Evolution PDEs and augmented eigenfunctions. I finite interval

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Abstract

The so-called unified method expresses the solution of an initial-boundary value problem for an evolution PDE in the finite interval in terms of an integral in the complex Fourier (spectral) plane. Simple initial-boundary value problems, which will be referred to as problems of type I, can be solved via a classical transform pair. For example, the Dirichlet problem of the heat equation can be solved in terms of the transform pair associated with the Fourier sine series. Such transform pairs can be constructed via the spectral analysis of the associated spatial operator. For more complicated initial-boundary value problems, which will be referred to as problems of type II, there does *not* exist a classical transform pair and the solution *cannot* be expressed in terms of an infinite series. Here we pose and answer two related questions: first, does there exist a (non-classical) transform pair capable of solving a type II problem, and second, can this transform pair be constructed via spectral analysis? The answer to both of these questions is positive and this motivates the introduction of a novel class of spectral entities. We call these spectral entities augmented eigenfunctions, to distinguish them from the generalised eigenfunctions introduced in the sixties by Gel'fand and his co-authors.

MSC: 35P10 (primary), 35C15, 35G16, 47A70 (secondary).

1 Introduction

Consider the following initial-boundary value problems for the linearized Korteweg-de Vries (LKdV) equation:

Problem 1

$$q_t(x,t) + q_{xxx}(x,t) = 0 \qquad (x,t) \in (0,1)\times(0,T), \tag{1.1a}$$
$$q(x,0) = f(x) \qquad x \in [0,1], \tag{1.1b}$$
$$q(0,t) = q(1,t) = 0 \qquad t \in [0,T], \tag{1.1c}$$
$$q_x(1,t) = q_x(0,t)/2 \qquad t \in [0,T]. \tag{1.1d}$$

Problem 2

$$q_t(x,t) + q_{xxx}(x,t) = 0 \qquad (x,t) \in (0,1)\times(0,T), \tag{1.2a}$$
$$q(x,0) = f(x) \qquad x \in [0,1], \tag{1.2b}$$
$$q(0,t) = q(1,t) = q_x(1,t) = 0 \qquad t \in [0,T]. \tag{1.2c}$$

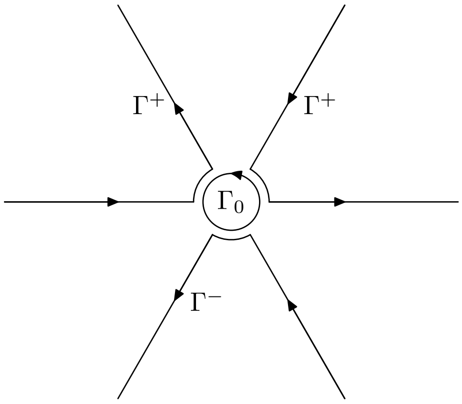

Figure 1: Contours for the linearized KdV equation.

It is shown in [4, 9, 10, 11] that these problems are well-posed and that their solutions can be expressed in the form

$$q(x,t) = \frac{1}{2\pi}\left\{\int_{\Gamma^+} + \int_{\Gamma_0}\right\} e^{i\lambda x + i\lambda^3 t}\frac{\zeta^+(\lambda)}{\Delta(\lambda)}\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{\Gamma^-} e^{i\lambda(x-1)+i\lambda^3 t}\frac{\zeta^-(\lambda)}{\Delta(\lambda)}\,\mathrm{d}\lambda, \tag{1.3}$$

where, Γ_0 is the circular contour of radius $\frac{1}{2}$ centred at 0, $\Gamma^\pm$ are the boundaries of the domains $\{\lambda \in \mathbb{C}^\pm : \mathrm{Im}(\lambda^3) > 0 \text{ and } |\lambda| > 1\}$ as shown on figure 1, α is the root of unity $e^{2\pi i/3}$, $\hat{f}(\lambda)$ is the Fourier transform

$$\int_0^1 e^{-i\lambda x} f(x)\,\mathrm{d}x, \qquad \lambda \in \mathbb{C} \tag{1.4}$$

and $\zeta^\pm(\lambda)$, $\Delta(\lambda)$ are defined as follows for all $\lambda \in \mathbb{C}$:

Problem 1

$$\Delta(\lambda) = e^{i\lambda} + \alpha e^{i\alpha\lambda} + \alpha^2 e^{i\alpha^2\lambda} + 2(e^{-i\lambda} + \alpha e^{-i\alpha\lambda} + \alpha^2 e^{-i\alpha^2\lambda}), \tag{1.5a}$$

$$\begin{aligned}\zeta^+(\lambda) = \hat{f}(\lambda)(e^{i\lambda} + 2\alpha e^{-i\alpha\lambda} + 2\alpha^2 e^{-i\alpha^2\lambda}) + \hat{f}(\alpha\lambda)(\alpha e^{i\alpha\lambda} - 2\alpha e^{-i\lambda})\\ + \hat{f}(\alpha^2\lambda)(\alpha^2 e^{i\alpha^2\lambda} - 2\alpha^2 e^{-i\lambda}),\end{aligned} \tag{1.5b}$$

$$\zeta^-(\lambda) = -\hat{f}(\lambda)(2 + \alpha^2 e^{-i\alpha\lambda} + \alpha e^{-i\alpha^2\lambda}) - \alpha\hat{f}(\alpha\lambda)(2 - e^{-i\alpha^2\lambda}) - \alpha^2\hat{f}(\alpha^2\lambda)(2 - e^{-i\alpha\lambda}). \tag{1.5c}$$

Problem 2

$$\Delta(\lambda) = e^{-i\lambda} + \alpha e^{-i\alpha\lambda} + \alpha^2 e^{-i\alpha^2\lambda}, \tag{1.6a}$$

$$\zeta^+(\lambda) = \hat{f}(\lambda)(\alpha e^{-i\alpha\lambda} + \alpha^2 e^{-i\alpha^2\lambda}) - (\alpha\hat{f}(\alpha\lambda) + \alpha^2\hat{f}(\alpha^2\lambda))e^{-i\lambda}, \tag{1.6b}$$

$$\zeta^-(\lambda) = -\hat{f}(\lambda) - \alpha\hat{f}(\alpha\lambda) - \alpha^2\hat{f}(\alpha^2\lambda). \tag{1.6c}$$

For evolution PDEs defined in the finite interval, $x \in [0,1]$, one may expect that the solution can be expressed in terms of an infinite series. However, it is shown in [9, 10] that for generic boundary conditions this is *impossible*. The solution *can* be expressed in the form of an infinite series only for a particular class of boundary value problems; this class is characterised explicitly in [10]. In particular, problem 2 does *not* belong to this class, in contrast to problem 1 for which there exists the following alternative representation:

$$q(x,t) = \frac{1}{2\pi}\sum_{\substack{\sigma\in\overline{\mathbb{C}^+}:\\ \Delta(\sigma)=0}}\int_{\Gamma_\sigma} e^{i\lambda x + i\lambda^3 t}\frac{\zeta^+(\lambda)}{\Delta(\lambda)}\,\mathrm{d}\lambda + \frac{1}{2\pi}\sum_{\substack{\sigma\in\mathbb{C}^-:\\ \Delta(\sigma)=0}}\int_{\Gamma_\sigma} e^{i\lambda(x-1)+i\lambda^3 t}\frac{\zeta^-(\lambda)}{\Delta(\lambda)}\,\mathrm{d}\lambda, \tag{1.7}$$

where Γ_σ is a circular contour centred at σ with radius $\frac{1}{2}$; the asymptotic formula for σ is given in [11]. By using the residue theorem, it is possible to express the right hand side of equation (1.7) in terms of an infinite series over σ.

We note that even for problems for which there does exist a series representation (like problem 1), the integral representation (1.3) has certain advantages. In particular, it provides an efficient numerical evaluation of the solution [3].

Generic initial-boundary value problems for which there does *not* exist an infinite series representation will be referred to as problems of type II, in contrast to those problems whose solutions possess both an integral and a series representation, which will be referred to as problems of type I,

existence of a series representation: type I
existence of only an integral representation: type II.

Transform pair

Simple initial-boundary value problems for linear evolution PDEs can be solved via an appropriate transfrom pair. For example, the Dirichlet and Neumann problems of the heat equation on the finite interval can be solved with the transform pair associated with the Fourier-sine and the Fourier-cosine series, respectively. Similarly, the series that can be constructed using the residue calculations of the right hand side of equation (1.7) can be obtained directly via a classical transform pair, which in turn can be constructed via standard spectral analysis.

It turns out that the unified method provides an algorithmic way for constructing a transform pair tailored for a given initial-boundary value problem. For example, the integral representation (1.3) gives rise to the following transform pair tailored for solving problems 1 and 2:

$$f(x) \mapsto F(\lambda): \qquad F_\lambda(f) = \begin{cases} \int_0^1 \phi^+(x,\lambda) f(x)\,\mathrm{d}x & \text{if } \lambda \in \Gamma^+ \cup \Gamma_0, \\ \int_0^1 \phi^-(x,\lambda) f(x)\,\mathrm{d}x & \text{if } \lambda \in \Gamma^-, \end{cases} \tag{1.8a}$$

$$F(\lambda) \mapsto f(x): \qquad f_x(F) = \left\{ \int_{\Gamma_0} + \int_{\Gamma^+} + \int_{\Gamma^-} \right\} e^{i\lambda x} F(\lambda)\,\mathrm{d}\lambda, \qquad x \in [0,1], \tag{1.8b}$$

where for problems 1 and 2 respectively, $\phi^\pm$ are given by

$$\begin{aligned} \phi^+(x,\lambda) = \frac{1}{2\pi\Delta(\lambda)} \Big[& e^{-i\lambda x}(e^{i\lambda} + 2\alpha e^{-i\alpha\lambda} + 2\alpha^2 e^{-i\alpha^2\lambda}) + e^{-i\alpha\lambda x}(\alpha e^{i\alpha\lambda} - 2\alpha e^{-i\lambda}) \\ & + e^{-i\alpha^2\lambda x}(\alpha^2 e^{i\alpha^2\lambda} - 2\alpha^2 e^{-i\lambda}) \Big], \end{aligned} \tag{1.8c}$$

$$\begin{aligned} \phi^-(x,\lambda) = \frac{-e^{-i\lambda}}{2\pi\Delta(\lambda)} \Big[& e^{-i\lambda x}(2 + \alpha^2 e^{-i\alpha\lambda} + \alpha e^{-i\alpha^2\lambda}) + \alpha e^{-i\alpha\lambda x}(2 - e^{-i\alpha^2\lambda}) \\ & + \alpha^2 e^{-i\alpha^2\lambda x}(2 - e^{-i\alpha\lambda}) \Big] \end{aligned} \tag{1.8d}$$

and

$$\phi^+(x,\lambda) = \frac{1}{2\pi\Delta(\lambda)} \left[e^{-i\lambda x}(\alpha e^{-i\alpha\lambda} + \alpha^2 e^{-i\alpha^2\lambda}) - (\alpha e^{-i\alpha\lambda x} + \alpha^2 e^{-i\alpha^2\lambda x}) e^{-i\lambda} \right], \tag{1.8e}$$

$$\phi^-(x,\lambda) = \frac{-e^{-i\lambda}}{2\pi\Delta(\lambda)} \left[e^{-i\lambda x} + \alpha e^{-i\alpha\lambda x} + \alpha^2 e^{-i\alpha^2\lambda x} \right]. \tag{1.8f}$$

The alternative representation (1.7) gives rise to the following alternative transform pair tailored for solving problem 1:

$$F(\lambda) \mapsto f(x): \qquad f_x^\Sigma(F) = \sum_{\substack{\sigma\in\mathbb{C}: \\ \Delta(\sigma)=0}} \int_{\Gamma_\sigma} e^{i\lambda x} F(\lambda)\,\mathrm{d}\lambda, \tag{1.9}$$

where $F_\lambda(f)$ is defined by equations (1.8a), (1.8c) and (1.8d) and Γ_σ is defined below (1.7).

