the lemma is complete.

Similarly we can show

Corrolarry 2.2. There exists a small ε_2 and a $C > 0$ such that, if $0 < \varepsilon < \varepsilon_2$ and $T \gg 1$ satisfy (1.5), given $z \in C[0,T]$, there exists a unique solution $v = M_{\varepsilon,T}[z]$ of

$$\begin{cases} \varepsilon^2\ddot{v} + v = \varepsilon z(t), & 0 \le t \le T, \\ \dot{v}(0) = 0, \quad \dot{v}(T) = -\dot{y}_1(T). \end{cases}$$

Moreover,

$$\|v\| + \|\varepsilon\dot{v}\| \le CT\|z\| + \varepsilon^{10}.$$

We are now ready for the

Proof of Theorem 1.2 In view of (A2), and Corollaries 2.1 and 2.2, we have to find a small fixed point, in the convex set

$$X_T = \left\{(u,v) \in C^1[0,T] \times C^1[0,T] \ : \ \dot{u}(0) = 0, \ \dot{u}(T) = -\dot{\phi}(T), \ \dot{v}(0) = 0, \ \dot{v}(T) = -\dot{y}_1(T)\right\},$$

of the compact mapping $(u,v) \to (\bar{u},\bar{v})$, defined by

$$\begin{aligned} \bar{u} &= L_T\left[-h(\phi+u) + h(\phi) + h'(\phi)u + f(\phi+u, \dot{\phi}+\dot{u}, y_1+v, \varepsilon(\dot{y}_1+\dot{v}), \varepsilon)\right], \\ \bar{v} &= M_{\varepsilon,T}\left[\varepsilon\left(g(\phi+u, \dot{\phi}+\dot{u}, y_1+v, \varepsilon(\dot{y}_1+\dot{v}), \varepsilon) - g(\phi, \dot{\phi}, y_1, \varepsilon\dot{y}_1, \varepsilon)\right) - \varepsilon^{-1}E_1\right], \end{aligned} \tag{2.12}$$

(the mapping is compact thanks to the well known Arczela-Ascoli theorem). Indeed, recalling (A3) and (2.4), such functions $\phi + u$ and $y_1 + v$ fashion solutions of (1.1) in $(0,T)$ with Neumann boundary conditions on the boundary, and are close to $(\phi, y_1) \approx (\phi, 0)$ on $[0,T]$. Then, in view of (A2), we can extend them, via consecutive even reflections with respect to the lines $t = (2m+1)T$, $m \in \mathbb{Z}$, to $2T$-periodic even solutions of (1.1) (see [11], [13] for more details).

By Corollaries 2.1–2.2, (A1), (1.5), (2.2), and (2.3), recalling that $h \in C^2$, we can estimate

$$\|\dot{\bar{u}}\| + \|\bar{u}\| \le C(\|u\|^2 + \|v\| + \|\varepsilon\dot{v}\| + \varepsilon),$$

$$\|\varepsilon\dot{\bar{v}}\| + \|\bar{v}\| \le C\varepsilon^{\delta}(\|u\| + \|\dot{u}\| + \|v\| + \|\varepsilon\dot{v}\| + \varepsilon^2).$$

Now, it follows readily that Schauder's fixed point theorem (see for instance Theorem 3.4.7 in [7]) can be applied, for small ε, to produce a fixed point in the closed convex subset of X_T that is defined by

$$B_\varepsilon = \left\{(u,v) \in X_T \ : \ \|u\| + \|\dot{u}\| \le M\varepsilon, \quad \|v\| + \|\varepsilon\dot{v}\| \le N\varepsilon^{1+\delta}\right\},$$

for some large constants $M, N > 1$ that are independent of ε and T. Indeed, if $(u, v) \in B_\varepsilon$, we have

$$\|\dot{\bar{u}}\| + \|\bar{u}\| \leq C(M^2\varepsilon^2 + N\varepsilon^{1+\delta} + \varepsilon),$$

$$\|\varepsilon\dot{\bar{v}}\| + \|\bar{v}\| \leq C\varepsilon^\delta(M\varepsilon + N\varepsilon^{1+\delta} + \varepsilon^2),$$

with C independent of ε, M and N. We choose $M = 2C$, $N = 4C^2$, and find that

$$\|\dot{\bar{u}}\| + \|\bar{u}\| \leq C(4C^2\varepsilon^2 + 4C^2\varepsilon^{1+\delta} + \varepsilon),$$

$$\|\varepsilon\dot{\bar{v}}\| + \|\bar{v}\| \leq C\varepsilon^\delta(2C\varepsilon + 4C^2\varepsilon^{1+\delta} + \varepsilon^2).$$

Consequently, decreasing ε further, if necessary, we obtain that $(\bar{u}, \bar{v}) \in B_\varepsilon$, as is required for applying Schauder's theorem.

