ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ - ΤΜΗΜΑ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
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Another construction of BV solutions to rate-independent systems

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ANOTHER CONSTRUCTION OF BV SOLUTIONS TO RATE-INDEPENDENT SYSTEMS

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Abstract. We study one kind of weak solutions to rate-independent systems, which is constructed by using the local minimality in a small neighborhood of order ε and then taking the limit $\varepsilon \to 0$. We show that the resulting solution satisfies both the weak local stability and the new energy-dissipation balance, similarly to the BV solutions constructed by vanishing viscosity introduced recently by Mielke, Rossi and Savaré.

1. Introduction

A rate-independent system is a specific case of quasistatic systems. It is time-dependent but its behavior is slow enough that the inertial effects can be ignored and the systems are affected only by external loadings. Some specific rate-independent systems were studied by many authors including Francfort, Marigo, Larsen, Dal Maso and Lazzaroni on brittle fractures [9, 8, 11, 6], Dal Maso, DeSimone and Solombrino on the Cam-Clay model [5], Dal Maso, DeSimone, Mora, Morini on plasticity with softening [3, 4], Mielke on elasto-plasticity [13, 14], Mielke, Theil and Levitas on shape-memory alloys [21, 22, 23], Müller, Schmid and Mielke on super-conductivity [24, 26], and Alberti and DeSimone on capillary drops [1]. We refer to the surveys [16, 15, 17, 18] by Mielke for the study in abstract setting as well as for further references.

In this work, for simplicity we consider an evolution $u : [0, T] \to \mathbb{R}^d$, subject to a force defined by an energy functional $\mathscr{E} : [0, T] \times \mathbb{R}^d \to [0, +\infty)$, which is of class C^1, and a dissipation function $\Psi(x) := |x|$. Let an initial position $x_0 \in \mathbb{R}^d$ such that x_0 is a local minimizer for the functional $x \mapsto \mathscr{E}(0, x) + |x - x_0|$. We say that u is a solution to the rate-independent system $(\mathscr{E}, x_0)$ if $u(0) = x_0$ and the following inclusion holds true,

$$0 \in (\partial_x |\cdot|)(\dot{u}(t)) + \nabla_x \mathscr{E}(t, u(t)) \text{ for a.e. } t \in (0, T). \tag{1}$$

In general, strong solutions to (1) may not exist [27]. Hence, the question on defining some *weak solutions* arises naturally.

A widely-used weak solution is *energetic solution*, which was first introduced by Mielke and Theil [21] (see [22, 12, 10, 16] for further studies). A function $u : [0, T] \to \mathbb{R}^d$ is called an energetic solution to the rate-independent system $(\mathscr{E}, x_0)$, if it satisfies

(i) the initial condition $u(0) = x_0$,

Date: Octorber 14, 2012.

1991 *Mathematics Subject Classification.* Primary: 49M99; Secondary: 49J20.

Key words and phrases. Rate-independent systems, BV solutions, local minimizers, energy-dissipation balance.

The author is partially supported by the PRIN 2008 grant "Optimal mass transportation, Geometric and Functional Inequalities and Applications" and the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation".

(ii) the *global stability* that for $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\mathscr{E}(t,u(t)) \le \mathscr{E}(t,x) + |x - u(t)|, \tag{2}$$

(iii) and the *energy-dissipation balance* that for all $0 \le t_1 < t_2 \le T$,

$$\mathscr{E}(t_2,u(t_2)) - \mathscr{E}(t_1,u(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s,u(s))\,ds - \mathscr{D}iss(u;[t_1,t_2]). \tag{3}$$

Here we used the notion of dissipation

$$\mathscr{D}iss(u(t);[t_1,t_2]) := \sup\left\{ \sum_{i=1}^{N} |u(s_{i-1}) - u(s_i)| \;\middle|\; N \in \mathbb{N}, t_1 \le s_0 < s_1 < \cdots < s_N \le t_2 \right\}.$$

Note that in the case that energy functional is not convex, the global minimality (2) makes the energetic solutions jump sooner than they should, and hence fail to describe the related physical phenomena (see Examples 2 below). Hence, some weak solutions based on *local minimality* are of interests.

Recently, an elegant weak solution based on vanishing viscosity method was introduced by Mielke, Rossi and Savaré [19, 20]. Their idea is to add a small viscosity term to the dissipation functional Ψ. This results in a new dissipation functional Ψ_ε, e.g. $\Psi_\varepsilon(x) = |x| + \varepsilon|x|^2$, which has super-linear growth at infinity and which converges to Ψ as ε tends to zero in an appropriate sense. They showed that the modified system $(\mathscr{E}, x_0)$ with $|\cdot|$ replaced by Ψ_ε admits a solution u_ε. The limit u of a subsequence u_ε as $\varepsilon \to 0$, called *BV solution*, enjoys the following properties

(i) the initial condition $u(0) = x_0$,

(ii) the *weak local stability* that for all $t \in [0,T]\backslash J$,

$$|\nabla_x \mathscr{E}(t,u(t))| \le 1, \tag{4}$$

(iii) and the *new energy-dissipation balance* that for all $0 \le t_1 \le t_2 \le T$,

$$\mathscr{E}(t_2,u(t_2)) - \mathscr{E}(t_1,u(t_1)) = \int_{t_1}^{t_2} \partial_t \mathscr{E}(s,u(s))\,ds - \mathscr{D}iss_{new}(u;[t_1,t_2]). \tag{5}$$

Here we denote the *jump set* by

$$J := \{t \in [0,T] \mid u(\cdot) \text{ is not continuous at } t\}$$

and the *new dissipation* by

$$\begin{aligned} \mathscr{D}iss_{new}(u;[t_1,t_2]) \;&:=\; \mathscr{D}iss(u;[t_1,t_2]) + \sum_{t\in J} \left(\Delta_{new}(t,x(t^-),x(t)) + \Delta_{new}(t,x(t),x(t^+))\right) \\ &\quad - \sum_{t \in J} \left(|u(t^-) - u(t)| + |u(t) - u(t^+)|\right), \end{aligned}$$

where $\Delta_{new}(t;a,b)$ depends also on the energy functional $\mathscr{E}$, and is defined by

$$\inf\left\{ \int_0^1 |\dot\gamma(s)| \cdot \max\{1, |\nabla_x \mathscr{E}(t,\gamma(s))|\,ds\} \mid \gamma \in AC([0,1];\mathbb{R}^d), \gamma(0) = a, \gamma(1) = b \right\}.$$

The new energy-dissipation balance is a deeply insight observation, which contains the information at the jump points. Indeed, it was shown in [20] if the BV solution u jumps at time t, then there exists an absolutely continuous path $\gamma : [0,1] \to \mathbb{R}^d$, which called an *optimal transition* between $u(t^-)$ and $u(t^+)$, such that

(i) $\gamma(0) = u(t^-)$ and $\gamma(1) = u(t^+)$,

(ii) $|\nabla_x \mathscr{E}(t,\gamma(s))| \geq 1$ for all $s \in [0,1]$,
(iii) and $\mathscr{E}(t,u(t^-)) - \mathscr{E}(t,u(t^+)) = \int_0^1 |\nabla_x \mathscr{E}(t,\gamma(s))| \cdot |\gamma'(s)|\, ds$.

An inconvenience of the BV solution constructed by vanishing viscosity is that it depends on the choice of the viscosity, and the solution obtained by some viscosity does not have expected behavior (see Examples 2).

In this work, we shall study another weak solution which is constructed by the local minimality in a small neighborhood. The idea is to consider the minimization problem (2) in a small neighborhood of order ε and obtain a solution u_ε, and then take $\varepsilon \to 0$ to get a limit u, which called BV solution constructed by *epsilon-neighborhood* method. The epsilon-neighborhood approach was first suggested in [14, Section 6] for one dimensional case when ε is chosen proportional to the square root of the time-step and the weak local stability was then obtained in [7].

Roughly speaking, this approach is a special case of vanishing viscosity approach when viscosity term is chosen as follows

$$\Psi_0(v) := \begin{cases} 0 & \text{if } |v| \leq 1, \\ +\infty & \text{if } |v| > 1. \end{cases}$$

However, Ψ_0 does not quite satisfy the requirement to become a viscosity in vanishing viscosity in [20, Section 2.3].