The validity of these transform pairs is established in section 2. The solution of problems 1 and 2 is then given by

$$q(x,t) = f_x\left(e^{i\lambda^3 t}F_\lambda(f)\right). \tag{1.10}$$

Spectral representation

The basis for the classical transform pairs used to solve initial-boundary value problems for linear evolution PDEs is the expansion of the initial datum in terms of appropriate eigenfunctions of the spatial differential operator. The transform pair diagonalises the associated differential operator in the sense of the classical spectral theorem. The main goal of this paper is to show that the unified method yields an integral representation, like (1.3), which in turn gives rise to a transform pair like (1.8), and furthermore the elucidation of the spectral meaning of such new transform pairs leads to new results in spectral theory.

In connection with this, we recall that Gel'fand and coauthors have introduced the concept of generalised eigenfunctions [6] and have used these eigenfunctions to construct the spectral representations of self-adjoint differential operators [5]. This concept is inadequate for analysing the IBVPs studied here because our problems are in general non-self-adjoint. Although the given formal differential operator is self-adjoint, the boundary conditions are in general not self-adjoint.

In what follows, we introduce the notion of *augmented eigenfunctions*. Actually, in order to analyse type I and type II IBVPs, we introduce two types of augmented eigenfunctions. Type I are a slight generalisation of the eigenfunctions introduced by Gel'fand and Vilenkin and are also related with the notion of pseudospectra [2]. However, it appears that type II eigenfunctions comprise a new class of spectral functionals.

Definition 1.1. *Let C be a linear topological space with subspace Φ and let $L:\Phi\to C$ be a linear operator. Let γ be an oriented contour in $\mathbb{C}$ and let $E=\{E_\lambda:\lambda\in\gamma\}$ be a family of functionals $E_\lambda\in C'$. Suppose there exist corresponding* remainder *functionals $R_\lambda\in\Phi'$ and* eigenvalues *$z:\gamma\to\mathbb{C}$ such that*

$$E_\lambda(L\phi) = z(\lambda)E_\lambda(\phi) + R_\lambda(\phi), \qquad \forall\,\phi\in\Phi,\ \forall\,\lambda\in\gamma. \tag{1.11}$$

If

$$\int_\gamma e^{i\lambda x}R_\lambda(\phi)\,\mathrm{d}\lambda = 0, \qquad \forall\,\phi\in\Phi,\ \forall\,x\in[0,1], \tag{1.12}$$

then we say E is a family of type I augmented eigenfunctions *of L up to integration along γ.*

If

$$\int_\gamma \frac{e^{i\lambda x}}{z(\lambda)}R_\lambda(\phi)\,\mathrm{d}\lambda = 0, \qquad \forall\,\phi\in\Phi,\ \forall\,x\in(0,1), \tag{1.13}$$

then we say E is a family of type II augmented eigenfunctions *of L up to integration along γ.*

We note that the class of families of augmented eigenfunctions of a given operator is closed under union.

Recall that in the theory of pseudospectra it is required that the norm of the functional $R_\lambda(\phi)$ is finite, whereas in our definition it is required that the integral of $\exp(i\lambda x)R_\lambda(\phi)$ along the contour γ vanishes. Recall that the inverse transform of the relevant transform pair is defined in terms of a contour integral, thus the above definition is sufficient for our needs.

It will be shown in Section 4 that $\{F_\lambda:\lambda\in\Gamma_\sigma\ \exists\,\sigma\in\mathbb{C}:\Delta(\sigma)=0\}$ is a family of type I augmented eigenfunctions of the differential operator representing the spatial part of problem 1 with eigenvalue λ^3. Similarly $\{F_\lambda:\lambda\in\Gamma_0\}$ is a family of type I augmented eigenfunctions of the spatial operator in problem (1.2). However, $\{F_\lambda:\lambda\in\Gamma^+\cup\Gamma^-\}$ is a family of type II augmented eigenfunctions.

Diagonalisation of the operator

Our definition of augmented eigenfunctions, in contrast to the generalized eigenfunctions of Gel'fand and Vilenkin [6, Section 1.4.5], allows the occurence of remainder functionals. However, the contribution of these remainder functionals is eliminated by integrating over γ. Hence, integrating equation (1.11) over γ gives rise to a non-self-adjoint analogue of the spectral representation of an operator.

Definition 1.2. *We say that* $E = \{E_\lambda : \lambda \in \gamma\}$ *is a* complete *family of functionals* $E_\lambda \in C'$ *if*

$$\phi \in \Phi \text{ and } E_\lambda \phi = 0 \ \forall \ \lambda \in \gamma \quad \Rightarrow \quad \phi = 0. \tag{1.14}$$

We now define a spectral representation of the non-self-adjoint differential operators we study in this paper.

Definition 1.3. *Suppose that* $E = \{E_\lambda : \lambda \in \gamma\}$ *is a system of type* I *augmented eigenfunctions of* L *up to integration over* γ, *and that*

$$\int_\gamma e^{i\lambda x} E_\lambda L\phi \, \mathrm{d}\lambda \text{ converges } \ \forall \ \phi \in \Phi, \ \forall \ x \in (0,1). \tag{1.15}$$

Furthermore, assume that E *is a complete system. Then* E *provides a* spectral representation *of* L *in the sense that*

$$\int_\gamma e^{i\lambda x} E_\lambda L\phi \, \mathrm{d}\lambda = \int_\gamma e^{i\lambda x} z(\lambda) E_\lambda \phi \, \mathrm{d}\lambda \qquad \forall \ \phi \in \Phi, \ \forall \ x \in (0,1). \tag{1.16}$$

Definition 1.4. *Suppose that* $E^{(\mathrm{I})} = \{E_\lambda : \lambda \in \gamma^{(\mathrm{I})}\}$ *is a system of type* I *augmented eigenfunctions of* L *up to integration over* $\gamma^{(\mathrm{I})}$ *and that*

$$\int_{\gamma^{(\mathrm{I})}} e^{i\lambda x} E_\lambda L\phi \, \mathrm{d}\lambda \text{ converges } \ \forall \ \phi \in \Phi, \ \forall \ x \in (0,1). \tag{1.17}$$

Suppose also that $E^{(\mathrm{II})} = \{E_\lambda : \lambda \in \gamma^{(\mathrm{II})}\}$ *is a system of type* II *augmented eigenfunctions of* L *up to integration over* $\gamma^{(\mathrm{II})}$ *and that*

$$\int_{\gamma^{(\mathrm{II})}} e^{i\lambda x} E_\lambda \phi \, \mathrm{d}\lambda \text{ converges } \ \forall \ \phi \in \Phi, \ \forall \ x \in (0,1). \tag{1.18}$$

Furthermore, assume that $E = E^{(\mathrm{I})} \cup E^{(\mathrm{II})}$ *is a complete system. Then* E *provides a* spectral representation *of* L *in the sense that*

$$\int_{\gamma^{(\mathrm{I})}} e^{i\lambda x} E_\lambda L\phi \, \mathrm{d}\lambda = \int_{\gamma^{(\mathrm{I})}} z(\lambda) e^{i\lambda x} E_\lambda \phi \, \mathrm{d}\lambda \qquad \forall \ \phi \in \Phi, \ \forall \ x \in (0,1), \tag{1.19a}$$

$$\int_{\gamma^{(\mathrm{II})}} \frac{1}{z(\lambda)} e^{i\lambda x} E_\lambda L\phi \, \mathrm{d}\lambda = \int_{\gamma^{(\mathrm{II})}} e^{i\lambda x} E_\lambda \phi \, \mathrm{d}\lambda \qquad \forall \ \phi \in \Phi, \ \forall \ x \in (0,1). \tag{1.19b}$$

According to Definition 1.3, the operator L is diagonalised (in the traditional sense) by the complete transform pair

$$\left(E_\lambda, \int_\gamma e^{i\lambda x} \cdot \, \mathrm{d}\lambda \right). \tag{1.20}$$

Hence, augmented eigenfunctions of type I provide a natural extension of the generalised eigenfunctions of Gel'fand & Vilenkin. This form of spectral representation is sufficient to describe the transform pair associated with problem 1. However, the spectral interpretation of the transform pair used to solve problem 2 gives rise to augmented eigenfunctions of type II, which are clearly quite different from the generalised eigenfunctions of Gel'fand & Vilenkin.

Definition 1.4 describes how an operator may be written as the sum of two parts, one of which is diagonalised in the traditional sense, whereas the other possesses a diagonalised inverse.

Theorem 1.5. *The transform pairs* (F_λ, f_x) *defined in* (1.8a)–(1.8d) *and* (1.8a), (1.8b), (1.8e) *and* (1.8f) *provide spectral representations of the spatial differential operators associated with problems 1 and 2 respectively in the sense of Definition 1.4.*

Theorem 1.6. *The transform pair* (F_λ, f_x^Σ) *defined in* (1.8a), (1.8c), (1.8d) *and* (1.9) *provides a spectral representation of the spatial differential operator associated with problem 1 in the sense of Definition 1.3.*

Remark 1. Both problems 1 and 2 involve homogeneous boundary conditions. It is straightforward to extend the above analysis for problems with inhomogeneous boundary conditions, see Section 3.1.