The proof of the theorem is complete.

Remark 2.4. At the end of the corresponding proof in [11, 13], it is remarked that if $h \in C^2$ (as is the case here) then we can apply the well known uniform contraction mapping principle to (2.12), and get local uniqueness of the obtained solutions. However, in view of the first equation in (1.4) for example, this point is not obvious to us.

2.1. **Acknowledgements.** The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7-REGPOT-2009-1) under grant agreement n^o 245749.

References

[1] Alikakos ND, Bates PW, Fusco G. Solutions to the nonautonomous bistable equation with specified Morse index. Part I: Existence. Trans. Amer. Math. Soc. 1993; **340**: 641-654.
[2] Amick CJ, Toland JF. Solitary waves with surface tension I: Trajectories homoclinic to periodic orbits in four dimensions. Arch. Rational Mech. Anal. 1992; **118**: 37–69.
[3] Angenent S. A variational interpretation of Melnikov's function and exponentially small separatrix splitting. Sympletic Geometry, London Mathematical Society Lecture Note Series **192**, Cambridge University Press: Cambridge; 1994: 5–36.
[4] Arnol'd VI. *Ordinary differential equations*. Springer-Verlag: Berlin; 1992.
[5] Bates PW, Zhang C. Traveling pulses for the Klein–Gordon equation on a lattice or continuum with long–range interaction. Discrete Contin. Dyn. Syst. 2006; **16**: 235-252.
[6] Berestycki H, Lions PL. Nonlinear scalar field equations I: Existence of a ground state. Arch. Rat. Mech. Anal. 1983; **82**: 313–347.
[7] Chang K-C. *Methods in nonlinear analysis*. Springer-Verlag: Berlin; 2005.
[8] del Pino M, Kowalczyk M, Wei J. Concentration on curves for nonlinear Schrödinger equations. Comm. Pure Appl. Math. 2007; **60**: 113–146.
[9] del Pino M, Kowalczyk M, Pacard F, Wei J. The Toda system and multiple-end solutions of autonomous planar elliptic problems. Adv. Math. 2010; **224**: 1462–1516.
[10] Lu N, Zeng C. Normally elliptic singular perturbations and persistence of homoclinic orbits. J. Differential Equations. 2011; **250**: 4124-4176.
[11] Fečkan M, Rothos V. Bifurcations of periodics from homoclinics in singular O.D.E.: Applications to discretizations of travelling waves of P.D.E.. Communnications in Pure and Applied Analysis. 2002; **1**: 475–483.
[12] Fečkan M, Rothos V. Kink–like periodic travelling waves for lattice equations with on–site and inter–site potentials. Dynam. Part. Differ. Eq. 2005; **2**: 357–370.
[13] Fečkan M. *Topological degree approach to bifurcation problems*. Springer; 2008.
[14] Guckenheimer J, Holmes P. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Springer: New York-Berlin; 1983.

[15] Jones CKRT. Geometric singular perturbation theory. Dynamical Systems, Lecture Notes in Mathematics **1609**, Springer-Verlag: Berlin; 1995: 44-118.
[16] Lombardi E. *Oscillatory integrals and phenomena beyond all algebraic orders*. Lecture Notes in Mathematics **1741**, Springer-Verlag: Berlin; 2000.
[17] Malchiodi A. Adiabatic limits of closed orbits for some Newtonian systems in $\mathbb{R}^n$. Asymptot. Anal. 2001; **25**: 149-181.
[18] Mazzeo R, Pacard F. Constant curvature foliations in asymptotically hyperbolic spaces. Rev. Mat. Iberoamericana. 2011; **27**: 303–333.
[19] Shatah J, Zeng C. Periodic solutions for Hamiltonian systems under strong constraining forces. J. Differential Equations. 2002; **186**: 572-585.
[20] Walter W. *Ordinary differential equations*. Springer-Verlag: New York; 1998.

Department of Applied Mathematics and Department of Mathematics, University of Crete, GR–714 09 Heraklion, Crete, Greece.
E-mail address: csourdis@tem.uoc.gr