In this article, we shall show that the BV solutions constructed by epsilon-neighborhood method u indeed satisfies both the weak local stability and the new energy-dissipation balance, similarly to the BV solutions introduced by Mielke, Rossi and Savaré [19, 20].

2. Main results

For simplicity, we shall consider the case when $X = \mathbb{R}^d$ and $\Psi(x) = |x|$. Moreover, we assume that the energy functional $\mathscr{E}(t,x) : [0,T] \times \mathbb{R}^d \to [0,\infty)$ is C^1, and satisfies the following technical assumption: there exists $\lambda = \lambda(\mathscr{E})$ such that

$$|\partial_t \mathscr{E}(s,x)| \leq \lambda\, \mathscr{E}(s,x) \text{ for all } (s,x) \in [0,T] \times \mathbb{R}^d. \tag{6}$$

Remark. The condition (6) was proposed in [18]. The condition (6) together with Gronwall's inequality imply that

$$\mathscr{E}(r,x) \leq \mathscr{E}(s,x)\, e^{\lambda|r-s|},\ |\partial_t \mathscr{E}(r,x)| \leq \lambda\, \mathscr{E}(s,x)\, e^{\lambda|r-s|} \tag{7}$$

for any r, s in $[0,T]$.

Definition (Construction of discrete solution). Let $\varepsilon > 0$, $\tau > 0$ and let $N \in \mathbb{N}$ satisfy $T \in [\tau N, \tau(N+1))$. We define a sequence $\{x^{\varepsilon,\tau}\}_{i=0}^N$ by $x_0^{\varepsilon,\tau} = x_0$ (initial position) and

$$x_i^{\varepsilon,\tau} \in \operatorname{argmin}\{\mathscr{E}(t_i,x) + |x - x_{i-1}^{\varepsilon,\tau}| \mid |x - x_{i-1}^{\varepsilon,\tau}| \leq \varepsilon\} \text{ for every } i \in \{1,\dots,N\}.$$

We define the discretized solution $x^{\varepsilon,\tau}(\cdot)$ by interpolation

$$x^{\varepsilon,\tau}(t) := x_{i-1}^{\varepsilon,\tau} \text{ for every } t \in [t_{i-1},t_i), i \in \{1,\dots,N\}.$$

Our main result is as follows.

Theorem 1 (BV solutions constructed by epsilon-neighborhood method)**.** *Let $\mathscr{E} : [0,T] \times \mathbb{R}^d \to [0,+\infty]$ be of class C^1 and satisfy (6). Let an initial datum $x_0 \in \mathbb{R}^d$ be such that x_0 is a local minimizer for the functional $x \mapsto \mathscr{E}(0,x) + |x - x_0|$. Then the following statements hold true.*

(i) *(Discrete solution) For any $\varepsilon > 0$ and $\tau > 0$, there exists a discretized solution $t \mapsto x^{\varepsilon,\tau}(\cdot)$ as described above.*

(ii) *(Epsilon-neighborhood solution) For any $\varepsilon > 0$ fixed, there exists a subsequence $\tau_n \to 0$ such that $x^{\varepsilon,\tau_n}(\cdot)$ converges pointwise to some limit $x^\varepsilon(\cdot)$. The function $x^\varepsilon(\cdot)$ satisfies*

- *(Epsilon local stability) If $x^\varepsilon(\cdot)$ is right-continuous at t, namely $\lim_{t'\to t^+} x^\varepsilon(t') = x^\varepsilon(t)$, then $x^\varepsilon(t)$ satisfies the epsilon local stability*

$$\mathscr{E}(t, x^\varepsilon(t)) \le \mathscr{E}(t, x) + |x - x^\varepsilon(t)| \text{ for all } |x - x^\varepsilon(t)| \le \varepsilon. \tag{eps-LS}$$

- *(Energy-dissipation inequalities) We have $\mathscr{D}iss(x^\varepsilon; [0, T]) \le C$ (independent of ε), $\partial_t\mathscr{E}(\cdot, x^\varepsilon(\cdot)) \in L^1(0, T)$ and for all $0 \le s \le t \le T$,*

$$-\mathscr{D}iss_{new}(x^\varepsilon; [s, t]) \le \mathscr{E}(t, x^\varepsilon(t)) - \mathscr{E}(s, x^\varepsilon(s)) - \int_s^t \partial_t\mathscr{E}(r, x^\varepsilon(r))\, dr \le -\mathscr{D}iss(x^\varepsilon; [s, t]).$$

(iii) *(BV solution constructed by epsilon-neighborhood) There exists a subsequence $\varepsilon_n \to 0$ such that x^{ε_n} converges pointwise to some BV function u. The function u satisfies*

- *(Weak local stability) If $t \mapsto u(t)$ is continuous at t, then*

$$|\partial_x\mathscr{E}(t, u(t))| \le 1.$$

- *(New energy-dissipation balance) For all $0 \le s \le t \le T$, one has*

$$\mathscr{E}(t, u(t)) - \mathscr{E}(s, u(s)) = \int_s^t \partial_t\mathscr{E}(r, u(r))\, dr - \mathscr{D}iss_{new}(u; [s, t]).$$

The proof of Theorem 1 is provided in the next sections.

An example. An explicit example is given below (a detail explanation can be found in Appendix).

Example 2. Consider the case $X = \mathbb{R}$, $\Psi(x) = |x|$, $x_0 = 0$ and the energy functional

$$\mathscr{E}(t, x) := x^2 - x^4 + 0.3\, x^6 + t\,(1 - x^2) - x, \ t \in [0, 2].$$

(i) The energetic solution constructed by time-discretization satisfies

$$x(t) = 0 \ \text{ if } \ t < \frac{1}{6}\ ,\ x(1/6) \in \{0, \sqrt{5/3}\} \ \text{ and } x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \ \text{ if } t > \frac{1}{6}.$$

The solution jumps at $t = 1/6$, from $x = 0$ to $x = \sqrt{5/3}$, but this jump is not reasonable (see Fig. 1 below). The energetic solution satisfies the energy-dissipation balance but it does not satisfies the new energy-dissipation balance.

(ii) The BV solution corresponding to the viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon x^2$ is

$$x(t) = 0 \text{ for all } t \in [0, 2].$$

(iii) The BV solution constructed by epsilon-neighborhood method satisfies

$$x(t) = 0 \ \text{ if } \ t < 1 \ \text{ and } \ x(t) = \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3} \ \text{ if } t > 1.$$

This solution jumps at $t = 1$ which is reasonable (see Fig. 2 below). This solution satisfies the new energy-dissipation balance but it does not satisfy the energy-dissipation balance.

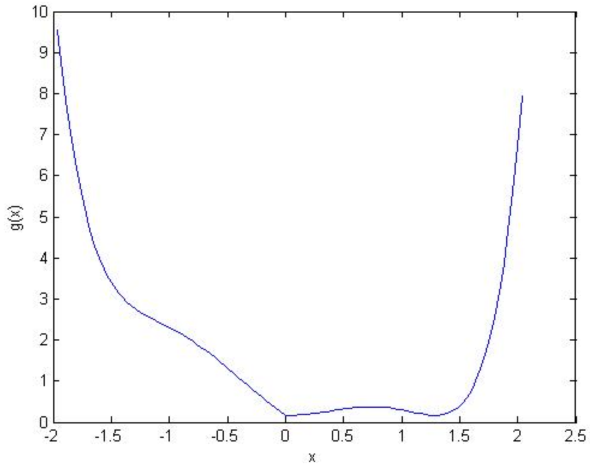

Figure 1. Function $\mathscr{E}(t,x)+|x|$ with $t=1/6$ in Example 2.