2 Validity of transform pairs

In section 2.1 we will derive the validity of the transform pairs defined by equations (1.8). In section 2.2 we derive an analogous transform pair for a general IBVP.

2.1 Linearized KdV

Proposition 2.1. *Let* $F_\lambda(f)$ *and* $f_x(F)$ *be given by equations* (1.8a)–(1.8d). *For all* $f \in C^\infty[0,1]$ *such that* $f(0) = f(1) = 0$ *and* $f'(0) = 2f'(1)$ *and for all* $x \in (0,1)$, *we have*

$$f_x(F_\lambda(f)) = f(x). \tag{2.1}$$

Let $F_\lambda(f)$ *and* $f_x(F)$ *be given by equations* (1.8a), (1.8b), (1.8e) *and* (1.8f). *For all* $f \in C^\infty[0,1]$ *such that* $f(0) = f(1) = f'(1) = 0$ *and for all* $x \in (0,1)$,

$$f_x(F_\lambda(f)) = f(x). \tag{2.2}$$

Proof. The definition of the transform pair (1.8a)–(1.8d) implies

$$f_x(F_\lambda(f)) = \frac{1}{2\pi}\left\{\int_{\Gamma^+} + \int_{\Gamma_0}\right\} e^{i\lambda x}\frac{\zeta^+(\lambda)}{\Delta(\lambda)}\,\mathrm{d}\lambda + \frac{1}{2\pi}\int_{\Gamma^-} e^{i\lambda(x-1)}\frac{\zeta^-(\lambda)}{\Delta(\lambda)}\,\mathrm{d}\lambda, \tag{2.3}$$

where $\zeta^\pm$ and Δ are given by equations (1.5) and the contours Γ^+, Γ^- and Γ_0 are shown in figure 1.

The fastest-growing exponentials in the sectors exterior to $\Gamma^\pm$ are indicated on figure 2a. Each of these exponentials occurs in Δ and integration by parts shows that the fastest-growing-terms in $\zeta^\pm$ are the exponentials shown on figure 2a multiplied by λ^{-2}. Hence the ratio $\zeta^+(\lambda)/\Delta(\lambda)$ decays for large λ within the sector $\pi/3 \leqslant \arg\lambda \leqslant 2\pi/3$ and the ratio $\zeta^-(\lambda)/\Delta(\lambda)$ decays for large λ within the sectors $-\pi \leqslant \arg\lambda \leqslant -2\pi/3$, $-\pi/3 \leqslant \arg\lambda \leqslant 0$. The relevant integrands are meromorphic functions with poles only at the zeros of Δ. The distribution theory of zeros of exponential polynomials [7] implies that the only poles occur within the sets bounded by $\Gamma^\pm$.

The above observations and Jordan's lemma allow us to deform the relevant contours to the contour γ shown on figure 2b; the red arrows on figure 2a indicate the deformation direction. Hence equation (2.3) simplifies to

$$f_x(F_\lambda(f)) = \frac{1}{2\pi}\int_\gamma \frac{e^{i\lambda x}}{\Delta(\lambda)}\left(\zeta^+(\lambda) - e^{-i\lambda}\zeta^-(\lambda)\right)\mathrm{d}\lambda. \tag{2.4}$$

Equations (1.5) imply,

$$\left(\zeta^+(\lambda) - e^{-i\lambda}\zeta^-(\lambda)\right) = \hat{f}(\lambda)\Delta(\lambda), \tag{2.5}$$

where $\hat{f}$ is the Fourier transform of a piecewise smooth function supported on $[0,1]$. Hence the integrand on the right hand side of equation (2.4) is an entire function, so we can deform the contour onto the real axis. The usual Fourier inversion theorem completes the proof.

The proof for the transform pair (1.8a), (1.8b), (1.8e) and (1.8f) is similar. □

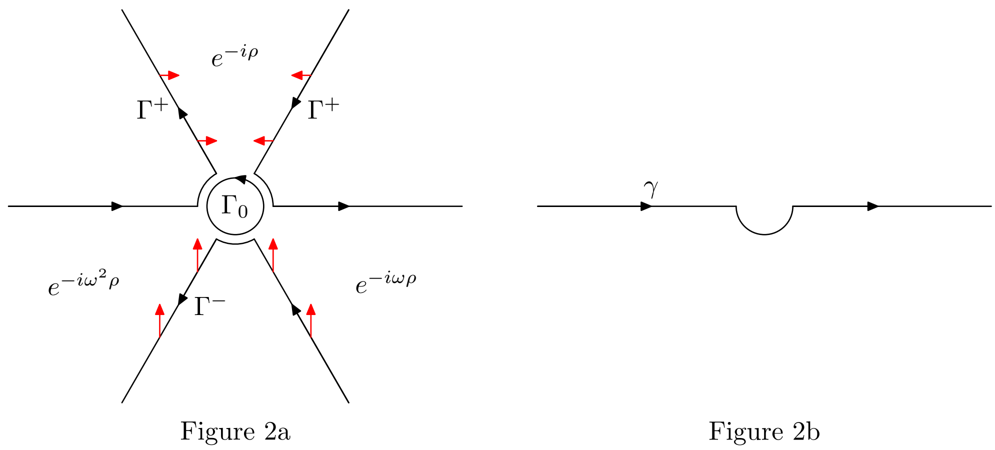

Figure 2: Contour deformation for the linearized KdV equation.

Although

$$f_0(F_\lambda(\phi)) \neq f(0), \qquad f_1(F_\lambda(\phi)) \neq f(1), \tag{2.6}$$

the values at the endpoints can be recovered by taking apporpriate limits in the interior of the interval.

2.2 General

Spatial differential operator

Let $C = C^\infty[0,1]$ and $B_j : C \to \mathbb{C}$ be the following linearly independent boundary forms

$$B_j\phi = \sum_{k=0}^{n-1} \left(b_{j\,k}\phi^{(j)}(0) + \beta_{j\,k}\phi^{(j)}(1) \right), \quad j \in \{1, 2, \dots, n\}. \tag{2.7}$$

Let $\Phi = \{\phi \in C : B_j\phi = 0 \,\forall\, j \in \{1, 2, \dots, n\}\}$ and $\{B_j^\star : j \in \{1, 2, \dots, n\}\}$ be a set of adjoint boundary forms with adjoint boundary coefficients $b^\star_{j\,k}$, $\beta^\star_{j\,k}$. Let $S : \Phi \to C$ be the differential operator defined by

$$S\phi(x) = (-i)^n \frac{\mathrm{d}^n\phi}{\mathrm{d}x^n}(x). \tag{2.8}$$

Then S is formally self-adjoint but, in general, does not admit a self-adjoint extension because, in general, $B_j \neq B_j^\star$. Indeed, adopting the notation

$$[\phi\psi](x) = (-i)^n \sum_{j=0}^{n-1} (-1)^j (\phi^{(n-1-j)}(x)\overline{\psi}^{(j)}(x)), \tag{2.9}$$

of [1, Section 11.1] and using integration by parts, we find

$$((-i\,\mathrm{d}/\,\mathrm{d}x)^n\phi, \psi) = [\phi\psi](1) - [\phi\psi](0) + (\phi, (-i\,\mathrm{d}/\,\mathrm{d}x)^n\psi), \qquad \forall\, \phi, \psi \in C^\infty[0,1]. \tag{2.10}$$

If $\phi \in \Phi$, then ψ must satisfy the adjoint boundary conditions in order for $[\phi\psi](1) - [\phi\psi](0) = 0$ to be valid.

Initial-boundary value problem

Associated with S and constant $a \in \mathbb{C}$, we define the following homogeneous initial-boundary value problem:

$$(\partial_t + aS)q(x,t) = 0 \qquad \forall\, (x,t) \in (0,1)\times(0,T), \tag{2.11a}$$
$$q(x,0) = f(x) \qquad \forall\, x \in [0,1], \tag{2.11b}$$
$$q(\cdot,t) \in \Phi \qquad \forall\, t \in [0,1], \tag{2.11c}$$

where $f \in \Phi$ is arbitrary. Only certain values of a are permissible. Clearly $a = 0$ is nonsensical and a reparametrisation ensures there is no loss of generality in assuming $|a| = 1$. The problem is guaranteed to be ill-posed (for the same reason as the reverse-time heat equation is ill-posed) without the following further restrictions on a: if n is odd then $a = \pm i$ and if n is even then $\operatorname{Re}(a) \geqslant 0$ [4, 10].

A full characterisation of well-posedness for all problems (2.11) is given in [8, 10, 11]; For even-order problems, well-posedess depends upon the boundary conditions only, but for odd-order it is often the case that a problem is well-posed for $a = i$ and ill-posed for $a = -i$ or vice versa. Both problems (1.1) and (1.2) are well-posed.

Definition 2.2. *We classify the IBVP* (2.11) *into three classes using the definitions of [11]:*

type I: *if the problem for* (S, a) *is well-posed and the problem for* $(S, -a)$ *is well-conditioned.*

type II: *if the problem for* (S, a) *is well-posed but the problem for* $(S, -a)$ *is ill-conditioned.*

ill-posed *otherwise.*

We will refer to the operators S *associated with cases* I *and* II *as operators of* type I *and* type II *respectively.*

The spectral theory of type I operators is well understood in terms of an infinite series representation. Here, we provide an alternative spectral representation of the type I operators and also provide a suitable spectral representation of the type II operators.

Transform pair

Let $\alpha = e^{2\pi i/n}$. We define the entries of the matrices $M^{\pm}(\lambda)$ entrywise by

$$M^{+}_{k\,j}(\lambda) = \sum_{r=0}^{n-1} (-i\alpha^{k-1}\lambda)^r b^{\star}_{j\,r}, \tag{2.12a}$$
$$M^{-}_{k\,j}(\lambda) = \sum_{r=0}^{n-1} (-i\alpha^{k-1}\lambda)^r \beta^{\star}_{j\,r}. \tag{2.12b}$$

Then the matrix $M(\lambda)$, defined by

$$M_{k\,j}(\lambda) = M^{+}_{k\,j}(\lambda) + M^{-}_{k\,j}(\lambda) e^{-i\alpha^{k-1}\lambda}, \tag{2.13}$$

is a realization of Birkhoff's *adjoint characteristic matrix.*

We define $\Delta(\lambda) = \det M(\lambda)$. From the theory of exponential polynomials [7], we know that the only zeros of Δ are of finite order and are isolated with positive infimal separation 5ε, say. We define $X^{l\,j}$ as the $(n-1)\times(n-1)$ submatrix of M with $(1,1)$ entry the $(l+1, j+1)$ entry of M.

