ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ - ΤΜΗΜΑ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
Archimedes Center for Modeling, Analysis & Computation
UNIVERSITY OF CRETE - DEPARTMENT OF APPLIED MATHEMATICS
Archimedes Center for Modeling, Analysis & Computation

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D. C. Antonopoulou and D. Blomker and Georgia D Karali

Original Citation:

Antonopoulou, D. C. and Blomker, D. and Karali, Georgia D

(2012)

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SIAM Journal on Mathematical Analysis.

(In Press)

FRONT MOTION IN THE ONE-DIMENSIONAL STOCHASTIC CAHN-HILLIARD EQUATION

D.C. ANTONOPOULOU[†¶], D. BLÖMKER[‡], G.D. KARALI[†¶]

ABSTRACT. In this paper, we consider the one-dimensional Cahn-Hilliard equation perturbed by additive noise, and study the dynamics of interfaces for the stochastic model. The noise is smooth in space and defined as a Fourier series with independent Brownian motions in time. Motivated by the work of Bates & Xun on slow manifolds for the integrated Cahn-Hilliard equation, our analysis reveals the significant difficulties and differences in comparison to the deterministic problem. New higher order terms that we estimate appear due to Itô calculus and stochastic integration and dominate the exponentially slow deterministic dynamics. Using a local coordinate system and defining the admissible interface positions as a multi-dimensional diffusion process, we derive a first order linear system of stochastic ordinary differential equations approximating the equations of front motion. Furthermore, we prove stochastic stability of the approximate slow manifold of solutions over a very long time scale and evaluate the noise effect.

Keywords: 1-D Stochastic Cahn-Hilliard equation, slow manifold, interface motion, additive noise, dynamics, stability.

1. Introduction

1.1. **The problem.** The standard Cahn-Hilliard equation is a simple model for the phase separation of a binary alloy at a fixed temperature proposed in [18, 19]. This model was extended by Cook [25, 43] in order to incorporate thermal fluctuations in the form of an additive noise. In this paper, we consider the one-dimensional Cahn-Hilliard equation posed on $(0,1)$ with an additive stochastic term:

$$u_t = (-\varepsilon^2 u_{xx} + f(u))_{xx} + \partial_x \dot{W}_\varepsilon, \quad 0 < x < 1, \quad t > 0, \tag{SC-H}$$

with no-flux boundary conditions of Neumann type:

$$u_x = u_{xxx} = 0 \text{ at } x = 0, 1. \tag{1.1}$$

The nonlinearity $f = f(u)$ is the derivative of a smooth double equal-well potential F taking its global minimum value 0 at $u = \pm 1$ [1], with non-degenerate minima. A typical example is $F(u) := \frac{1}{4}(u^2-1)^2$ with $f(u) := u^3 - u$. The parameter $\varepsilon > 0$ is a small atomistic interaction length modeling the width of layers that develop during the initial phase separation of spinodal decomposition (cf. [13, 14]). In the later stages of the separation process ε measures the width of transitions between the pure phases $u = \pm 1$. Here, $\dot{W}_\varepsilon$ is a space-time noise smooth in space, and defined as the formal derivative of an ε-dependent Wiener process W_ε. As it is common in stochastic phase-field models, the noise scales with ε. See for example the work of Funaki [35] or Shardlow [47] on the stochastic Allen-Cahn equation. Here the noise strength is controlled by ε, more specifically it is bounded by $\mathcal{O}(\varepsilon^\delta)$ for some $\delta > 9/2$. For details see Assumption 2.3 later.

A characteristic feature of the Cahn-Hilliard equation model is the conservation of total mass $\int_0^1 u(t,x)dx$, which we now fix to be $M \in (-1,1)$. Substituting $\tilde{u}(t,x) := \int_0^x u(t,y)dy$ we obtain the equivalent integrated stochastic Cahn-Hilliard equation:

$$\tilde{u}_t = -\varepsilon^2 \tilde{u}_{xxxx} + (f(\tilde{u}_x))_x + \dot{W}_\varepsilon, \quad 0 < x < 1, \quad t > 0, \tag{ISC-H}$$

1991 *Mathematics Subject Classification.* 35K55, 35K40, 60H30, 60H15.