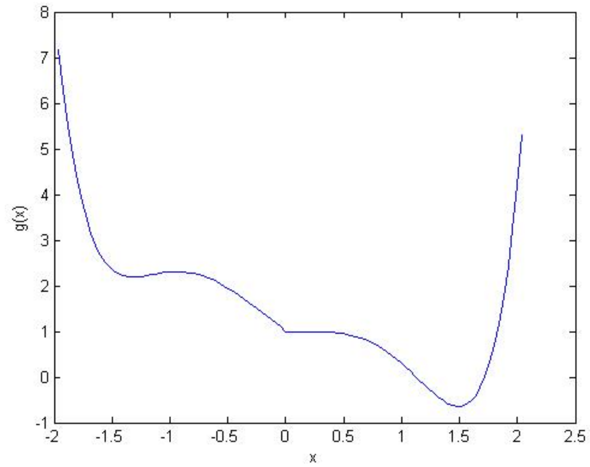

Figure 2. Function $\mathscr{E}(t,x)+|x|$ with $t=1$ in Example 2.

3. Epsilon-neighborhood solution x^{ε}

We start by considering the discrete solution.

Lemma 3 (Discretized solution)**.** *For any given initial state x_0, any $\varepsilon>0$ and $\tau>0$ and any partition $0=t_0<t_1<\cdots<t_N\le T$ of $[0,T]$ such that $t_n-t_{n-1}=\tau$ and $T\in[\tau N,\tau(N+1))$, there exists a sequence $\{x_i^{\varepsilon,\tau}\}_{i=0}^N$ such that $x_0^{\varepsilon,\tau}=x_0$ and for every $i=1,2,\dots,N$, $x_i^{\varepsilon,\tau}$ minimizes the functional*

$$x\mapsto \mathscr{E}(t_i,x)+|x_{i-1}^{\varepsilon,\tau}-x|$$

over $x\in\mathbb{R}^d$, $|x-x_{i-1}^{\varepsilon,\tau}|\le\varepsilon$.

Moreover, the function $t\mapsto x^{\varepsilon,\tau}(t)$ defined by the interpolation $x^{\varepsilon,\tau}(t)=x_{i-1}^{\varepsilon,\tau}$ if $t\in[t_{i-1},t_i)$, $i\in\{1,...,N\}$ satisfies the following energy estimates.

(i) *(Discrete bound) For any $n\in\{1,\dots,N\}$ we have*

$$\mathscr{E}(t_n,x_n^{\varepsilon,\tau})\le\mathscr{E}(0,x_0)\,e^{\lambda t_n}\text{ and }\mathscr{E}(0,x_n^{\varepsilon,\tau})\le\mathscr{E}(0,x_0)\,e^{2\lambda t_n}.$$

(ii) *(Integral bound) For all* $0 \le s \le t \le T$, *it holds that* $\mathscr{D}iss(x^{\varepsilon,\tau};[s,t]) < \infty$, $\partial_t\mathscr{E}(\cdot, x^{\varepsilon,\tau}(\cdot)) \in L^1(0,T)$ *and*

$$\mathscr{E}(t, x^{\varepsilon,\tau}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau}(s)) \le \int_s^t \partial_t\mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr - \mathscr{D}iss(x^{\varepsilon,\tau};[s,t]).$$

Proof. Since $x \mapsto \mathscr{E}(t_n, x) + |x - x_{i-1}^{\varepsilon,\tau}|$ is continuous, this functional has a minimizer $x_i^{\varepsilon,\tau}$ in the compact set $|x - x_{i-1}^{\varepsilon,\tau}| \le \varepsilon$. The energy estimates can be proved similarly for energetic solution (see e.g. [16]). A detailed proof can be found in the Appendix. □

Lemma 4 (Epsilon-neighborhood solution)**.** *Given any initial data* $x_0 \in \mathbb{R}^d$ *such that* $\mathscr{E}(0,x_0) < \infty$ *and* x_0 *is a local minimizer for the functional* $x \mapsto \mathscr{E}(0,x) + |x - x_0|$. *Let* $x^{\varepsilon,\tau}$ *be as in Lemma 3. Then there exists a subsequence* $\tau_n \to 0$ *such that* $x^{\varepsilon,\tau_n}(t) \to x^{\varepsilon}(t)$ *for all* $t \in [0,T]$. *Moreover, the epsilon-neighborhood solution* $x^{\varepsilon}(\cdot)$ *satisfies the following properties:*

(i) *(Epsilon local stability) If* $x^{\varepsilon}(\cdot)$ *is right-continuous at* t, *namely* $\lim_{t' \to t^+} x^{\varepsilon}(t') = x^{\varepsilon}(t)$, *then* $x^{\varepsilon}(t)$ *satisfies the epsilon local stability*

$$\mathscr{E}(t, x^{\varepsilon}(t)) \le \mathscr{E}(t,x) + |x - x^{\varepsilon}(t)| \text{ for all } |x - x^{\varepsilon}(t)| \le \varepsilon.$$

(ii) *(Energy-dissipation inequalities) We have* $\mathscr{D}iss(x^{\varepsilon};[0,T]) \le C$ *(independent of* ε*),* $\partial_t\mathscr{E}(\cdot, x^{\varepsilon}(\cdot)) \in L^1(0,T)$ *and for all* $0 \le s \le t \le T$,

$$-\mathscr{D}iss_{new}(x^{\varepsilon};[s,t]) \le \mathscr{E}(t, x^{\varepsilon}(t)) - \mathscr{E}(s, x^{\varepsilon}(s)) - \int_s^t \partial_t\mathscr{E}(r, x^{\varepsilon}(r))\, dr \le -\mathscr{D}iss(x^{\varepsilon};[s,t]).$$

Proof. **Step 1. Existence.** By the Integral bound in Lemma 3, the fact that $\mathscr{E}$ is non-negative, and condition (E1), we have

$$\begin{aligned}\mathscr{D}iss(x^{\varepsilon,\tau};[0,T]) &\le \mathscr{E}(0,x_0) - \mathscr{E}(T, x^{\varepsilon,\tau}(T)) + \int_0^T \partial_t\mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr \\ &\le \mathscr{E}(0,x_0) + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \lambda\, \mathscr{E}(t_{i-1}, x_{i-1}^{\varepsilon,\tau})\, e^{\lambda(r - t_{i-1})}\, dr.\end{aligned}$$

Here we denote T by t_{N+1}. Then, using the Discrete bound in Lemma 3, we get

$$\begin{aligned}\mathscr{D}iss(x^{\varepsilon,\tau};[0,T]) &\le \mathscr{E}(0,x_0) + \int_0^T \lambda\, \mathscr{E}(0,x_0)\, e^{\lambda r}\, dr \\ &= \mathscr{E}(0,x_0)\, e^{\lambda T}.\end{aligned}$$

Thus, $\{x^{\varepsilon,\tau}(\cdot)\}$ has uniformly bounded variation and it is uniformly bounded. Therefore, applying Helly's selection principle [12, 1, 25], we can find a subsequence $\tau_n \to 0$ and a BV function $x^{\varepsilon}(\cdot)$ such that $x^{\varepsilon,\tau_n}(t) \to x^{\varepsilon}(t)$ as $n \to \infty$ for all $t \in [0,T]$.

Step 2. A consequence of the right-continuity. Let us denote by $\{t_i^n\}_{i=0}^{N_n}$ the partition corresponding to τ_n and assume that $t \in [t_{i-1}^n, t_i^n)$. It is obvious that

$$x_{i-1}^{\varepsilon,\tau_n} = x^{\varepsilon,\tau_n}(t) \to x^{\varepsilon}(t)$$

as $n \to \infty$. Now we show that if $x^{\varepsilon}(\cdot)$ is right-continuous at t, then

$$x_i^{\varepsilon,\tau_n} = x^{\varepsilon,\tau_n}(t_i^n) \to x^{\varepsilon}(t).$$

Let $t' > t$. Due to the Integral bound in Lemma 3, we have

$$\mathscr{E}(t', x^{\varepsilon,\tau_n}(t')) - \mathscr{E}(t, x^{\varepsilon,\tau_n}(t)) + \mathscr{D}iss(x^{\varepsilon,\tau_n}; [t,t']) \leq \int_t^{t'} \partial_t \mathscr{E}(r, x^{\varepsilon,\tau_n}(r))\, dr \leq C|t'-t|,$$

here the last inequality due to the continuity of $\partial_t \mathscr{E}$ and the fact that x^{ε,τ_n} is bounded on $(0,T)$. For n large enough, we have $t < t_i^n < t'$. Therefore,

$$|x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \leq \mathscr{D}iss(x^{\varepsilon,\tau_n}; [t,t']).$$

Moreover, when $n \to \infty$, we have

$$x^{\varepsilon,\tau_n}(t) \to x^{\varepsilon}(t) \text{ and } x^{\varepsilon,\tau_n}(t') \to x^{\varepsilon}(t').$$

Thus it follows from the above integral bound that

$$\mathscr{E}(t', x^{\varepsilon}(t')) - \mathscr{E}(t, x^{\varepsilon}(t)) + \limsup_{n\to\infty} |x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \leq C|t'-t|.$$

Since this inequality holds for all $t' > t$, we can take $t' \to t$ and use the assumption $x^{\varepsilon}(t^+) = x^{\varepsilon}(t)$ to obtain

$$\limsup_{n\to\infty} |x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \leq 0.$$

Since we have already known that $x_{i-1}^{\varepsilon,\tau_n} \to x(t)$, we can conclude that $x_i^{\varepsilon,\tau_n} \to x(t)$.