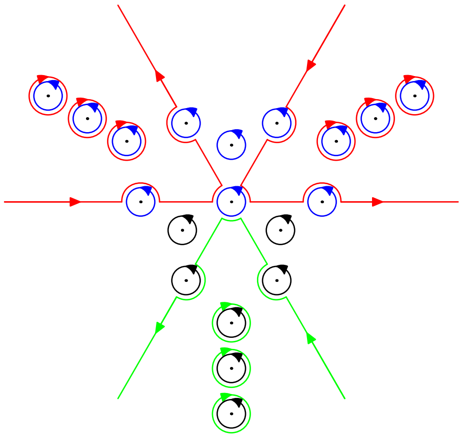

Figure 3: Definition of the contour Γ.

The transform pair is given by

$$f(x) \mapsto F(\lambda): \qquad F_\lambda(f) = \begin{cases} F_\lambda^+(f) & \text{if } \lambda \in \Gamma_0^+ \cup \Gamma_a^+, \\ F_\lambda^-(f) & \text{if } \lambda \in \Gamma_0^+ \cup \Gamma_a^+, \end{cases} \tag{2.14a}$$

$$F(\lambda) \mapsto f(x): \qquad f_x(F) = \int_\Gamma e^{i\lambda x} F(\lambda)\,\mathrm{d}\lambda, \qquad x \in [0,1], \tag{2.14b}$$

where, for $\lambda \in \mathbb{C}$ such that $\Delta(\lambda) \neq 0$,

$$F_\lambda^+(f) = \frac{1}{2\pi\Delta(\lambda)} \sum_{l=1}^n \sum_{j=1}^n \det X^{l\,j}(\lambda) M_{1\,j}^+(\lambda) \int_0^1 e^{-i\alpha^{l-1}\lambda x} f(x)\,\mathrm{d}x, \tag{2.15a}$$

$$F_\lambda^-(f) = \frac{-e^{-i\lambda}}{2\pi\Delta(\lambda)} \sum_{l=1}^n \sum_{j=1}^n \det X^{l\,j}(\lambda) M_{1\,j}^-(\lambda) \int_0^1 e^{-i\alpha^{l-1}\lambda x} f(x)\,\mathrm{d}x, \tag{2.15b}$$

and the various contours are defined by

$$\Gamma = \Gamma_0 \cup \Gamma_a, \tag{2.16a}$$

$$\Gamma_0 = \Gamma^+ \cup \Gamma^-, \tag{2.16b}$$

$$\Gamma_0^+ = \bigcup_{\substack{\sigma \in \overline{\mathbb{C}^+}: \\ \Delta(\sigma)=0}} C(\sigma, \varepsilon), \tag{2.16c}$$

$$\Gamma_0^- = \bigcup_{\substack{\sigma \in \mathbb{C}^-: \\ \Delta(\sigma)=0}} C(\sigma, \varepsilon), \tag{2.16d}$$

$$\Gamma_a = \Gamma_a^+ \cup \Gamma_a^-, \tag{2.16e}$$

$\Gamma_a^\pm$ is the boundary of the domain

$$\left\{\lambda \in \mathbb{C}^\pm : \mathrm{Re}(a\lambda^n) > 0\right\} \setminus \bigcup_{\substack{\sigma \in \mathbb{C}: \\ \Delta(\sigma)=0}} D(\sigma, 2\varepsilon). \tag{2.16f}$$

Figure 3 shows the position of the contours for some hypothetical Δ with zeros at the black dots. The contour Γ_0^+ is shown in blue and the contour Γ_0^- is shown in black. The contours Γ_a^+

and Γ_a^- are shown in red and green respectively. This case corresponds to $a = -i$. The figure indicates the possibility that there may be infinitely many zeros lying in the interior of the sectors bounded by Γ_a. For such a zero, Γ_a has a circular component enclosing this zero with radius 2ε.

The validity of the transform pairs is expressed in the following proposition:

Proposition 2.3. *Let S be a type* I *or type* II *operator. Then for all $f \in \Phi$ and for all $x \in (0,1)$,*

$$f_x(F_\lambda(f)) = \left\{\int_{\Gamma_0^+} + \int_{\Gamma_a^+}\right\} e^{i\lambda x} F_\lambda^+(f)\,\mathrm{d}\lambda + \left\{\int_{\Gamma_0^-} + \int_{\Gamma_a^-}\right\} e^{i\lambda x} F_\lambda^-(f)\,\mathrm{d}\lambda = f(x). \tag{2.17}$$

Proof. A simple calculation yields

$$\forall\, f \in C,\ \forall\, S, \qquad F_\lambda^+(f) - F_\lambda^-(f) = \frac{1}{2\pi}\hat{f}(\lambda). \tag{2.18}$$

As shown in [10], the well-posedness of the initial-boundary value problem implies $F_\lambda^\pm(f) = O(\lambda^{-1})$ as $\lambda \to \infty$ within the sectors exterior to $\Gamma_a^\pm$. The only singularities of $F_\lambda^\pm(f)$ are isolated poles hence, by Jordan's Lemma,

$$\begin{aligned}
&\left\{\int_{\Gamma_0^+} + \int_{\Gamma_a^+}\right\} e^{i\lambda x} F_\lambda^+(f)\,\mathrm{d}\lambda + \left\{\int_{\Gamma_0^-} + \int_{\Gamma_a^-}\right\} e^{i\lambda x} F_\lambda^-(f)\,\mathrm{d}\lambda \\
&= \sum_{\substack{\sigma\in\mathbb{C}:\\ \mathrm{Im}(\sigma)>\varepsilon,\\ \Delta(\sigma)=0}} \left\{\int_{C(\sigma,\varepsilon)} - \int_{C(\sigma,2\varepsilon)}\right\} e^{i\lambda x} F_\lambda^+(f)\,\mathrm{d}\lambda + \sum_{\substack{\sigma\in\mathbb{C}:\\ \mathrm{Im}(\sigma)<\varepsilon,\\ \Delta(\sigma)=0}} \left\{\int_{C(\sigma,\varepsilon)} - \int_{C(\sigma,2\varepsilon)}\right\} e^{i\lambda x} F_\lambda^-(f)\,\mathrm{d}\lambda \\
&\qquad + \int_\gamma e^{i\lambda x}\left(F_\lambda^+(f) - F_\lambda^-(f)\right)\mathrm{d}\lambda,
\end{aligned} \tag{2.19}$$

where γ is a contour running along the real line in the increasing direction but perturbed along circular arcs in such a way that it is always at least ε away from each pole of Δ. The series on the right hand side of equation (2.19) yield a zero contribution. As $f \in \Phi$, its Fourier transform $\hat{f}$ is an entire function hence, by statement (2.18), the integrand in the final term on the right hand side of equation (2.19) is an entire function and we may deform γ onto the real line. The validity of the usual Fourier transform completes the proof. ☐

3 True integral transform method for IBVP

In section 3.1 we will prove equation (1.10) for the transform pairs (1.8). In section 3.2, we establish equivalent results for general type I and type II initial-boundary value problems.

3.1 Linearized KdV

Proposition 3.1. *The solution of problem 1 is given by equation* (1.10)*, with $F_\lambda(f)$ and $f_x(F)$ defined by equations* (1.8a)–(1.8d)*.*

The solution of problem 2 is given by equation (1.10)*, with $F_\lambda(f)$ and $f_x(F)$ defined by equations* (1.8a)*,* (1.8b)*,* (1.8e) *and* (1.8f)*.*

Proof. We present the proof for problem 2. The proof for problem 1 is very similar.

Suppose $q \in C^\infty([0,1] \times [0,T])$ is a solution of the problem (1.2). Applying the forward transform to q yields

$$F_\lambda(q(\cdot,t)) = \begin{cases} \int_0^1 \phi^+(x,\lambda) q(x,t)\,\mathrm{d}x & \text{if } \lambda \in \overline{\mathbb{C}^+}, \\ \int_0^1 \phi^-(x,\lambda) q(x,t)\,\mathrm{d}x & \text{if } \lambda \in \mathbb{C}^-. \end{cases} \tag{3.1}$$

The PDE and integration by parts imply the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F_\lambda(q(\cdot,t)) &= \int_0^1 f^\pm(x,\lambda)q_{xxx}(x,t)\,\mathrm{d}x \\ &= -\partial_x^2 q(1,t)\phi^\pm(1,\lambda) + \partial_x^2 q(0,t)\phi^\pm(0,\lambda) + \partial_x q(1,t)\partial_x\phi^\pm(1,\lambda) - \partial_x q(0,t)\partial_x\phi^\pm(0,\lambda) \\ &\qquad - q(1,t)\partial_{xx}\phi^\pm(1,\lambda) + q(0,t)\partial_{xx}\phi^\pm(0,\lambda) + i\lambda^3 F_\lambda(q(\cdot,t)).\end{aligned} \tag{3.2}$$

Rearranging, multiplying by $e^{-i\lambda^3 t}$ and integrating, we find

$$F_\lambda(q(\cdot,t)) = e^{i\lambda^3 t}F_\lambda(f) + e^{i\lambda^3 t}\sum_{j=0}^{2}(-1)^j\left[\partial_x^{2-j}\phi^\pm(0,\lambda)Q_j(0,\lambda) - \partial_x^{2-j}\phi^\pm(1,\lambda)Q_j(1,\lambda)\right], \tag{3.3}$$

where

$$Q_j(x,\lambda) = \int_0^t e^{-i\lambda^3 s}\partial_x^j q(x,s)\,\mathrm{d}s. \tag{3.4}$$

Evaluating $\partial_x^j\phi^\pm(0,\lambda)$ and $\partial_x^j\phi^\pm(1,\lambda)$, we obtain

$$\begin{aligned}F_\lambda(q(\cdot,t)) = e^{i\lambda^3 t}F_\lambda(f) + \frac{e^{i\lambda^3 t}}{2\pi}\Bigg[&Q_1(1,\lambda)i\lambda(\alpha-\alpha^2)\frac{e^{i\alpha\lambda}-e^{i\alpha^2\lambda}}{\Delta(\lambda)} \\ &+Q_0(0,\lambda)\lambda^2\frac{2e^{-i\lambda}-\alpha e^{-i\alpha\lambda}-\alpha^2 e^{-i\alpha^2\lambda}}{\Delta(\lambda)} \\ &+Q_0(1,\lambda)\lambda^2\frac{(1-\alpha^2)e^{i\alpha\lambda}+(1-\alpha)e^{i\alpha^2\lambda}}{\Delta(\lambda)} + Q_2(0,\lambda) + Q_1(0,\lambda)i\lambda\Bigg],\end{aligned} \tag{3.5}$$

for all $\lambda \in \overline{\mathbb{C}^+}$ and

$$\begin{aligned}F_\lambda(q(\cdot,t)) = e^{i\lambda^3 t}F_\lambda(f) + \frac{e^{i\lambda^3 t}}{2\pi}\Bigg[&Q_1(1,\lambda)i\lambda\frac{e^{-i\lambda}+\alpha^2 e^{-i\alpha\lambda}+\alpha e^{-i\alpha^2\lambda}}{\Delta(\lambda)} \\ &+Q_0(0,\lambda)\lambda^2\frac{3}{\Delta(\lambda)} - Q_0(1,\lambda)\lambda^2\frac{e^{-i\lambda}+e^{-i\alpha\lambda}+e^{-i\alpha^2\lambda}}{\Delta(\lambda)} + Q_2(1,\lambda)e^{-i\lambda}\Bigg],\end{aligned} \tag{3.6}$$

for all $\lambda \in \mathbb{C}^-$.