‡ Institut für Mathematik Universität Augsburg, D-86135 Augsburg.
† Department of Applied Mathematics, University of Crete, GR–714 09 Heraklion, Greece.
¶ Institute of Applied and Computational Mathematics, FO.R.T.H., GR–711 10 Heraklion, Greece.

associated with the boundary conditions:

$$\begin{aligned}\tilde{u}(t,0)=0,\;\; \tilde{u}(t,1)=M,\\ \tilde{u}_{xx}(t,0)=\tilde{u}_{xx}(t,1)=0.\end{aligned}\tag{1.2}$$

J. Carr and R. Pego in [22, 23] presented a detailed analysis of the slow evolution of patterns of the singularly perturbed Ginzburg-Landau equation. They proved existence and persistence of metastable patterns and analyzed the equations governing their motion. These metastable states have been characterized in terms of the global unstable manifolds of equilibria. In [8, 9], P.W. Bates and J. Xun extended their argument and studied the dynamics of the one-dimensional C-H equation in a neighborhood of an equilibrium having $N+1$ transition layers, using several estimates presented in [22, 23]. They determined the exponentially slow speed of the layer motion and described precisely the layer motion directions. In addition, they established existence of an N-dimensional unstable invariant manifold attracting solutions exponentially fast uniformly in ε. Related works in this direction are [10, 36, 45].

Motivated by the work of Bates and Xun for the deterministic problem, we study dynamics for the stochastic model. Due to stochastic integration, new higher order terms appear that we estimate using techniques and ideas of [8, 9, 22, 23]. In the sequel we shall refer frequently to some important definitions and results presented in the aforementioned articles; therefore, we give some details concerning our notation. Following [22, 23], we use the letter f for the nonlinearity in (SC-H), and denote by F the double equal well potential. In [8, 9] the symbol W' is used in place of f; we avoided such a notation since we denote by the standard symbol $\dot{W}$ the additive noise.

1.2. **The effect of noise.** The stochastic Cahn-Hilliard equation being one of the important examples of the nonlinear Langevin equations is based on a field-theoretic approach to the non-equilibrium dynamics of metastable states (see for example [25, 40, 43]). The multi-dimensional generalized stochastic Cahn-Hilliard equation associated with Neumann boundary conditions posed on bounded domains contains a time dependent noise in the chemical potential and an additive noise defined as the formal derivative of a Wiener process. The chemical potential noise describes external fields [38, 40, 42], while the free-energy independent noise may describe thermal fluctuations or external mass supply [25, 38, 40, 43].

Existence and uniqueness of solution for the stochastic problem was first studied in [26], where the nonlinearity f is a polynomial of odd degree and the problem is posed on multi-dimensional rectangular domains. Further, in [20], the author proved existence of solution and of its density for the stochastic Cahn-Hilliard equation with additive noise (in the sense of Walsh, cf. [48]) posed on cubic domains. When the trace of the Wiener process is finite, existence was analyzed in [30]. In [5], existence for the generalized stochastic Cahn-Hilliard equation was derived for general convex or Lipschitz domains; the main novelty was the derivation of space-time Hölder estimates for the Green's kernel of the stochastic problem, by using the domain's geometry, which can be very useful in many other circumstances. The polynomial nonlinearity which forces the solution to stay between the pure phases ± 1 has been analyzed in [13, 14, 20, 21, 26, 30], while in [29, 28, 37] a stochastic Cahn-Hilliard equation with reflection was considered.

In [13, 14] (see [15] for a review), the effect of noise on evolving interfaces during the initial stage of phase separation is analyzed. The evolution of these interfaces is stochastic and not yet fully understood. In [13], the authors show that for a solution starting at the homogeneous state, the probability of staying near a certain finite-dimensional space of pattern is high as long the solution stays within the distance of the homogeneous state. Further, in [14], the dynamics of a nonlinear partial differential equation perturbed by additive noise are considered. Under the assumption that the underlying deterministic equation has an unstable equilibrium, the authors show that the nonlinear stochastic partial differential equation exhibits essentially linear dynamics even far from equilibrium.