Step 3. Stability. We show that for all $t \in [0,T]$, if $x^{\varepsilon}(\cdot)$ is right-continuous at t, then

$$\mathscr{E}(t, x^{\varepsilon}(t)) \leq \mathscr{E}(t,z) + |z - x^{\varepsilon}(t)| \text{ for all } |z - x^{\varepsilon}(t)| \leq \varepsilon.$$

First, we prove the result for $z \in \mathbb{R}^d$ such that $|z - x^{\varepsilon}(t)| < \varepsilon$. Since $\lim_{n\to\infty} x^{\varepsilon,\tau_n}(t) = x^{\varepsilon}(t)$, we get

$$|z - x^{\varepsilon,\tau_n}(t)| < \varepsilon$$

for n large enough. Using the notation in Step 2. Since $t \in [t_{i-1}^n, t_i^n)$, we get $x^{\varepsilon,\tau_n}(t) = x_{i-1}^{\varepsilon,\tau_n}$. From the definition of x_i^{ε,τ_n} and condition $|z - x_{i-1}^{\varepsilon,\tau_n}| < \varepsilon$, we obtain

$$\mathscr{E}(t_i^n, x_i^{\varepsilon,\tau_n}) + |x_i^{\varepsilon,\tau_n} - x_{i-1}^{\varepsilon,\tau_n}| \leq \mathscr{E}(t_i^n, z) + |z - x_{i-1}^{\varepsilon,\tau_n}|.$$

Taking the limit as $n \to \infty$ and using the fact that both $x_{i-1}^{\varepsilon,\tau_n}$ and x_i^{ε,τ_n} converge to $x^{\varepsilon}(t)$ (see Step 2), we obtain

$$\mathscr{E}(t, x^{\varepsilon}(t)) \leq \mathscr{E}(t,z) + |z - x^{\varepsilon}(t)| \text{ for all } |z - x^{\varepsilon}(t)| < \varepsilon. \tag{8}$$

Now for any z such that $|z - x^{\varepsilon}(t)| = \varepsilon$, we can choose a sequence z_n converges to z such that $|z_n - x^{\varepsilon}(t)| < \varepsilon$. Applying (8) for z_n, we get

$$\mathscr{E}(t, x^{\varepsilon}(t)) \leq \mathscr{E}(t, z_n) + |z_n - x^{\varepsilon}(t)|. \tag{9}$$

Notice that the mappings $z \mapsto \mathscr{E}(t,z)$ and $z \mapsto |z - x^{\varepsilon}(t)|$ are continuous. Hence, we can take the limit in (9) and get the result also for $|z - x^{\varepsilon}(t)| = \varepsilon$.

Step 4. Energy-dissipation inequalities.

By the Integral bound in Lemma 3, we have for all $0 \leq s \leq t \leq T$,

$$\mathscr{E}(t, x^{\varepsilon,\tau_n}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau_n}(s)) \leq \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau_n}(r))\, dr - \mathscr{D}iss(x^{\varepsilon,\tau_n}; [s,t]).$$

Since $x^{\varepsilon,\tau_n}(r) \to x^{\varepsilon}(r)$ for all $r \in [0,T]$, we have

$$\mathscr{E}(t, x^{\varepsilon,\tau_n}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau_n}(s)) \to \mathscr{E}(t, x^{\varepsilon}(t)) - \mathscr{E}(s, x^{\varepsilon}(s))$$

and

$$\int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau_n}(r))\, dr \to \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon}(r))\, dr$$

as $n \to \infty$. Moreover, one has

$$\liminf_{n\to\infty} \mathscr{D}iss(x^{\varepsilon,\tau_n}; [s,t]) \geq \mathscr{D}iss(x^{\varepsilon}; [s,t]).$$

Thus we can derive one energy-dissipation inequality

$$\mathscr{E}(t, x^{\varepsilon}(t)) - \mathscr{E}(s, x^{\varepsilon}(s)) \leq \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon}(r))\, dr - \mathscr{D}iss(x^{\varepsilon}; [s,t]).$$

We shall use Lemma 5 to obtain the other energy-dissipation inequality,

$$\mathscr{E}(t, x^{\varepsilon}(t)) - \mathscr{E}(s, x^{\varepsilon}(s)) \geq \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon}(r))\, dr - \mathscr{D}iss_{new}(x^{\varepsilon}; [s,t]).$$

It is suffices to verify that $|\nabla_x \mathscr{E}(t, x^{\varepsilon}(t))| \leq 1$ for a.e. $t \in (0,T)$. In fact, for every $t \in [0,T]$ such that $x^{\varepsilon}(\cdot)$ is right-continuous at t, we have proved in Step 3 the ε-stability

$$\mathscr{E}(t, x^{\varepsilon}(t)) \leq \mathscr{E}(t, x) + |x - x^{\varepsilon}(t)| \text{ for all } |x - x^{\varepsilon}(t)| \leq \varepsilon.$$

This inequality implies that $|\nabla_x \mathscr{E}(t, x^{\varepsilon}(t))| \leq 1$. On the other hand, since $x^{\varepsilon}(\cdot)$ is a BV function, it is continuous except at most countably many points. Therefore, the desired inequality follows from the following result. □

Lemma 5 (Lower bound of the new energy-dissipation balance). *For any BV function $u : [0,T] \to \mathbb{R}^d$, for any energy functional $\mathscr{E} \in C^1([0,T] \times \mathbb{R}^d)$ satisfying the constraint $|\nabla_x \mathscr{E}(t, u(t))| \leq 1$ for a.e. $t \in (0,T)$, it holds that*

$$\mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_0, u(t_0)) \geq \int_{t_0}^{t_1} \partial_t \mathscr{E}(s, u(s))\, ds - \mathscr{D}iss_{new}(u; [t_0, t_1]).$$

This result is due to Mielke, Rossi and Savaré [20, Theorem 4.7]. For the reader convenience, a proof of Lemma 5 is included in Appendix.

4. BV solution constructed by epsilon-neighborhood method

Lemma 6 (Limit of epsilon-neighborhood solution). *Let be given an initial datum $x_0 \in \mathbb{R}^d$ such that $\mathscr{E}(0, x_0) < \infty$ and x_0 is a local minimizer for the functional $x \mapsto \mathscr{E}(0,x) + |x - x_0|$. Let x^{ε} be as in Lemma 4. Then there exists a subsequence $\varepsilon_n \to 0$ and a BV function u such that $x^{\varepsilon_n}(t) \to u(t)$ for all $t \in [0,T]$. Moreover, the function u satisfies the following properties*

(i) *(Local stability) If $t \mapsto u(t)$ is continuous at t, then*

$$|\nabla_x \mathscr{E}(t, u(t))| \leq 1.$$

(ii) *(New energy-dissipation balance) For all $0 \leq s \leq t \leq T$, one has*

$$\mathscr{E}(t, u(t)) - \mathscr{E}(s, u(s)) = \int_s^t \partial_t \mathscr{E}(r, u(r))\, dr - \mathscr{D}iss_{new}(u; [s,t]).$$

Proof. **Step 1. Existence.** Since $\mathscr{D}iss(x^{\varepsilon}; [0,T]) \leq C$ independent of ε, by Helly's selection principle we can find a subsequence $\varepsilon_n \to 0$ and a BV function u such that $x^{\varepsilon_n}(t) \to u(t)$ as $n \to \infty$ for all $t \in [0,T]$.