Hence, the validity of the transform pair (Proposition 2.1) implies

$$
\begin{aligned}
q(x,t) = &\left\{\int_{\Gamma_0} + \int_{\Gamma^+} + \int_{\Gamma^-}\right\} e^{i\lambda x + i\lambda^3 t} F_\lambda(f)\,\mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\left\{\int_{\Gamma_0} + \int_{\Gamma^+}\right\} e^{i\lambda x + i\lambda^3 t}\left[Q_1(1,\lambda) i\lambda(\alpha-\alpha^2)\frac{e^{i\alpha\lambda} - e^{i\alpha^2\lambda}}{\Delta(\lambda)}\right. \\
&\left. + Q_0(0,\lambda)\lambda^2 \frac{2e^{-i\lambda} - \alpha e^{-i\alpha\lambda} - \alpha^2 e^{-i\alpha^2\lambda}}{\Delta(\lambda)} + Q_0(1,\lambda)\lambda^2\frac{(1-\alpha^2)e^{i\alpha\lambda} + (1-\alpha)e^{i\alpha^2\lambda}}{\Delta(\lambda)}\right]\mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\int_{\Gamma^-} e^{i\lambda x + i\lambda^3 t}\left[Q_1(1,\lambda) i\lambda \frac{e^{-i\lambda} + \alpha^2 e^{-i\alpha\lambda} + \alpha e^{-i\alpha^2\lambda}}{\Delta(\lambda)}\right. \\
&\left. + Q_0(0,\lambda)\lambda^2\frac{3}{\Delta(\lambda)} - Q_0(1,\lambda)\lambda^2 \frac{e^{-i\lambda} + e^{-i\alpha\lambda} + e^{-i\alpha^2\lambda}}{\Delta(\lambda)}\right]\mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\left\{\int_{\Gamma_0} + \int_{\Gamma^+}\right\} e^{i\lambda x + i\lambda^3 t}\left[Q_2(0,\lambda) + Q_1(0,\lambda) i\lambda\right]\mathrm{d}\lambda \\
&+ \frac{1}{2\pi}\int_{\Gamma^-} e^{i\lambda(x-1) + i\lambda^3 t} Q_2(1,\lambda)\,\mathrm{d}\lambda. \qquad (3.7)
\end{aligned}
$$

Integration by parts yields

$$Q_j(x,t) = O(\lambda^{-3}), \qquad (3.8)$$

as $\lambda \to \infty$ within the region enclosed by $\Gamma\pm$. Hence, by Jordan's lemma, the final two lines of equation (3.7) vanish. The boundary conditions imply

$$Q_0(0,\lambda) = Q_0(1,\lambda) = Q_1(1,\lambda) = 0, \qquad (3.9)$$

so the second, third, fourth and fifth lines of equation (3.7) vanish. Hence

$$q(x,t) = \left\{\int_{\Gamma_0} + \int_{\Gamma^+} + \int_{\Gamma^-}\right\} e^{i\lambda x + i\lambda^3 t} F_\lambda(f)\,\mathrm{d}\lambda. \qquad (3.10)$$

□

The above proof also demonstrates how the transform pair may be used to solve a problem with inhomogeneous boundary conditions: consider the problem,

$$
\begin{aligned}
q_t(x,t) + q_{xxx}(x,t) &= 0 & (x,t) &\in (0,1)\times(0,T), & (3.11\text{a})\\
q(x,0) &= \phi(x) & x &\in [0,1], & (3.11\text{b})\\
q(0,t) &= h_1(t) & t &\in [0,T], & (3.11\text{c})\\
q(1,t) &= h_2(t) & t &\in [0,T], & (3.11\text{d})\\
q_x(1,t) &= h_3(t) & t &\in [0,T], & (3.11\text{e})
\end{aligned}
$$

for some given boundary data $h_j \in C^\infty[0,1]$. Then $Q_0(0,\lambda)$, $Q_0(1,\lambda)$ and $Q_1(1,\lambda)$ are nonzero, but they are known quantities, namely t-transforms of the boundary data. Substituting these values into equation (3.7) yields an explicit expression for the solution.

3.2 General

Proposition 3.2. *The solution of a type* I *or type* II *initial-boundary value problem is given by*

$$q(x,t) = f_x\left(e^{-a\lambda^n t} F_\lambda(f)\right). \qquad (3.12)$$

Lemma 3.3. *Let $f \in \Phi$. Then there exist polynomials $P_f^{\pm}$ of degree at most $n-1$ such that*

$$F_\lambda^+(Sf) = \lambda^n F_\lambda^+(f) + P_f^+(\lambda), \tag{3.13a}$$

$$F_\lambda^-(Sf) = \lambda^n F_\lambda^-(f) + P_f^-(\lambda)e^{-i\lambda}. \tag{3.13b}$$

Proof. Let (ϕ, ψ) be the usual inner product $\int_0^1 \phi(x)\overline{\psi}(x)\,\mathrm{d}x$. For any $\lambda \in \Gamma$, we can represent $F_\lambda^{\pm}$ as the inner product $F_\lambda^{\pm}(f) = (f, \phi_\lambda^{\pm})$, for the function $\phi_\lambda^{\pm}(x)$, smooth in x and meromorphic in λ, defined by

$$\overline{\phi_\lambda^+}(x) = \frac{1}{2\pi\Delta(\lambda)} \sum_{l=1}^{n} \sum_{j=1}^{n} \det X^{l\,j}(\lambda) M_{1\,j}^+(\lambda) e^{-i\alpha^{l-1}\lambda x}, \tag{3.14a}$$

$$\overline{\phi_\lambda^-}(x) = \frac{-e^{-i\lambda}}{2\pi\Delta(\lambda)} \sum_{l=1}^{n} \sum_{j=1}^{n} \det X^{l\,j}(\lambda) M_{1\,j}^-(\lambda) e^{-i\alpha^{l-1}\lambda x}. \tag{3.14b}$$

As $\phi_\lambda^{\pm}$, $Sf \in C^\infty[0,1]$ and $\alpha^{(l-1)n} = 1$, equation (2.10) yields

$$F_\lambda^{\pm}(Sf) = \lambda^n F_\lambda^{\pm}(f) + [f\phi_\lambda^{\pm}](1) - [f\phi_\lambda^{\pm}](0). \tag{3.15}$$

If B, $B^\star : C^\infty[0,1] \to \mathbb{C}^n$, are the real vector boundary forms

$$B = (B_1, B_2, \dots, B_n), \qquad B^\star = (B_1^\star, B_2^\star, \dots, B_n^\star), \tag{3.16}$$

then the boundary form formula [1, Theorem 11.2.1] guarantees the existance of complimentary vector boundary forms B_c, $B_c^\star$ such that

$$[f\phi_\lambda^{\pm}](1) - [f\phi_\lambda^{\pm}](0) = Bf \cdot B_c^\star \phi_\lambda^{\pm} + B_c f \cdot B^\star \phi_\lambda^{\pm}, \tag{3.17}$$

where $\cdot$ is the sesquilinear dot product. We consider the right hand side of equation (3.17) as a function of λ. As $Bf = 0$, this expression is a linear combination of the functions $B_k^\star \overline{\phi^{\pm}}_\lambda$ of λ, with coefficients given by the complementary boundary forms.

The definitions of $B_k^\star$ and ϕ_λ^+ imply

$$\begin{aligned} B_k^\star \overline{\phi_\lambda^+} &= \frac{1}{2\pi\Delta(\lambda)} \sum_{l=1}^{n} \sum_{j=1}^{n} \det X^{l\,j}(\lambda) M_{1\,j}^+(\lambda) B_k^\star(e^{-i\alpha^{l-1}\lambda\cdot}) \\ &= \frac{1}{2\pi\Delta(\lambda)} \sum_{l=1}^{n} \sum_{j=1}^{n} \det X^{l\,j}(\lambda) M_{1\,j}^+(\lambda) M_{l\,k}(\lambda). \end{aligned}$$

But

$$\sum_{l=1}^{n} \det X^{l\,j}(\lambda) M_{l\,k}(\lambda) = \Delta(\lambda)\delta_{j\,k}, \tag{3.18}$$

so

$$B_k^\star \overline{\phi_\lambda^+} = \frac{1}{2\pi} M_{1\,k}^+(\lambda). \tag{3.19a}$$

Similarly,

$$B_k^\star \overline{\phi_\lambda^-} = \frac{-e^{-i\lambda}}{2\pi} M_{1\,k}^-(\lambda). \tag{3.19b}$$

Finally, by equations (2.12), $M_{1\,k}^{\pm}$ are polynomials of order at most $n-1$. □

of Proposition 3.2. Let q be the solution of the problem. Then, since q satisfies the partial differential equation (2.11a),

$$\frac{\mathrm{d}}{\mathrm{d}t} F^+_\lambda(q(\cdot,t)) = -aF^+_\lambda(S(q(\cdot,t))) = -a\lambda^n F^+_\lambda(q(\cdot,t)) - aP^+_{q(\cdot,t)}(\lambda), \tag{3.20}$$

where, by Lemma 3.3, $P^+_{q(\cdot,t)}$ is a polynomial of degree at most $n-1$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{a\lambda^n t} F^+_\lambda(q(\cdot,t))\right) = -ae^{a\lambda^n t} P^+_{q(\cdot,t)}(\lambda). \tag{3.21}$$