On the other hand interface motion has been studied for many related models like Allen-Cahn or Ginzburg Landau and phase-field models, cf. for example [4, 12, 16] for a rigorous analysis or the results of [32] for formal arguments, which describe the interfaces as interacting Brownian motions. Numerical results for interface motion are presented in [39, 47]. The problem of singular perturbation for a reaction-diffusion stochastic partial differential equation of Ginzburg-Landau type is investigated in [34]. The motion of interfaces for Cahn-Hilliard equation was only studied in an unpublished note by S. Brassesco in 2003, where she studied a solution with a single interface on $\mathbb{R}$. When properly rescaled, the interface is driven by non-Markovian dynamics (cf. [12] for a similar result). In [46], the authors present a numerical study of the late stages of spinodal decomposition with noise.

The deterministic Cahn-Hilliard equation was proposed by Cahn and Hilliard ([18, 17]) as a model for the phase separation of a binary alloy at a fixed temperature, with $u(t,x)$ defining the mass concentration of one of the phases at a point x at time t. For a more physical background, derivation and discussion of the deterministic Cahn-Hilliard equation and related equations, we refer to [7, 17, 18, 31, 33] and the references therein. Results for the noisy Cahn-Hilliard equation are of great interest for the studying of Ostwald ripening [2, 3, 41] and nucleation [11]. For a survey, including numerical results and conjectures concerning the nucleation problem, see [15].

1.3. **The approximate slow manifold.** The space-time noise that we introduce is smooth in space allowing for the application of Itô-formula. For our study of the dynamics of transition layers for the stochastic model, we closely follow the approach of Bates & Xun and Carr & Pego based on the analysis of an approximate invariant manifold $\mathcal{M}$. Although constructed in a different way, it can be thought of as piecing together a rescaled one kink (or front) of steady state solutions on the whole real-line. The elements of the manifold are parametrized by the position of the fronts given by $h \in \mathbb{R}^{N+1}$. Nevertheless, in our case the dependency on time is stochastic. This fact leads to the very interesting and difficult problem of further investigating the properties of $\mathcal{M}$ by means of deriving higher order estimates related to the stationary problem.

Let us present first the details necessary for the steady state solutions ϕ, the parameters h and the manifold $\mathcal{M}$. Given $\varepsilon > 0$, we consider a such that $f'(u) > 0$ for all u satisfying $|u \pm 1| < a$. Then, cf. [22], there exists $\rho > 0$ such that if ℓ satisfies $\frac{\varepsilon}{\ell} < \rho$ then a unique solution $\phi = \phi(x, \ell, \pm 1)$ exists for the following stationary Dirichlet problem

$$\begin{aligned} &\varepsilon^2 \phi_{xx} - f(\phi) = 0, \quad -\ell/2 < x < \ell/2, \\ &\phi = 0, \quad x = \pm \ell/2, \end{aligned} \tag{1.3}$$

that satisfies:

(a) $\phi(x, \ell, +1) > 0$ for $|x| < \ell/2$ and $|\phi(0) - 1| < a$, (b) $\phi(x, \ell, -1) < 0$ for $|x| < \ell/2$ and $|\phi(0) + 1| < a$.

For sufficiently small $\varepsilon > 0$, it is known that $\phi \approx \pm 1$ with transition layers of order $\mathcal{O}(\varepsilon)$ near $x = \pm \ell/2$.

Following [9], we consider the slowly evolving solutions with $N+1$ layers well separated and bounded away from the boundary $x = 0, 1$ and define the set of admissible positions h of the interfaces

$$\Omega_\rho := \Big\{ h \in R^{N+1} : 0 < h_1 < \cdots < h_{N+1} < 1, \text{ and } \frac{\varepsilon}{\rho} < h_j - h_{j-1}, \ \ j = 1, \dots, N+2 \Big\}, \tag{1.4}$$

with $h_0 := -h_1$, $h_{N+2} := 2 - h_{N+1}$. These interfaces evolve in time, and we expect them to have a width of order ε. Thus, the distance is bounded below by ε/ρ for some small ρ. Later we fix $\rho = \varepsilon^\kappa$ for any small $\kappa > 0$.