Step 2. Stability. Let

$$A := \{t \in [0,T] \,|\, x^{\varepsilon_n}(\cdot) \text{ is right continuous at } t \text{ for all } n \geq 1\}.$$

Then $[0,T]\backslash A$ is at most countable. Moreover, for $t \in A$, by Lemma 4 we have

$$\mathscr{E}(t, x^{\varepsilon_n}(t)) \leq \mathscr{E}(t,z) + |z - x^{\varepsilon_n}(t)| \text{ for all } |z - x^{\varepsilon_n}(t)| \leq \varepsilon_n$$

for all $n \geq 1$. Therefore,

$$|\nabla_x \mathscr{E}(t, x^{\varepsilon_n}(t))| \leq 1 \text{ for all } n \geq 1.$$

Taking $n \to \infty$, we obtain

$$|\nabla_x \mathscr{E}(t, u(t))| \leq 1$$

for all $t \in A$.

Moreover, by continuity, we also get $|\nabla_x \mathscr{E}(t, u(t))| \leq 1$ provided u is continuous at t.

Step 3. New energy-dissipation balance. First, similarly to the proof of energy inequalities in Lemma 4, we have

$$-\mathscr{D}iss_{new}(u;[s,t]) \leq \mathscr{E}(t,u(t)) - \mathscr{E}(s,u(s)) - \int_s^t \partial_t \mathscr{E}(r,u(r))\,dr \leq -\mathscr{D}iss(u;[s,t]).$$

(More precisely, the second inequality is a consequence of the corresponding inequality of x^ε in Lemma 4 and Fatou's lemma, while the first inequality follows from Lemma 5.)

Notice that if the solution $t \mapsto u(t)$ is continuous on $[a,b] \subset [0,T]$, then $\mathscr{D}iss(u;[a,b]) = \mathscr{D}iss_{new}(u;[a,b])$. Thus, we have immediately the energy-dissipation balance

$$\mathscr{E}(b,u(b)) - \mathscr{E}(a,u(a)) - \int_a^b \partial_t \mathscr{E}(r,u(r))\,dr = -\mathscr{D}iss(u;[a,b]) = -\mathscr{D}iss_{new}(u;[a,b]).$$

Therefore, it remains to consider jump points. More precisely, we need to show that if u jumps at $t \in (0,T)$, namely $u(t^-) \neq u(t^+)$, then

$$\mathscr{E}(t,u(t^+)) - \mathscr{E}(t,u(t^-)) = -\Delta_{new}(t,u(t^-),u(t)) - \Delta_{new}(t,u(t),u(t^+)).$$

This fact follows from Lemma 7 and 8 below. □

To prove the upper bound, we start by showing that the discretized solution $x^{\varepsilon,\tau}$ is "almost" an optimal transition.

Lemma 7 (Approximate optimal transition)**.** *For the discretized solution $x^{\varepsilon,\tau}$, if we write $x_j := x^{\varepsilon,\tau}(t_j)$, then*

$$-\nabla_x \mathscr{E}(t_i,x_i) \cdot (x_i - x_{i-1}) = \max\{1, |\nabla_x \mathscr{E}(t_i,x_i)|\} \cdot |x_i - x_{i-1}|.$$

Consequently, if $\delta \geq \varepsilon + |t - t_i|$ and we denote by $v : [a,b] \to \mathbb{R}^d$ the linear curve connecting x_{i-1} and x_i, namely

$$v(s) = x_{i-1} + \frac{s-a}{b-a}(x_i - x_{i-1}),$$

then

$$\int_a^b \max\{1, |\nabla_x \mathscr{E}(t,v(s))|\} \cdot |\dot{v}(s)|\,ds \leq \mathscr{E}(t,x_{i-1}) - \mathscr{E}(t,x_i) + g(\delta) \cdot |x_i - x_{i-1}|$$

where $g(\delta) \to 0$ as $\delta \to 0$.

Proof. **Step 1.** Recall that x_i is a minimizer for

$$\inf_{|z-x_{i-1}|\le\varepsilon} h(z) = \inf_{|z-x_{i-1}|\le\varepsilon} \{\mathscr{E}(t_i,z)+|z-x_{i-1}|\}\,.$$

Denote $c := |x_i - x_{i-1}|$; then x_i is also a minimizer for

$$\inf_{|z-x_{i-1}|=c} h(z).$$

By Lagrange multiplier, there exists $\lambda \in \mathbb{R}$ such that

$$\nabla_x \mathscr{E}(t_i,x_i) = \lambda(x_i - x_{i-1}).$$

Step 2. Moreover, since the function $h_1(t) = h(x_{i-1}+t(x_i-x_{i-1}))$ satisfies $h_1(t)\ge h_1(1)$ for all $t\in[0,1]$, we obtain

$$0 \ge \left[\frac{dh_1}{dt}\right]_{t=1} = \nabla_x\mathscr{E}(t_i,x_i)\cdot(x_i-x_{i-1}) + |x_i-x_{i-1}|.$$

Thus either $x_i = x_{i-1}$, or $|\nabla_x\mathscr{E}(t_i,x_i)|\ge 1$ and $\lambda<0$. Therefore, we can conclude that

$$\nabla_x\mathscr{E}(t_i,x_i)\cdot(x_i-x_{i-1}) = -\max\{1,|\nabla_x\mathscr{E}(t_i,x_i)|\}\cdot|x_i-x_{i-1}|.$$

Step 3. Consequently, using $|t-t_i|\le\delta$, $|x_{i-1}-x_i|\le\varepsilon\le\delta$ and the fact that $\nabla_x\mathscr{E}(\cdot,\cdot)$ is continuous on compact sets, we obtain that

$$-\nabla_x\mathscr{E}(t,v(s))\cdot\dot v(s) \ge \max\{1,|\nabla_x\mathscr{E}(t,v(s))|\}\cdot\dot v(s) - g(\delta)\,|\dot v(s)|$$

for every $s\in[a,b]$. Therefore,

$$\begin{aligned}\mathscr{E}(t,x_{i-1})-\mathscr{E}(t,x_i) &= -\int_a^b \nabla_x\mathscr{E}(t,v(s))\cdot\dot v(s)\,ds\\ &\ge \int_a^b \max\{1,|\nabla_x\mathscr{E}(t,v(s))|\}\cdot\dot v(s)\,ds - g(\delta)\cdot|x_i-x_{i-1}|.\end{aligned}$$

□

Now we prove the new energy-dissipation upper bound.

Lemma 8 (Upper bound)**.** *Let u be the function as in Lemma 6. If $u(t^-)\ne u(t)$, then*

$$\Delta_{new}(t,u(t^-),u(t)) \le \mathscr{E}(t,u(t^-)) - \mathscr{E}(t,u(t)).$$

Proof. Let $0\ll\tau\ll\varepsilon\ll\delta\ll1$. By the definition of the discretized solution $x^{\varepsilon,\tau}$, for every $t\in(0,T)$ we have

$$x^{\varepsilon,\tau}(t-\delta) = x^{\varepsilon,\tau}(t_i) \text{ and } x^{\varepsilon,\tau}(t) = x^{\varepsilon,\tau}(t_{i+k})$$

for $t_i, t_{i+k}\in[t-2\delta,t+\delta]$.