Integrating with respect to t and applying the initial condition (2.11b), we find

$$F^+_\lambda(q(\cdot,t)) = e^{-a\lambda^n t} F^+_\lambda(f) - ae^{-a\lambda^n t}\int_0^t e^{a\lambda^n s} P^+_{q(\cdot,s)}(\lambda)\,\mathrm{d}s. \tag{3.22}$$

Similarly,

$$F^-_\lambda(q(\cdot,t)) = e^{-a\lambda^n t} F^-_\lambda(f) - ae^{-i\lambda-a\lambda^n t}\int_0^t e^{a\lambda^n s} P^-_{q(\cdot,s)}(\lambda)\,\mathrm{d}s, \tag{3.23}$$

where $P^-_{q(\cdot,t)}$ is another polynomial of degree at most $n-1$. The validity of the type II transform pair, Proposition 2.3, implies

$$\begin{aligned} q(x,t) = \int_{\Gamma^+} e^{i\lambda x - a\lambda^n t} F^+_\lambda(f)\,\mathrm{d}\lambda + \int_{\Gamma^-} e^{i\lambda(x-1) - a\lambda^n t} F^-_\lambda(f)\,\mathrm{d}\lambda \\ - a\int_{\Gamma^+_0} e^{i\lambda x - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^+_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right)\mathrm{d}\lambda \\ - a\int_{\Gamma^-_0} e^{i\lambda(x-1) - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^-_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right)\mathrm{d}\lambda \\ - a\int_{\Gamma^+_a} e^{i\lambda x - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^+_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right)\mathrm{d}\lambda \\ - a\int_{\Gamma^-_a} e^{i\lambda(x-1) - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^-_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right)\mathrm{d}\lambda. \end{aligned} \tag{3.24}$$

As $P^\pm_{q(\cdot,s)}$ are polynomials, the integrands

$$e^{i\lambda x - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^+_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right) \text{ and } e^{i\lambda(x-1) - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^-_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right)$$

are both entire functions of λ. Hence the third and fourth terms of equation (3.24) vanish. Integration by parts yields

$$\begin{aligned} e^{i\lambda x - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^+_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right) &= O(\lambda^{-1}) \text{ as } \lambda\to\infty \quad \begin{array}{c}\text{within the region}\\ \text{enclosed by } \Gamma^+_a,\end{array} \\ e^{i\lambda(x-1) - a\lambda^n t}\left(\int_0^t e^{a\lambda^n s} P^-_{q(\cdot,s)}(\lambda)\,\mathrm{d}s\right) &= O(\lambda^{-1}) \text{ as } \lambda\to\infty \quad \begin{array}{c}\text{within the region}\\ \text{enclosed by } \Gamma^-_a.\end{array} \end{aligned}$$

Hence, by Jordan's Lemma, the final two terms of equation (3.24) vanish. □

Remark 2. The same method may be used to solve initial-boundary value problems with inhomogeneous boundary conditions. The primary difference is that statement (2.18) must be replaced with [10, Lemma 4.1].

4 Analysis of the transform pair

In this section we analyse the spectral properties of the transform pairs using the notion of augmented eigenfunctions.

4.1 Linearized KdV

Augmented Eigenfunctions

Let $S^{(\mathrm{I})}$ and $S^{(\mathrm{II})}$ be the differential operators representing the spatial parts of the IBVPs 1 and 2, respectively. Each operator is a restriction of the same formal differential operator, $(-i\,\mathrm{d}/\,\mathrm{d}x)^3$ to the domain of initial data compatible with the boundary conditions of the problem:

$$\mathcal{D}(S^{(\mathrm{I})}) = \{f \in C^\infty[0,1] : f(0) = f(1) = 0,\ f'(0) = 2f'(1)\}, \tag{4.1}$$

$$\mathcal{D}(S^{(\mathrm{II})}) = \{f \in C^\infty[0,1] : f(0) = f(1) = f'(1) = 0\}. \tag{4.2}$$

A simple calculation reveals that $\{F_\lambda : \lambda \in \Gamma_0, \Delta(\sigma) = 0\}$ (where F_λ is defined by equations (1.8a), (1.8c) and (1.8d)) is a family of type I augmented eigenfunctions of $S^{(\mathrm{I})}$. Indeed, integration by parts yields

$$F_\lambda(S^{(\mathrm{I})}f) = \begin{cases} \lambda^3 F_\lambda(f) + \left(-\dfrac{i}{2\pi}f''(0) + \dfrac{\lambda}{2\pi}f'(0)\right) & \lambda \in \overline{\mathbb{C}^+}, \\ \lambda^3 F_\lambda(f) + \left(-\dfrac{i}{2\pi}f''(1) + \dfrac{\lambda}{2\pi}f'(1)\right) & \lambda \in \mathbb{C}^-. \end{cases} \tag{4.3}$$

For any f, the remainder functional is an entire function of λ and Γ_0 is a closed, circular contour hence (1.12) holds.

In the same way $\{F_\lambda : \lambda \in \Gamma_0\}$ (where F_λ is defined by equations (1.8a), (1.8c) and (1.8d)) is a family of type I augmented eigenfunctions of $S^{(\mathrm{II})}$. Indeed

$$F_\lambda(S^{(\mathrm{II})}f) = \begin{cases} \lambda^3 F_\lambda(f) + \left(-\dfrac{i}{2\pi}f''(0) - \dfrac{\lambda}{2\pi}f'(0)\right) & \lambda \in \overline{\mathbb{C}^+}, \\ \lambda^3 F_\lambda(f) + \left(-\dfrac{i}{2\pi}f''(1)\right) & \lambda \in \mathbb{C}^-, \end{cases} \tag{4.4}$$

so the remainder functional is again entire.

Furthermore, the ratio of the remainder functionals to the eigenvalue is a rational function with no pole in the regions enclosed by $\Gamma^\pm$ and decaying as $\lambda \to \infty$. Jordan's lemma implies (1.13) hence $\{F_\lambda : \lambda \in \Gamma^+ \cup \Gamma^-\}$ is a family of type II augmented eigenfunctions of the corresponding $S^{(\mathrm{I})}$ or $S^{(\mathrm{II})}$.

Spectral representation of $S^{(\mathrm{II})}$

We have shown above that $\{F_\lambda : \lambda \in \Gamma_0\}$ is a family of type I augmented eigenfunctions and $\{F_\lambda : \lambda \in \Gamma^+ \cup \Gamma^-\}$ is a family of type II augmented eigenfunctions of $S^{(\mathrm{II})}$, each with eigenvalue λ^3. It remains to show that the integrals

$$\int_{\Gamma_0} e^{i\lambda x} F_\lambda(Sf)\,\mathrm{d}\lambda, \qquad \int_{\Gamma^+\cup\Gamma^-} e^{i\lambda x} F_\lambda(f)\,\mathrm{d}\lambda \tag{4.5}$$

converge.

A simple calculation reveals that $F_\lambda(\psi)$ has a removable singularity at $\lambda = 0$, for any $\psi \in C$. Hence the first integral not only converges but evaluates to 0. Thus, the second integral represents $f_x(F_\lambda(f)) = f$ and converges by Proposition 2.1.

This completes the proof of Theorem 1.5 for problem 2.

Spectral representation of $S^{(\mathrm{I})}$

By the above argument, it is clear that the transform pair (F_λ, f_x) defined by equations (1.3) provides a spectral representation of $S^{(\mathrm{I})}$ in the sense of Definition 1.4, verifying Theorem 1.5 for problem 1.

It is clear that $\{F_\lambda : \lambda \in \Gamma^\pm\}$ is not a family of type I augmented eigenfunctions, so the representation (1.3) does *not* provide a spectral representation of $S^{(\mathrm{I})}$ in the sense of Definition 1.3. However, equation (1.7) does provide a representation in the sense of Definition 1.3. Indeed, equation (1.7) implies that it is possible to deform the contours $\Gamma^\pm$ onto

$$\bigcup_{\substack{\sigma\in\mathbb{C}:\\ \Delta(\sigma)=0}} \Gamma_\sigma.$$

It is possible to make this deformation without any reference to the initial-boundary value problem. By an argument similar to that in the proof of Proposition 2.1, we are able to 'close' (whereas in the earlier proof we 'opened') the contours $\Gamma^\pm$ onto simple circular contours each enclosing a single zero of Δ. Thus, an equivalent inverse transform is given by (1.9). It is clear that, for each σ a zero of Δ, $\{F_\lambda : \lambda \in \Gamma_\sigma\}$ is a family of type I augmented eigenfunctions of $S^{(\mathrm{I})}$ up to integration over Γ_σ.

It remains to show that the series

$$\sum_{\substack{\sigma\in\mathbb{C}:\\ \Delta(\sigma)=0}} \int_{\Gamma_\sigma} e^{i\lambda x} F_\lambda(Sf)\,\mathrm{d}\lambda \tag{4.6}$$

converges. The validity of the transform pair (F_λ, f_x^Σ) defined by equations (1.8a), (1.8c), (1.8d) and (1.9) is insufficient to justify this convergence since, in general, Sf may not satisfy the boundary conditions, so Sf may not be a valid initial datum of the problem. Thus, we prove convergence directly.