Let $h \in \Omega_\rho$ be given as above, and denote the mid points between interfaces by $m_j := \frac{h_{j-1} + h_j}{2}$ for $j = 1, \dots, N+2$ with $m_0 = 0$ and $m_{N+1} = 1$. Moreover, we define the function $u^h : I_j := [m_j, m_{j+1}] \to \mathbb{R}$ for the interfaces h by

$$\begin{aligned} u^h(x) = &\Big[1 - \chi\left(\tfrac{x - h_j}{\varepsilon}\right) \Big] \cdot \phi\left(x - m_j, h_j - h_{j-1}, (-1)^j\right) \\ &+ \chi\left(\tfrac{x - h_j}{\varepsilon}\right) \cdot \phi\left(x - m_{j+1}, h_{j+1} - h_j, (-1)^{j+1}\right), \end{aligned} \tag{1.5}$$

where $\chi : \mathbb{R} \to [0,1]$ is a C^∞ cut-off function such that $\chi = 1$ on $[1, \infty)$ and $\chi = 0$ on $(-\infty, -1]$.

Definition 1.1 (**Approximate slow manifold).** *The first approximate manifold of the stochastic Cahn-Hilliard equation solution is defined by*

$$\mathcal{M}_1 := \Big\{ u^h : h \in \Omega_\rho \Big\}.$$

Fixing a mass $M \in (-1,1)$, we define as the second approximate manifold the submanifold $\mathcal{M}$ of $\mathcal{M}_1$ where mass conservation holds i.e.

$$\mathcal{M} := \Big\{ u^h \in \mathcal{M}_1 : \int_0^1 u^h dx = M \Big\}.$$

For the integrated equation, we consider the manifold

$$\tilde{\mathcal{M}} := \Big\{ \tilde{u}^h \;:\; u^h \in \mathcal{M},\; \tilde{u}^h(x) = \int_0^x u^h dx \Big\}.$$

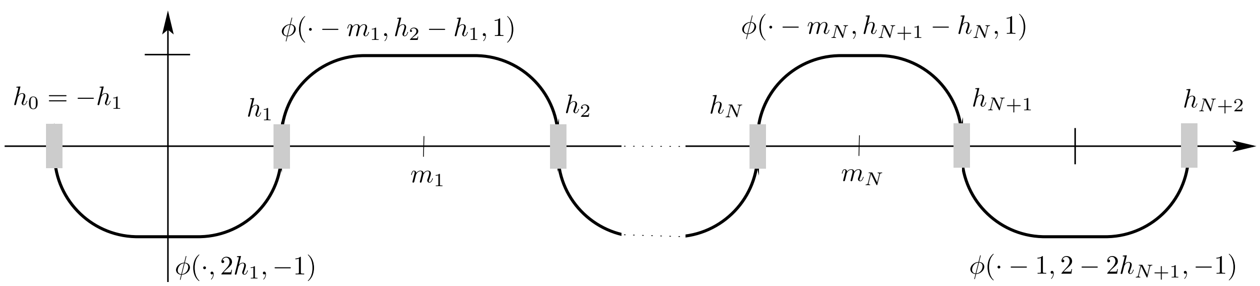

FIGURE 1.1. Gluing together positive and negative solutions of (1.3) to obtain $u^h \in \mathcal{M}$. Note that $m_1 = 0$, $m_{N+2} = 1$, and $I_j = [m_j, m_{j+1}]$.

Remark 1.2. *In view of the initial stochastic equation (SC-H), conservation of mass holds if and only if formally*

$$\int_0^1 \partial_x \dot{W}_\varepsilon dy = \dot{W}_\varepsilon(1) - \dot{W}_\varepsilon(0) = 0. \tag{1.6}$$

This is later assured by our assumptions on W_ε, which impose Dirichlet-boundary conditions for $\dot{W}_\varepsilon$ (cf. Definition 2.2 and Assumption 2.3). A very simple rigorous example is the following: consider $\dot{W}_\varepsilon := \delta_\varepsilon g(x)\dot{\beta}(t)$, where $\dot{\beta}(t)$ is a white noise in time and g a smooth function satisfying $g(1) = g(0)$, then by integrating in space the equation (SC-H) and using the fact that

$$\int_0^1 \partial_x \dot{W}_\varepsilon dy = \delta_\varepsilon \dot{\beta}(t) \int_0^1 g_x(y) dy = 0,$$

we obtain mass conservation even with the noise. This example extends to infinite series of such terms.