We can construct an absolutely continuous function $v:[0,1]\to\mathbb{R}^d$ by linearly interpolating the following $(k+3)$ points:

$$u(t^-), x^{\varepsilon,\tau}(t-\delta) = x^{\varepsilon,\tau}(t_i), x^{\varepsilon,\tau}(t_{i+1}),\ldots,x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t), u(t).$$

More precisely, we define

$$\begin{aligned}
z_0 &= u(t^-),\\
z_1 &= x^{\varepsilon,\tau}(t-\delta) = x^{\varepsilon,\tau}(t_i),\\
z_2 &= x^{\varepsilon,\tau}(t_{i+1}),\\
&\dots\\
z_{k+1} &= x^{\varepsilon,\tau}(t_{i+k}) = x^{\varepsilon,\tau}(t),\\
z_{k+2} &= u(t),
\end{aligned}$$

and denote $r := 1/(k+2)$ and

$$v(s) = z_j + \frac{s-jr}{r}(z_{j+1}-z_j) \text{ when } s \in [jr,(j+1)r],\ j = 0,1,\dots,k+1.$$

By the definition of the new dissipation, we have

$$\begin{aligned}
\Delta_{new}(t,u(t^-),u(t)) &\le \int_0^1 \max\{1, |\nabla_x \mathscr{E}(t,v(s))|\}\cdot|\dot v(s)|\,ds\\
&= \sum_{j=0}^{k+1}\int_{jr}^{(j+1)r} \max\{1, |\nabla_x \mathscr{E}(t,v(s))|\}\cdot|\dot v(s)|\,ds.
\end{aligned}$$

When $j=0$ and $j=k+1$, we estimate

$$\int_{jr}^{(j+1)r} \max\{1, |\nabla_x \mathscr{E}(t,v(s))|\}\cdot|\dot v(s)| \le C\int_{jr}^{(j+1)r}|\dot v(s)|\,ds = C|z_{j+1}-z_j|.$$

When $j = 1,2,\dots,k$, using Lemma 7, we obtain

$$\begin{aligned}
\int_{jr}^{(j+1)r} \max\{1, |\nabla_x \mathscr{E}(t,v(s))|\}\cdot|\dot v(s)|\,ds \le\ & \mathscr{E}(t,x^{\varepsilon,\tau}(t_{i+j-1})) - \mathscr{E}(t,x^{\varepsilon,\tau}(t_{i+j}))\\
&+g(\delta)\cdot|x^{\varepsilon,\tau}(t_{i+j}) - x^{\varepsilon,\tau}(t_{i+j-1})|
\end{aligned}$$

where $g(\delta)\to 0$ as $\delta \to 0$. Taking the sum over $j = 0,1,\dots,k+1$ and using the bound $\mathscr{D}iss(x^{\varepsilon,\tau};[0,T]) \le C$, we find that

$$\begin{aligned}
\Delta_{new}(t,u(t^-),u(t)) &\le \int_0^1 \max\{1, |\nabla_x \mathscr{E}(t,v(s))|\}\cdot|\dot v(s)|\,ds\\
&\le \mathscr{E}(t,x^{\varepsilon,\tau}(t-\delta)) - \mathscr{E}(t,x^{\varepsilon,\tau}(t)) + Cg(\delta)\\
&\quad + C|u(t^-) - x^{\varepsilon,\tau}(t-\delta)| + C|x^{\varepsilon,\tau}(t) - u(t)|.
\end{aligned}$$

Taking the limit $\tau\to 0$, then $\varepsilon \to 0$, then $\delta\to 0$, we conclude that

$$\Delta_{new}(t,u(t^-),u(t)) \le \mathscr{E}(t,u(t^-)) - \mathscr{E}(t,u(t)).$$

This finishes the proof. □

5. Appendix: Technical proofs

5.1. **Example 2.**

Part I. Energetic solution via time-discretization. Step 1. Fix a time step $\tau > 0$. To find the discretized solution $x^\tau(t)$, it suffices to calculate $x_i := x^\tau(t_i)$ where $0 = t_0 < \cdots < t_N \le 1$ and $t_i - t_{i-1} = \tau$ for all $i = 1, 2, \ldots, N$. Here $N \in \mathbb{N}$ satisfies $1 \in [\tau N, \tau(N+1))$.

We have $x_0 = 0$ and for all $i = 1, 2, \ldots, N$, x_i is a minimizer of the functional

$$x \in \mathbb{R} \mapsto \mathscr{E}(t_i, x) + |x - x_{i-1}|.$$

Step 2. Let us fix $t \in (0, 2]$ and consider the functional

$$F(x) := \mathscr{E}(t, x) + |x| = x^2 - x^4 + 0.3\,x^6 + t(1 - x^2) - x + |x|, \ x \in \mathbb{R}.$$

It is straightforward to see that

- When $t \le 1$, $F(x)$ has two local minimizers (see Fig. 1)

$$x = 0 \text{ and } x = y(t) := \frac{\sqrt{10 + \sqrt{10 + 90t}}}{3}.$$

  Moreover,

$$F(y(t)) - F(0) = \frac{1}{243}(10 + \sqrt{10 + 90t})(8 - 18t - \sqrt{10 + 90t}),$$

  which is positive if $t < 1/6$ and negative if $t > 1/6$. Hence F has a unique global minimizer $x = 0$ if $0 \le t < 1/6$, and then F has a unique global minimizer at $x = y(t)$ if $1/6 < t < 1$.
- When $t > 1$, $F(x)$ has a unique local (also global) minimizer at $x = y(t)$.

Step 3. By induction, we can show that if $t_{i_0} < 1/6 \le t_{i_0+1}$, then $x_i = 0$ for all $i = 1, 2, ..., i_0$, and either $x_{i_0+1} = y(t_{i_0+1})$, or $x_{i_0+1} = 0$ and $x_{i_0+2} = y(t_{i_0+2})$.

Next, we show that if $t_{i-1} \ge 1/6$ and $x_{i-1} = y(t_{i-1}) > 0$, then $x_i = y(t_i)$. Recall that x_i is a global minimizer for the functional

$$x \in \mathbb{R} \mapsto F_i(x) := \mathscr{E}(t_i, x) + |x - x_{i-1}| = x^2 - x^4 + 0.3\,x^6 + t_i(1 - x^2) - x + |x - x_{i-1}|.$$

By using the triangle inequality $-x + |x - x_{i-1}| \ge -x_{i-1}$ and the same analysis of F, we can conclude that $x_i = y(t_i)$.

Taking the limit as $\tau \to 0$, we obtain the energetic solution

$$x(t) = 0 \ \text{ if } \ t \in [0, 1/6), \ x(1/6) \in \{0, \sqrt{5/3}\}, \ x(t) = y(t) \ \text{ if } t \in [1/6, 2].$$

Step 4. Finally, we show that the energetic solution does not satisfies the new energy-dissipation balance. It suffices to show that at the jump point $t = 1/6$,

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) > -\Delta_{new}(t, x(t^-), x(t^+)).$$

In fact, a direct computation gives us at $t = 1/6$,

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = \mathscr{E}(1/6, \sqrt{5/3}) - \mathscr{E}(1/6, 0) = -\sqrt{5/3}.$$

On the other hand, at $t = 1/6$ we have

$$\Delta_{new}(t, x(t^-), x(t^+)) = \int_0^{\sqrt{15}/3} \max\left\{1, \left|\frac{2}{3}y - 4y^3 + 1.8y^5 - 1\right|\right\} dy = \frac{185}{486} + \sqrt{\frac{5}{3}}.$$

Thus,

$$\mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) > -\Delta_{new}(t, x(t^-), x(t^+)) \text{ at } t = 1/6.$$

Part II. BV solution constructed by the viscous dissipation $\Psi_\varepsilon(x) = |x| + \varepsilon x^2$. We construct the BV solution via vanishing viscosity with the viscous term εx^2 by the method in [20].

Let us briefly recall the construction of the BV solution. Let $\varepsilon > 0$ and $\tau > 0$ and denote $e := \varepsilon/\tau$ and let $0 = t_0 < \cdots < t_N \le T$ be a partition of $[0,T]$ satisfying $t_i - t_{i-1} = \tau$ for every $i \in \{1,\dots,N\}$ and $T - t_N < \tau$. The discretized problem now becomes to find a sequence $\{x^{\varepsilon,\tau}\}_{i=1}^N$ such that $x_0^{\varepsilon,\tau} = 0$ and $x_i^{\varepsilon,\tau}$ is a global minimizer for the functional

$$x \in \mathbb{R} \mapsto \{\mathscr{E}(t_i,x) + |x - x_{i-1}^{\tau,\varepsilon}| + e|x - x_{n-1}^{\tau,\varepsilon}|^2\}$$

for every $i = 1,2,...,N$ and $e = \varepsilon/\tau$. Then use interpolation and pass to the pointwise limit as $\tau \to 0, \varepsilon \to 0, e = \varepsilon/\tau \to \infty$ to obtain the BV solution.