The augmented eigenfunctions F_λ are meromorphic functions of λ, represented in their definition (1.8a), (1.8c), (1.8d) as the ratio of two entire functions, with singularities only at the zeros of the exponential polynomial Δ. The theory of exponential polynomials [7] implies that the only zeros of Δ are of finite order, so each integral in the series converges and is equal to the residue of the pole at σ. Furthermore, an asymptotic calculation reveals that these zeros are at 0, $\alpha^j\lambda_k$, $\alpha^j\mu_k$, for each $j \in \{0,1,2\}$ and $k \in \mathbb{N}$, where

$$\lambda_k = \left(2k - \frac{1}{3}\right)\pi + i\log 2 + O\left(e^{-\sqrt{3}k\pi}\right), \tag{4.7}$$

$$\mu_k = -\left(2k - \frac{1}{3}\right)\pi + i\log 2 + O\left(e^{-\sqrt{3}k\pi}\right). \tag{4.8}$$

Evaluating the first derivative of Δ at these zeros, we find

$$\Delta'(\lambda_k) = (-1)^{k+1}\sqrt{2}e^{i\frac{\sqrt{3}}{2}\log 2}e^{\sqrt{3}\pi(k-1/6)} + O(1), \tag{4.9}$$

$$\Delta'(\mu_k) = (-1)^{k}\sqrt{2}e^{-i\frac{\sqrt{3}}{2}\log 2}e^{\sqrt{3}\pi(k-1/6)} + O(1). \tag{4.10}$$

Hence, at most finitely many zeros of Δ are of order greater than 1. A straightforward calculation reveals that 0 is a removable singularity. Hence, via a residue calculation and integration by parts, we find that we can represent the tail of the series (4.6) in the form

$$
\begin{aligned}
i \sum_{k=N}^{\infty} \Bigg\{ \frac{1}{\lambda_k \Delta'(\lambda_k)} & \left[e^{i\lambda_k x} \left((Sf)(1) Y_1(\lambda_k) - (Sf)(0) Y_0(\lambda_k) \right) \right. \\
&\quad + \alpha^2 e^{i\alpha\lambda_k x} \left((Sf)(1) Y_1(\alpha\lambda_k) - (Sf)(0) Y_0(\alpha\lambda_k) \right) \\
&\quad \left. - \alpha e^{i\alpha^2\lambda_k (x-1)} \left((Sf)(1) Z_1(\alpha^2\lambda_k) - (Sf)(0) Z_0(\alpha^2\lambda_k) \right) \right] \\
+ \frac{1}{\mu_k \Delta'(\mu_k)} & \left[e^{i\mu_k x} \left((Sf)(1) Y_1(\mu_k) - (Sf)(0) Y_0(\mu_k) \right) \right. \\
&\quad - \alpha^2 e^{i\alpha\mu_k (x-1)} \left((Sf)(1) Z_1(\alpha\mu_k) - (Sf)(0) Z_0(\alpha\mu_k) \right) \\
&\quad \left. + \alpha e^{i\alpha^2\mu_k x} \left((Sf)(1) Y_1(\alpha^2\mu_k) - (Sf)(0) Y_0(\alpha^2\mu_k) \right) \right] + O(k^{-2}) \Bigg\}, \qquad (4.11a)
\end{aligned}
$$

where

$$Y_1(\lambda) = 3 + 2(\alpha^2 - 1) e^{i\alpha\lambda} + 2(\alpha - 1) e^{i\alpha^2\lambda}, \qquad (4.11b)$$

$$Y_0(\lambda) = e^{i\lambda} + e^{i\alpha\lambda} + e^{i\alpha^2\lambda} - 4e^{-i\lambda} + 2e^{-i\alpha\lambda} + 2e^{-i\alpha^2\lambda}, \qquad (4.11c)$$

$$Z_1(\lambda) = \alpha e^{i\alpha\lambda} + 2e^{-i\alpha\lambda} + \alpha^2 e^{i\alpha^2\lambda} + 2e^{-i\alpha^2\lambda}, \qquad (4.11d)$$

$$Z_0(\lambda) = 6 + (\alpha^2 - 1) e^{-i\alpha\lambda} + (\alpha - 1) e^{-i\alpha^2\lambda}. \qquad (4.11e)$$

As $Y_j,\ Z_j \in O(\exp(\sqrt{3}\pi k))$, the Riemann-Lebesgue lemma guarantees conditional convergence for all $x \in (0, 1)$.

This completes the proof of Theorem 1.6.

Remark 3. We observed above that 0 is removable singularity of F_λ defined by (1.8a), (1.8c) and (1.8d). The same holds for F_λ defined by (1.8a), (1.8e) and (1.8f). Hence, for both problems 1 and 2,

$$\int_{\Gamma_0} e^{i\lambda x} F_\lambda(f) \, \mathrm{d}\lambda = 0 \qquad (4.12)$$

and we could redefine the inverse transform (1.8b) as

$$F(\lambda) \mapsto f(x): \qquad f_x(F) = \left\{ \int_{\Gamma^+} + \int_{\Gamma^-} \right\} e^{i\lambda x} F(\lambda) \, \mathrm{d}\lambda, \qquad x \in [0, 1]. \qquad (4.13)$$

This permits spectral representations of both $S^{(\mathrm{I})}$ and $S^{(\mathrm{II})}$ via augmented eigenfunctions of type II *only*, that is spectral representations in the sense of Definition 1.4 but with $E^{(\mathrm{I})} = \emptyset$.

4.2 General

We will show that the transform pair (F_λ, f_x) defined by equations (2.14) represents spectral decomposition into type I and type II augmented eigenfunctions.

Theorem 4.1. *Let S be the spatial differential operator associated with a type* II *IBVP. Then the transform pair (F_λ, f_x) provides a spectral representation of S in the sense of Definition 1.4.*

The principal tools for constructing families of augmented eigenfunctions are Lemma 3.3, as well as the following lemma:

Lemma 4.2. *Let $F_\lambda^\pm$ be the functionals defined in equations* (2.15).

(i) Let γ be any simple closed contour. Then $\{F_\lambda^\pm : \lambda \in \gamma\}$ are families of type I *augmented eigenfunctions of S up to integration along γ with eigenvalues λ^n.*

(ii) Let γ be any simple closed contour which neither passes through nor encloses 0*. Then* $\{F_\lambda^\pm : \lambda \in \gamma\}$ *are families of type* II *augmented eigenfunctions of* S *up to integration along* γ *with eigenvalues* λ^n*.*

(iii) Let $0 \leqslant \theta < \theta' \leqslant \pi$ *and define* γ^+ *to be the boundary of the open set*

$$\{\lambda \in \mathbb{C} : |\lambda| > \varepsilon,\, \theta < \arg \lambda < \theta'\}; \tag{4.14}$$

similarly, γ^- *is the boundary of the open set*

$$\{\lambda \in \mathbb{C} : |\lambda| > \varepsilon,\, -\theta' < \arg \lambda < -\theta\}. \tag{4.15}$$

Both γ^+ *and* γ^- *have positive orientation. Then* $\{F_\lambda^\pm : \lambda \in \gamma^\pm\}$ *are families of type* II *augmented eigenfunctions of* S *up to integration along* $\gamma^\pm$ *with eigenvalues* λ^n*.*

Proof.

(i) & (ii) By Lemma 3.3, the remainder functionals are analytic in λ within the region bounded by γ. Cauchy's theorem yields the result.

(iii) The set enclosed by γ^+ is contained within the upper half-plane. By Lemma 3.3,

$$\int_{\gamma^+} e^{i\lambda x} \lambda^{-n} (F_\lambda^+(Sf) - \lambda^n F_\lambda^+(f))\, \mathrm{d}\lambda = \int_{\gamma^+} e^{i\lambda x} \lambda^{-n} P_f^+(\lambda)\, \mathrm{d}\lambda, \tag{4.16}$$

and the integrand is the product of $e^{i\lambda x}$ with a function analytic on the enclosed set and decaying as $\lambda \to \infty$. Hence, by Jordan's Lemma, the integral of the remainder functionals vanishes for all $x > 0$. For γ^-, the proof is similar.

□

Remark 4. If we restrict to the case $0 < \theta < \theta' < \pi$ then the functionals $F_\lambda^\pm$ form families of type I augmented eigenfunctions up to integration along the resulting contours but this is insufficient for our purposes. Indeed, an infinite component of Γ_a lies on the real axis, but

$$\int_{-\infty}^{\infty} e^{i\lambda x} P_f^+(\lambda)\, \mathrm{d}\lambda \tag{4.17}$$

diverges and can only be interpreted as a sum of delta functions and their derivatives.

Let (S, a) be such that the associated initial-boundary value problem is well-posed. Then there exists a complete system of augmented eigenfunctions associated with S, some of which are type I whereas the rest are type II. Indeed:

Proposition 4.3. *The system*

$$\mathcal{F}_0 = \{F_\lambda^+ : \lambda \in \Gamma_0^+\} \cup \{F_\lambda^- : \lambda \in \Gamma_0^-\} \tag{4.18}$$

is a family of type I *augmented eigenfunctions of* S *up to integration over* Γ_0*, with eigenvalues* λ^n*. The system*

$$\mathcal{F}_a = \{F_\lambda^+ : \lambda \in \Gamma_a^+\} \cup \{F_\lambda^- : \lambda \in \Gamma_a^-\} \tag{4.19}$$

is a family of type II *augmented eigenfunctions of* S *up to integration over* Γ_a*, with eigenvalues* λ^n*.*

Furthermore, if an initial-boundary value problem associated with S *is well-posed, then* $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_a$ *is a complete system.*

Proof. Considering $f \in \Phi$ as the initial datum of the homogeneous initial-boundary value problem and applying Proposition 3.2, we evaluate the solution of problem (2.11) at $t = 0$,

$$f(x) = q(x,0) = \int_{\Gamma_0^+} e^{i\lambda x} F_\lambda^+(f)\,\mathrm{d}\lambda + \int_{\Gamma_0^-} e^{i\lambda x} F_\lambda^-(f)\,\mathrm{d}\lambda. \tag{4.20}$$

Thus, if $F_\lambda^\pm(f) = 0$ for all $\lambda \in \Gamma_0$ then $f = 0$.

By Lemma 4.2 (i), $\mathcal{F}_0$ is a system of type I augmented eigenfunctions up to integration along $\Gamma_0^+ \cup \Gamma_0^-$.

Applying Lemma 3.3 to $\mathcal{F}_a$, we obtain

$$F_\lambda^\pm(Sf) = \lambda^n F_\lambda^\pm(f) + R_\lambda^\pm(f), \tag{4.21}$$

with

$$R_\lambda^+(f) = P_f^+(\lambda), \qquad R_\lambda^-(f) = P_f^-(\lambda)e^{-i\lambda}. \tag{4.22}$$

By Lemma 4.2 (ii), we can deform the contours $\Gamma_a^\pm$ onto the union of several contours of the form of the $\gamma^\pm$ appearing in Lemma 4.2 (iii). The latter result completes the proof. □

of Theorem 4.1. Proposition 4.3 establishes completeness of the augmented eigenfunctions and equations (1.19), under the assumption that the integrals converge. The series of residues

$$\int_{\Gamma_0} e^{i\lambda x} F_\lambda^\pm(Sf)\,\mathrm{d}\lambda = 2\pi i \sum_{\substack{\sigma\in\mathbb{C}:\\ \Delta(\sigma)=0}} e^{i\sigma x} \operatorname*{Res}_{\lambda=\sigma} F_\lambda^\pm(Sf), \tag{4.23}$$

whose convergence is guaranteed by the well-posedness of the initial-boundary value problem [11]. Indeed, a necessary condition for well-posedness is the convergence of this series for $Sf \in \Phi$. But then the definition of $F_\lambda^\pm$ implies

$$\operatorname*{Res}_{\lambda=\sigma} F_\lambda^\pm(f) = O(|\sigma|^{-j-1}), \text{ where } j = \max\{k : \ \forall\, f \in \Phi,\ f^{(k)}(0) = f^{(k)}(1) = 0\},$$

so $\operatorname{Res}_{\lambda=\sigma} F_\lambda(Sf) = O(|\sigma|^{-1})$ and the Riemann-Lebesgue lemma gives convergence. This verifies statement (1.17). Theorem 2.3 ensures convergence of the right hand side of equation (1.19b). Hence statement (1.18) holds. □

Remark 5. Suppose S is a type I operator.