Throughout the entire paper we assume that the additive noise in (SC-H) satisfies (1.6)*, and therefore the proposed stochastic model exhibits mass conservation.*

1.4. **The new coordinate system.** Along $\tilde{\mathcal{M}}$ the natural coordinate system would be to use the parameters $h \in \Omega_\rho$ for the position in $\tilde{\mathcal{M}}$ (where N of them are sufficient due to mass conservation), together with the orthogonal projection onto $\tilde{\mathcal{M}}$. In order to relate the coordinate system to the deterministic flow of (ISC-H), one approximates the tangent space of $\tilde{\mathcal{M}}$ by the span of some functions E_i^ξ, $i = 1, \dots, N$ related to eigenfunctions of the linearization to be defined in the sequel. Here, we follow [8].

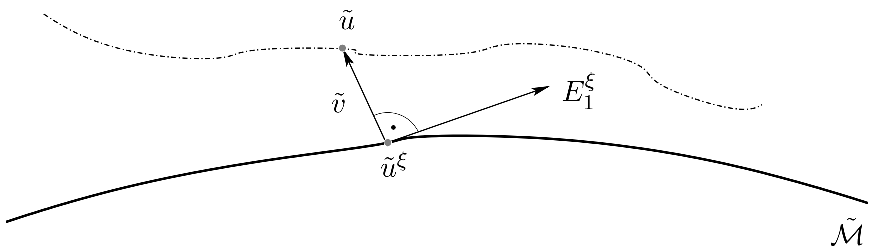

FIGURE 1.2. The local coordinate system $\tilde{u} = \tilde{u}^\xi + \tilde{v}$ around $\tilde{\mathcal{M}}$ for $N = 1$ (two interfaces). Note that $E_1^\xi \approx \tilde{u}_1^\xi$, which is the tangential vector along the manifold.

We denote the $L^2(0,1)$ inner product by $\langle u, v\rangle := \int_0^1 uv dx$, the induced L^2-norm by $\|\cdot\|$ and introduce the symbol $\tilde{g}(t,x) := \int_0^x g(t,y)dy$, for any g, which is spatially integrable.

Due to mass conservation, we reduce the parameter space Ω_ρ by one dimension. Define

$$\xi := (\xi_1, \dots, \xi_N) = (h_1, \dots, h_N),$$

and consider h_{N+1} as a function of ξ. Thus, for $\tilde{u}_j^h := \frac{\partial \tilde{u}^h}{\partial h_j}$ and $\tilde{u}_j^\xi := \frac{\partial \tilde{u}^\xi}{\partial \xi_j}$ we obtain

$$\tilde{u}_j^\xi = \frac{\partial \tilde{u}^h}{\partial h_{N+1}} \cdot \frac{\partial h_{N+1}}{\partial h_j} + \frac{\partial \tilde{u}^h}{\partial h_j} .$$

We use $\tilde{u} \to (\xi, \tilde{v})$ as coordinate system around $\tilde{\mathcal{M}}$. Let us split a solution $\tilde{u}$ of (ISC-H) into a sum of stochastic processes

$$\tilde{u}(t) := \tilde{u}^{\xi(t)} + \tilde{v}(t). \tag{1.7}$$

Here the position on $\tilde{\mathcal{M}}$ is given by $\tilde{u}^\xi \in \tilde{\mathcal{M}}$ while the distance from $\tilde{\mathcal{M}}$ is given by $\tilde{v}$ which is defined as the following projection such that

$$\langle \tilde{v}, E_j^\xi\rangle = 0 \quad \text{for} \quad j = 1, \dots, N . \tag{1.8}$$