Now coming back to our example, for $t \in (0,2]$, we consider the function

$$F(x) := \mathscr{E}(t,x) + |x| + e|x|^2 = t + (1+e-t)\,x^2 - x^4 + 0.3\,x^6 - x + |x|,\ x \in \mathbb{R}.$$

If e is large enough (such that $1 + e - t \ge 1$), one has

$$F(x) \ge t + x^2 - x^4 + 0.3\,x^6 = t + \frac{1}{6}x^2 + \left(\sqrt{\frac{5}{6}}x - \sqrt{\frac{3}{10}}x^3\right)^2 \ge t = F(0).$$

Thus F has a unique global minimizer at $x = 0$. Therefore, the discretized sequence $\{x_i^{\tau,\varepsilon}\}$ is identically equal to 0. Thus the BV solution is also identically equal to 0.

Part III. BV solution constructed by epsilon-neighborhood method. Step 1. Let $\varepsilon > 0$ and $\tau > 0$ be small. Let us compute $x_i := x^{\varepsilon,\tau}(t_i)$, where $t_i = i/N$ for $i = 0,1,\dots,N$. Here $N \in \mathbb{N}$ such that $1 \in [\tau N, \tau(N+1))$.

By definition, $x_0 = 0$ and x_i is a minimizer for the functional

$$F_i(x) := \mathscr{E}(t_i,x) + |x - x_{i-1}| = x^2 - x^4 + 0.3\,x^6 + t_i(1-x^2) - x + |x - x_{i-1}|$$

over $x \in [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$.

Step 2. In particular, if $x_{i-1} = 0$, then x_i is a minimizer for

$$F_i(x) := x^2 - x^4 + 0.3\,x^6 + t_i(1-x^2) - x + |x|$$

over $x \in [-\varepsilon, \varepsilon]$.

Recall that if $t_i < 1$, then $F_i(x)$ has two local minimizer at $x = 0$

$$x = y(t) = \frac{\sqrt{10 - \sqrt{10 + 90t_i}}}{3} = \frac{1}{3}\sqrt{\frac{100 - (10 + 90t_i)}{10 + \sqrt{10 + 90t_i}}} \ge \sqrt{\frac{1 - t_i}{2}}.$$

Therefore, if $\varepsilon < \sqrt{(1-t_i)/2}$, then $x = 0$ is the unique minimizer for $F_i(x)$ on $x \in [-\varepsilon, \varepsilon]$. By induction, we can conclude that if $t_i < 1 - 2\varepsilon^2$, then $x_i = 0$.

Step 3. We show that if $t_i \in [1 - 2\varepsilon^2, 1]$, then $x_i \le y(t_i)$. By induction, we can assume that $x_{i-1} \le y(t_{i-1})$. We assume by contradiction that $x_i > y(t_i)$.

Since $x_{i-1} \le y(t_{i-1}) < y(t_i) < x_i \le x_{i-1} + \varepsilon$, there exists $a \in (y(t_i), x_i) \cap [x_{i-1} - \varepsilon, x_{i-1} + \varepsilon]$. Then using the fact that the function $x \mapsto x^2 - x^4 + 0.3\,x^6 + t_i(1-x^2)$ is strictly increasing on $[y(t_i), \infty)$ and the triangle inequality $-x + |x - x_{i-1}| \ge -x_{i-1}$ we have

$$\begin{aligned} F_i(x_i) &= x_i^2 - x_i^4 + 0.3x_i^6 + t_i(1-x_i^2) - x + |x - x_{i-1}| \\ &> a^2 - a^4 + 0.3a^6 + t_i(1-a^2) - x_{i-1} = F_i(a). \end{aligned}$$

This contradicts to the assumption that x_i is a minimizer for $F_i(x)$ over $x \in [x_{i-1}-\varepsilon, x_{i-1}+\varepsilon]$. Thus we must have $x_i \le y(t_i)$.

Step 4. Now assume that $t_i \in (1,2]$ and $x_{i-1} \in [-y(t_{i-1}), y(t_{i-1})]$. It is straightforward to show that if $x_{i-1} = 0$, then $x_i \in \{\pm\varepsilon\}$; and if $x_{i-1} \in (0, y(t_{i-1})]$, then $x_i = \min\{x_{i-1}+\varepsilon, y(t_i)\}$.

Then taking the limit as $\tau \to 0$, we obtain that the epsilon-neighborhood solution $x^\varepsilon(\cdot)$ satisfies $x^\varepsilon(t) = 0$ if $t < 1 - 2\varepsilon^2$ and $x^\varepsilon(t) = y(t)$ for all $t \in (1, 2]$.

Taking the limit $\varepsilon \to 0$, we obtain that the BV solution constructed by epsilon-neighborhood method satisfies that $x(t) = 0$ if $t \in (0,1)$ and $x(t) = y(t)$ for $t \in (1,2)$.

Step 5. We show that the BV solution constructed by epsilon-neighborhood does not satisfy the energy-dissipation balance. At the jump point $t = 1$, one has

$$-|x(t^-) - x(t^+)| = -\frac{2\sqrt{5}}{3} > \mathscr{E}(t, x(t^+)) - \mathscr{E}(t, x(t^-)) = -\frac{400}{243} - \frac{\sqrt{20}}{3}.$$

5.2. **Proof of energy estimate in Lemma 3. Step 1.** By the minimality of $x_n^{\varepsilon,\tau}$ at time t_n, we have

$$\mathscr{E}(t_n, x_n^{\varepsilon,\tau}) + |x_{n-1}^{\varepsilon,\tau} - x_n^{\varepsilon,\tau}| \le \mathscr{E}(t_n, x_{n-1}^{\varepsilon,\tau}) = \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau}) + \int_{t_{n-1}}^{t_n} \partial_t \mathscr{E}(t, x_{n-1}^{\varepsilon,\tau})\, dt.$$

From the assumption (7),

$$\partial_t \mathscr{E}(t, x_{n-1}^{\varepsilon,\tau}) \le \lambda \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau})\, e^{\lambda(t-t_{n-1})} \text{ for all } t \in [t_{n-1}, t_n],$$

by Gronwall's inequality we obtain

$$\begin{aligned}
\mathscr{E}(t_n, x_n^{\varepsilon,\tau}) &\le \mathscr{E}(t_n, x_n^{\varepsilon,\tau}) + |x_{n-1}^{\varepsilon,\tau} - x_n^{\varepsilon,\tau}| \\
&\le \int_{t_{n-1}}^{t_n} \lambda \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau})\, e^{\lambda(t-t_{n-1})} dt + \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau}) \\
&= \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau})(e^{\lambda(t_n-t_{n-1})} - 1) + \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau}) = \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau})\, e^{\lambda(t_n-t_{n-1})}.
\end{aligned}$$

By induction,

$$\begin{aligned}
\mathscr{E}(t_n, x_n^{\varepsilon,\tau}) &\le \mathscr{E}(t_{n-1}, x_{n-1}^{\varepsilon,\tau})\, e^{\lambda(t_n-t_{n-1})} \le \mathscr{E}(t_{n-2}, x_{n-2}^{\varepsilon,\tau})\, e^{\lambda(t_{n-1}-t_{n-2})}\, e^{\lambda(t_n-t_{n-1})} \\
&\le \cdots \le \mathscr{E}(0, x_0)\, e^{\lambda(t_1-t_0)}\, e^{\lambda(t_2-t_1)} \dots e^{\lambda(t_n-t_{n-1})} = \mathscr{E}(0, x_0)\, e^{\lambda t_n}.
\end{aligned}$$

Finally, by (7) again,

$$\mathscr{E}(0, x_n^{\varepsilon,\tau}) \le \mathscr{E}(t_n, x_n^{\varepsilon,\tau})\, e^{\lambda t_n} \le \mathscr{E}(0, x_0)\, e^{2\lambda t_n}.$$