By the definition of a type I operator (more precisely, by the properties of an associated type I IBVP, see [11]), $F_\lambda^\pm(\phi) = O(\lambda^{-1})$ as $\lambda \to \infty$ within the sectors interior to $\Gamma_a^\pm$. Hence, by Jordan's Lemma,

$$\int_{\Gamma_a^+} e^{i\lambda x} F_\lambda^+(\phi)\,\mathrm{d}\lambda + \int_{\Gamma_a^-} e^{i\lambda x} F_\lambda^-(\phi)\,\mathrm{d}\lambda = 0. \tag{4.24}$$

Hence, it is possible to define an alternative inverse transform

$$F(\lambda) \mapsto f(x) : \qquad f_x^\Sigma(F) = \int_{\Gamma_0} e^{i\lambda x} F(\lambda)\,\mathrm{d}\lambda, \tag{4.25}$$

equivalent to f_x. The new transform pair (F_λ, f_x^Σ) defined by equations (2.14) and (4.25) may be used to solve an IBVP associated with S hence

$$\mathcal{F}_0 = \{F_\lambda^+ : \lambda \in \Gamma_0^+\} \cup \{F_\lambda^- : \lambda \in \Gamma_0^-\} \tag{4.26}$$

is a complete system of functionals on Φ.

Moreover, $\mathcal{F}_0$ is a family of type I augmented eigenfunctions *only*. Hence, $\mathcal{F}_0$ provides a spectral representation of S in the sense of Definition 1.3. Via a residue calculation at each zero of Δ, one obtains a classical spectral representation of S as a series of (generalised) eigenfunctions.

We emphasize that this spectral representation without type II augmented eigenfunctions is only possible for a type I operator.

Remark 6. By definition, the point $3\varepsilon/2$ is always exterior to the set enclosed by Γ. Therefore introducing a pole at $3\varepsilon/2$ does not affect the convergence of the contour integral along Γ. This means that, the system $\mathcal{F}' = \{(\lambda - 3\varepsilon/2)^{-n} F_\lambda : \lambda \in \Gamma\}$ is a family of type I augmented eigenfunctions, thus no type II augmented eigenfunctions are required; equation (1.16) holds for $\mathcal{F}'$ and the integrals converge. However, we cannot show that $\mathcal{F}'$ is complete, so we do not have a spectral representation of S through the system $\mathcal{F}'$.

Remark 7. There may be at infinitely many circular components of Γ_a, each corresponding to a zero of Δ which lies in the interior of a sector enclosed by the main component of Γ_a. It is clear that in equations (2.17) and (3.12), representing the validity of the transform pair and the solution of the initial-boundary value problem, the contributions of the integrals around these circular contours are cancelled by the contributions of the integrals around certain components of Γ_0, as shown in Figure 3. Hence, we could redefine the contours Γ_a and Γ_0 to exclude these circular components without affecting the validity of Propositions 2.3 and 3.2.

Our choice of Γ_a is intended to reinforce the notion that S is split into two parts by the augmented eigenfunctions. In Γ_0, we have chosen a contour which encloses each zero of the characterstic determinant individually, since each of these zeros is a classical eigenvalue, so $\mathcal{F}_0$ corresponds to the set of all generalised eigenfunctions. Hence $\mathcal{F}_a$ corresponds only to the additional spectral objects necessary to form a complete system.

Remark 8. As Γ_a encloses no zeros of Δ, we could choose a $R > 0$ and redefine $\Gamma^{\pm}_{a\,R}$ as the boundary of

$$\left\{\lambda \in \mathbb{C}^{\pm} : |\lambda| > R,\ \mathrm{Re}(a\lambda^n) > 0\right\} \setminus \bigcup_{\substack{\sigma\in\mathbb{C}:\\ \Delta(\sigma)=0}} D(\sigma, 2\varepsilon), \tag{4.27}$$

deforming Γ_a over a finite region. By considering the limit $R \to \infty$, we claim that $\mathcal{F}_a$ can be seen to represent *spectral objects with eigenvalue at infinity.*

Remark 9. By Lemma 4.2(ii), for all $\sigma \neq 0$ such that $\Delta(\sigma) = 0$, it holds that $\{F^{\pm}_\lambda : \lambda \in C(\sigma, \varepsilon)\}$ are families of type II augmented eigenfunctions. Hence, the only component of Γ_0 that may not be a family of type II augmented eigenfunctions is $C(0, \varepsilon)$. If

$$\gamma_a^+ = \Gamma_a^+ \cup \bigcup_{\substack{\sigma\in\overline{\mathbb{C}^+}:\\ \sigma\neq 0,\\ \Delta(\sigma)=0}} C(\sigma, \varepsilon), \tag{4.28a}$$

$$\gamma_a^- = \Gamma_a^- \cup \bigcup_{\substack{\sigma\in-\mathbb{C}^-:\\ \Delta(\sigma)=0}} C(\sigma, \varepsilon), \tag{4.28b}$$

$$\gamma_0 = C(0, \varepsilon), \tag{4.28c}$$

then

$$\mathcal{F}'_a = \{F^+\lambda : \lambda \in \gamma_a^+\} \cup \{F^-\lambda : \lambda \in \gamma_a^-\} \tag{4.29}$$

is a family of type II augmented eigenfunctions and

$$\mathcal{F}'_0 = \{F^+\lambda : \lambda \in \gamma_0\} \tag{4.30}$$

is a family of type I augmented eigenfunctions of S. For S type I or type II, $\mathcal{F}'_a \cup \mathcal{F}'_0$ provides a spectral representation of S in the sense of Definition 1.4, with minimal type I augmented eigenfunctions. (Note that it is possible to cancel certain circular components of $\gamma_a^{\pm}$.)

Assume that 0 is a removable singularity of F^+_λ. Then $\mathcal{F}'_a$ provides a spectral representation of S in the sense of Definition 1.4 with $E^{(\mathrm{I})} = \emptyset$. We have already identified the operators $S^{(\mathrm{I})}$ and $S^{(\mathrm{II})}$ for which this representation is possible (see Remark 3).

Remark 10. The validity of Lemmata 3.3 and 4.2 does not depend upon the class to which S belongs. Hence, even if all IBVPs associated with S are ill-posed, it is still possible to construct families of augmented eigenfunctions of S. However, without the well-posedness of an associated initial-boundary value problem, an alternative method is required in order to analyse the completeness of these families. Without completeness results, it is impossible to discuss the diagonalisation by augmented eigenfunctions.

5 Conclusion

In the classical separation of variables, one makes a particular assumption on the form of the solution. For evolution PDEs in one dimension, this is usually expressed as

> "Assume the solution takes the form $q(x,t) = \tau(t)\xi(x)$ for all $(x,t) \in [0,1] \times [0,T]$ for some $\xi \in C^\infty[0,1]$ and $\tau \in C^\infty[0,T]$."

However, when applying the boundary conditions, one superimposes infinitely many such solutions. So it would be more accurate to use the assumption

> "Assume the solution takes the form $q(x,t) = \sum_{m\in\mathbb{N}} \tau_m(t)\xi_m(x)$ for some sequences of functions $\xi_m \in C^\infty[0,1]$ which are eigenfunctions of the spatial differential operator, and $\tau_m \in C^\infty[0,T]$; assume that the series converges uniformly for $(x,t) \in [0,1]\times[0,T]$."

For this 'separation of variables' scheme to yield a result, we require completeness of the eigenfunctions $(\xi_m)_{m\in\mathbb{N}}$ in the space of admissible initial data.

The concept of generalized eigenfunctions, as presented by Gelfand and coauthors [5, 6] allows one to weaken the above assumption in two ways: first, it allows the index set to be uncountable, hence the series is replaced by an integral. Second, certain additional spectral functions, which are not genuine eigenfunctions, are admitted to be part of the series.

An integral expansion in generalized eigenfunctions is insufficient to describe the solutions of IBVPs obtained via the unified transform method for type II problems. In order to describe these IBVPs, we have introduced type II augmented eigenfunctions. Using these new eigenfunctions, the assumption is weakened further:

> "Assume the solution takes the form $q(x,t) = \int_{m\in\Gamma} \tau_m(t)\xi_m(x)\,\mathrm{d}m$ for some functions $\xi_m \in C^\infty[0,1]$, which are type I and II augmented eigenfunctions of the spatial differential operator, and $\tau_m \in C^\infty[0,T]$; assume that the integral converges uniformly for $(x,t) \in [0,1] \times [0,T]$."

It appears that it is *not* possible to weaken the above assumption any further. Indeed, it has been established in [4] that the unified method provides the solution of *all* well-posed problems. The main contribution of this paper is to replace the above assumption with the following theorem:

> "Suppose $q(x,t)$ is the C^∞ solution of a well-posed two-point linear constant-coefficient initial-boundary value problem. Then $q(x,t) = \int_{m\in\Gamma} \tau_m(t)\xi_m(x)\,\mathrm{d}m$, where $\xi_m \in C^\infty[0,1]$ are type I and II augmented eigenfunctions of the spatial differential operator and $\tau_m \in C^\infty[0,T]$ are some coefficient functions. The integral converges uniformly for $(x,t) \in [0,1] \times [0,T]$."

In summary, both type I and type II IBVPs admit integral representations like (1.3), which give rise to transform pairs associated with a combination of type I and type II augmented eigenfunctions. For type I IBVPs, it is possible (by appropriate contour deformations) to obtain alternative integral representations like (1.7), which give rise to transform pairs associated with only type I augmented eigenfunctions. Furthermore, in this case, a residue calculation yields a classical series representation, which can be associated with Gel'fand's generalised eigenfunctions.

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