It turns out that the functions E_j^ξ are good approximations to the first eigenfunctions of the linearized integrated Cahn-Hilliard operator, which in turn are good approximations to the tangent space of $\tilde{\mathcal{M}}$. They are defined as follows:

$$E_j^\xi(x) := \tilde{w}_j(x) - Q_j(x) , \qquad \tilde{w}_j(x) := \tilde{u}_j^h(x) + \tilde{u}_{j+1}^h(x) ,$$
$$Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\tilde{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\tilde{w}_{jxx}(1) + x\tilde{w}_j(1) , \quad j = 1, \dots, N .$$

The Q_j are exponentially small terms (cf. [8], pg. 437-439), taking care of the boundary values of E_j^ξ. More precisely, $\tilde{w}_j$ are good approximations of these eigenfunctions while $\tilde{w}_j(0) = 0$, and $\tilde{w}_j(1)$, $\tilde{w}_{jxx}(0)$, $\tilde{w}_{jxx}(1)$ are exponentially small quantities. Introducing the polynomial correction terms Q_j in the definition of $E_j^\xi(x)$ modifies the $\tilde{w}_j$ so that E_j^ξ are good approximations and satisfy exactly the boundary conditions of the linearized integrated Cahn-Hilliard operator, i.e.

$$E_j^\xi = E_{jxx}^\xi = 0 \quad \text{for} \quad x = 0, 1.$$

For short-hand notation, we also define higher derivatives using indices:

$$E_{il}^\xi := \frac{\partial E_i^\xi}{\partial \xi_l} , \qquad E_{ilk}^\xi := \frac{\partial^2 E_i^\xi}{\partial \xi_l \partial \xi_k} , \qquad \text{and} \qquad \tilde{u}_{kl}^\xi := \frac{\partial^2 \tilde{u}^\xi}{\partial \xi_k \partial \xi_l}. \tag{1.9}$$

1.5. **Assumptions on the noise and layers.** Throughout this paper the following three fundamental assumptions are considered for the noise and transition layers:

1) The noise is sufficiently regular in space and of small strength. As derived, the manifold used in this paper is stable and attractive for a long time-scale with high probability and thus consists a good approximation of the stochastic Cahn-Hilliard equation solution, if the noise $\dot{W}_\varepsilon$ is sufficiently regular in space (cf. Assumption 2.3) and its strength is bounded by $\mathcal{O}(\varepsilon^\delta)$ for some $\delta > 9/2$. The noise is presented in details at Definition 2.2 as the formal derivative of a Wiener process W_ε given by a Fourier series of independent Brownian motions in the sense of DaPrato and Zabzcyck, [27].

2) We analyze local solutions of the (ISC-H): The coordinates ξ of the projection onto the manifold perform a diffusion process. A main difference in stochastic dynamics of interfaces in comparison to the deterministic problem is that due to the noise the movements of the layers are co-related, and thus the resulting stochastic o.d.e. system given by (2.1) may be non-linear for a general noise definition.

In order to make the analysis tractable, when we derive the equations of motion for the interface we assume that the coordinates ξ of the projection onto the manifold perform a multi-dimensional diffusion process. By this natural assumption, we consider that the interfaces solve a very general stochastic ordinary equation driven by a Wiener process.

To be more precise, let $\tilde{u}$ be the solution of (ISC-H) where W_ε is an ε-dependent Wiener process defined in Definition 2.2. We assume that the projection coordinates $\xi(t)$ (positions of the interface) is a stochastic diffusion process in $\mathbb{R}^N$. Since the specific W_ε is introduced in (ISC-H) then the only underlying probability space is the Wiener space corresponding to W_ε. Therefore, diffusion is driven by W_ε and is defined for any $k = 1, \dots, N$ by

$$d\xi_k = b_k(\xi)dt + \langle \sigma_k(\xi), dW_\varepsilon \rangle,$$

for some unknown vector field $b : \mathbb{R}^N \to \mathbb{R}^N$ and some variance σ on $\mathbb{R}^N$. The unknown functions b, σ might not only depend on ξ, but also on time t and the distance from the manifold $\tilde{v}$.