Step 2. Now we prove the integral bound. Assume that $t_{i-1} < s \le t_i < t_{i+1} < \cdots < t_j \le t < t_{j+1}$, where $\{t_n\}$ is the partition corresponding to $x^{\varepsilon,\tau}$. We start by writing

$$\begin{aligned}
(10) \qquad \mathscr{E}(t, x^{\varepsilon,\tau}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau}(s)) &= \mathscr{E}(t, x^{\varepsilon,\tau}(t)) - \mathscr{E}(t_j, x^{\varepsilon,\tau}(t_j)) + \dots \\
&+ \mathscr{E}(t_j, x^{\varepsilon,\tau}(t_j)) - \mathscr{E}(t_{j-1}, x^{\varepsilon,\tau}(t_{j-1})) + \mathscr{E}(t_i, x^{\varepsilon,\tau}(t_i)) - \mathscr{E}(s, x^{\varepsilon,\tau}(s)).
\end{aligned}$$

By the minimality of $x_k := x^{\varepsilon,\tau}(t_k)$ at time t_k, we have

$$\begin{aligned}
\mathscr{E}(t_k, x_k) - \mathscr{E}(t_{k-1}, x_{k-1}) &\le \mathscr{E}(t_k, x_{k-1}) - |x_{k-1} - x_k| - \mathscr{E}(t_{k-1}, x_{k-1}) \\
&= \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(r, x_{k-1})\, dr - |x_{k-1} - x_k|.
\end{aligned}$$

Taking the sum for all k from $i+1$ to j and using $x^{\varepsilon,\tau}(r) = x_{k-1}$ for all $r \in [t_{k-1}, t_k)$, we get

$$(11)\sum_{k=i+1}^{j} [\mathscr{E}(t_k, x_k) - \mathscr{E}(t_{k-1}, x_{k-1})] \le \sum_{k=i+1}^{j} \int_{t_{k-1}}^{t_k} \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr - \sum_{k=i+1}^{j} |x_k - x_{k-1}|.$$

Moreover, since $t_{i-1} < s \le t_i$ and $t_j \le t < t_{j+1}$, we have

$$\begin{aligned}
\mathscr{E}(t_i, x^{\varepsilon,\tau}(t_i)) - \mathscr{E}(s, x^{\varepsilon,\tau}(s)) &= \mathscr{E}(t_i, x_i) - \mathscr{E}(s, x_{i-1}) \\
&\le \mathscr{E}(t_i, x_{i-1}) - |x_{i-1} - x_i| - \mathscr{E}(s, x_{i-1}) \\
&= \int_s^{t_i} \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr - |x^{\varepsilon,\tau}(s) - x_i|. \qquad (12) \\
\mathscr{E}(t, x^{\varepsilon,\tau}(t)) - \mathscr{E}(t_j, x^{\varepsilon,\tau}(t_j)) &= \mathscr{E}(t, x_j) - \mathscr{E}(t_j, x_j) \\
&= \int_{t_j}^{t} \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr - |x^{\varepsilon,\tau}(t) - x_j|, \qquad (13)
\end{aligned}$$

From (10), (11), (13) and (12), we get

$$\begin{aligned}
\mathscr{E}(t, x^{\varepsilon,\tau}(t)) - \mathscr{E}(s, x^{\varepsilon,\tau}(s)) &\le \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr \\
&\quad - \left(|x^{\varepsilon,\tau}(t) - x_j| + \sum_{k=i+1}^{j} |x_k - x_{k-1}| + |x^{\varepsilon,\tau}(s) - x_i| \right) \\
&= \int_s^t \partial_t \mathscr{E}(r, x^{\varepsilon,\tau}(r))\, dr - \mathscr{D}iss(x^{\varepsilon,\tau}; [s,t]).
\end{aligned}$$

5.3. **Proof of Lemma 5.**

Proof. Since u is BV, the distributional derivative Du can be split into three parts: the absolutely continuous part w.r.t. Lebesgue measure $D^a u$, the jump part $D^j u$ and the Cantor part $D^c u$. Now we denote $u'_{co} = D^a u + D^c u$, then applying the chain rule formula for $\mathscr{E} \in C^1$ and $u \in BV$ (see [2]), we get

$$\begin{aligned}
&\mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_0, u(t_0)) \\
&= \int_{t_0}^{t_1} \partial_t \mathscr{E}(s, u(s))\, ds + \int_{t_0}^{t_1} \langle \nabla_x \mathscr{E}(s, u(s)), u'_{co}(s) \rangle\, ds \\
&+ \sum_{t \in J \cap (t_0,t_1)} \big[\mathscr{E}(t, u(t)) - \mathscr{E}(t, u(t^-)) \big] + \sum_{t \in J \cap (t_0,t_1)} \big[\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t)) \big] \\
&+ \mathscr{E}(t_0, u(t_0^+)) - \mathscr{E}(t_0, u(t_0)) + \mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_1, u(t_1^-)) \\
\ge\ & \int_{t_0}^{t_1} \partial_t \mathscr{E}(s, u(s))\, ds - \int_{t_0}^{t_1} |u'_{co}(s)|\, ds \\
&- \sum_{t \in J \cap (t_0,t_1)} |\mathscr{E}(t, u(t)) - \mathscr{E}(t, u(t^-))| - \sum_{t \in J \cap (t_0,t_1)} |\mathscr{E}(t, u(t^+)) - \mathscr{E}(t, u(t))| \\
&- |\mathscr{E}(t_0, u(t_0^+)) - \mathscr{E}(t_0, u(t_0))| - |\mathscr{E}(t_1, u(t_1)) - \mathscr{E}(t_1, u(t_1^-))|.
\end{aligned}$$

Notice that

$$\int_{t_0}^{t_1} |u'_{co}(s)|\,ds = \mathscr{D}iss(u;[t_0,t_1]) - \sum_{t\in J\cap(t_0,t_1)} |u(t)-u(t^-)| - \sum_{t\in J\cap(t_0,t_1)} |u(t^+)-u(t)|$$
$$-|u(t_0^+)-u(t_0)| - |u(t_1)-u(t_1^-)|. \tag{14}$$

Moreover, for every absolutely continuous curve v in $AC([0,1];\mathbb{R}^d)$ such that $v(0)=u(t^-)$, $v(1)=u(t)$ we have

$$\begin{aligned} |\mathscr{E}(t,u(t))-\mathscr{E}(t,u(t^-))| &= \left|\int_0^1 \nabla_x\mathscr{E}(t,v(s))\cdot\dot v(s)\,ds\right| \\ &\le \int_0^1 \max\{1,|\nabla_x\mathscr{E}(t,v(s))|\}\cdot|\dot v(s)|\,ds. \end{aligned}$$

Therefore,

$$|\mathscr{E}(t,u(t))-\mathscr{E}(t,u(t^-))| \le \Delta_{new}(t,u(t^-),u(t)). \tag{15}$$

Similarly,

$$|\mathscr{E}(t,u(t^+))-\mathscr{E}(t,u(t))| \le \Delta_{new}(t,u(t),u(t^+)). \tag{16}$$

Thus it follows from (14), (15) and (16)

$$\begin{aligned} \mathscr{E}(t_1,u(t_1))-\mathscr{E}(t_0,u(t_0)) &\ge \int_{t_0}^{t_1} \partial_t\mathscr{E}(s,u(s))\,ds - \mathscr{D}iss(u;[t_0,t_1]) \\ &\quad + \sum_{t\in J\in(t_0,t_1)} |u(t^-)-u(t)| + \sum_{t\in J\in(t_0,t_1)} |u(t)-u(t^+)| \\ &\quad +|u(t_0)-u(t_0^+)| + |u(t_1^-)-u(t_1)| \\ &\quad - \sum_{t\in J\cap(t_0,t_1)} \Delta_{new}(t,u(t^-),u(t)) - \sum_{t\in J\cap(t_0,t_1)} \Delta_{new}(t,u(t),u(t^+)) \\ &\quad -\Delta_{new}(t_0,u(t_0),u(t_0^+)) - \Delta_{new}(t_1,u(t_1^-),u(t_1)) \\ &= \int_{t_0}^{t_1} \partial_t\mathscr{E}(s,u(s))\,ds - \mathscr{D}iss_{new}(u;[t_0,t_1]). \end{aligned}$$

This ends the proof of Lemma 5. □

Acknowledgments. I warmly thank Professor Giovanni Alberti for proposing to me the problem and giving many helpful directions.

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