As a result, we apply Itô calculus to the general system (2.1) in order to calculate explicitly the co-relations of layers movements and derive finally closed forms of b and σ. The assumption of ξ being a diffusion process is justified later in Theorem 3.2 after the derivation of the SDE for the motion of the interfaces. More specifically, the diffusion process ξ exists locally as a solution of the SDE defined up to a stopping time since the nonlinearities are only locally Lipschitz. It is possible to continue solutions, until they leave the domain of definition of the equations close to $\tilde{\mathcal{M}}$. In addition, as long as ξ is well defined and $\|\tilde{v}\|$ sufficiently small, then $\tilde{u}$ given by $\tilde{u} := \tilde{u}^\xi + \tilde{v}$ is well defined and solves the initial (ISC-H) equation (cf. Theorem 3.2). Further, by attractivity and stochastic stability, we derive that the time of existence is with high probability larger than the exit time from some slow channel (neighborhood of the approximate manifold), in which we study the stability result. So, local solutions of the form $\tilde{u} := \tilde{u}^\xi + \tilde{v}$ for ξ given as the solution of a diffusion process for the specific σ and b defined by (3.12) and (3.13) respectively, exist and solve (ISC-H). Local solutions of (ISC-H) of this type until some stopping time $\tau^* \leq T_\varepsilon$ are analyzed and approximated in this paper.

3) The number of transition layers is fixed. This is a natural assumption, which is also present in the work of Carr & Pego and Bates & Xun in [22, 8, 9]. Suggested by Fusco and Hale in [36] and further analyzed in [22, 8, 9], a geometric method was adopted and developed for the construction of a slow manifold of functions approximating a metastable state. This construction is valid for a fixed number of transition layers.

In [8], the study of dynamics of the one-dimensional Cahn-Hilliard equation considers the slow evolution of patterns in a neighborhood of an equilibrium having $N+1$ transition layers. Further, in the aforementioned paper, the authors constructed an N-dimensional approximate invariant manifold consisting of states with a fixed number of $N+1$ transition layers and a narrow tubular neighborhood or channel around this manifold. Solutions starting nearby approach this channel exponentially fast. In addition, [9] verifies the existence of an N-dimensional invariant manifold and all solutions inside the slow channel are attracted exponentially fast to this invariant manifold. The change of numbers of layers is only possible either by a rare stochastic event or when the solution leaves the slow channel after moving slowly along the manifold.

In our analysis, we study the dynamics for the stochastic problem locally in time i.e. as long as the number of transition layers is fixed and thus indeed the layer locations are well separated and bounded away from the boundary points 0, 1 (cf. [22, 8, 9] for the deterministic problem). This is also justified by the fact that, as we prove, for a sufficiently bounded noise strength stability and attractivity of the manifold hold in the stochastic case also, at least for a very long time scale and with high probability. Of course the solution can leave the manifold at the boundary by a layer breaking down.

Moreover, due to rare stochastic events an extra 'bump' (layer) could be formed. In our case this interesting event is rather unlikely, since the strength of the additive noise is sufficiently small so that the manifold $\tilde{\mathcal{M}}$ is stable and attractive with high probability. Apart from large deviation results the rigorous mathematical study on extra layers generation is highly not trivial. See for example the work of Xinfu Chen [24] on generation, propagation, and annihilation of metastable patterns for the deterministic Allen-Cahn equation. Therefore, this is not analyzed in the present paper.

2. Main results

The SDE (Stochastic Differential Equation) system for the motion of fronts is given by the projection onto the manifold $\tilde{\mathcal{M}}$, using the coordinate system of Section 1.4. We then prove that $\tilde{\mathcal{M}}$ is locally exponentially attracting and show that solutions stay with high probability in a small slow tube around $\tilde{\mathcal{M}}$, until large times or until one of the layers becomes small. The flow along $\tilde{\mathcal{M}}$ is well described by the SDE for the interfaces ξ. Depending on the strength of the noise, we investigate how the equation of motion of the fronts looks like and evaluate the noise effect. In addition, we study extensively the case $N = 1$ where the motion of the second interface is determined by the first. Here the motion is given by the Wiener process $\dot{W}_\epsilon$ projected onto $\tilde{\mathcal{M}}$. Finally, the case of space-time white noise is discussed. In the last section, we present the proofs of the estimates used in our analysis concerning all the higher order terms that appear in the stochastic setting. These are technical results that are independent of the other sections.

Let us first explain briefly how the equations of motions along $\tilde{\mathcal{M}}$ are derived in Section 3. For details we refer to Subsection 3.2. If $\tilde{u}$ is the solution of (ISC-H), then applying the Itô-formula in differentiating with respect to t , we get for $i = 1, \cdots, N$ the following system in $d\xi_1, \cdots, d\xi_N$ for the stochastic Cahn-Hilliard equation:

$$
\begin{aligned}
\sum_j \Big[\langle \tilde{u}^\xi_j, E^\xi_i\rangle - \langle \tilde{v}, E^\xi_{ij}\rangle\Big] d\xi_j =& \langle -\varepsilon^2(\tilde{u}^\xi_{xxxx} + \tilde{v}_{xxxx}) + (f(\tilde{u}^\xi_x + \tilde{v}_x))_x, E^\xi_i\rangle dt \\
&+ \sum_{l,k}\Big[\tfrac{1}{2}\langle \tilde{v}, E^\xi_{ilk}\rangle - \tfrac{1}{2}\langle \tilde{u}^\xi_{kl}, E^\xi_i\rangle - \langle \tilde{u}^\xi_k, E^\xi_{il}\rangle\Big] d\xi_l d\xi_k \\
&+ \sum_j \langle dW_\varepsilon, E^\xi_{ij}\rangle d\xi_j \\
&+ \langle E^\xi_i, dW_\varepsilon\rangle\ .
\end{aligned} \tag{2.1}
$$

We note that the last three additive terms above at the right-hand side are not present in [8, 9] where the deterministic Cahn-Hilliard equation was studied.

Remark 2.1. *In view of* (2.1)*, we observe that the analysis of the stochastic dynamics is a much more complicated and difficult problem compared to the deterministic one.*

(1) Deterministic case: *The system is linear in $d\xi_j$, therefore by estimating the inverse matrix on the left-hand side (possibly close to $\tilde{\mathcal{M}}$) and the right-hand side terms, the motion of interfaces is obtained, see* [9]*.*

(2) Stochastic case: *Obviously, for a general noise definition the system is non-linear due to the appearance of $d\xi_l d\xi_k$, which as we shall prove will dominate the exponentially small deterministic dynamics. In the sequel, in order to get rid of the co-relations $d\xi_l d\xi_k$, we make the ansatz that ξ performs a diffusion process, which is justified later. Further, we need estimates for the additional higher order terms E^ξ_{ij}, E^ξ_{ilk}, and $\tilde{u}^\xi_{kl}$. Therefore, we need to improve the estimates of* [8]*.*

The sufficiently regular noise $\dot{W}_\varepsilon$ is the formal derivative of a Wiener process W_ε defined as follows.

Definition 2.2 (The Wiener process W_ε). *Let W_ε be a $\mathcal{Q}_\varepsilon$-Wiener process in the underlying Hilbert-space $H = L^2(0,1)$, $\mathcal{Q}_\varepsilon$ a symmetric operator and $(e_k)_{k\in\mathbb{N}}$ an orthonormal basis with corresponding eigenvalues $\alpha^2_{\varepsilon,k}$, such that*

$$\mathcal{Q}_\varepsilon e_k = \alpha^2_{\varepsilon,k} e_k \quad \textit{and} \quad W_\varepsilon(t) = \sum_{k=1}^{\infty} \alpha_{\varepsilon,k}\beta_k(t) e_k,$$

for a sequence of independent real-valued standard Brownian motions $\{\beta_k(t)\}_{t\geq 0}$ (cf. DaPrato, Zabzcyck [27]).

We always rely on the following assumption, which implies mass conservation and regularity.

Assumption 2.3. *Suppose that the e_k are the eigenfunctions of the Dirichlet-Laplacian. Moreover, assume for some $\delta_\varepsilon > 0$*

$$(1)\ \|\mathcal{Q}_\varepsilon\| < C\delta_\varepsilon^2, \qquad (2)\ \sum_{k=1}^{\infty} \alpha^2_{\varepsilon,k} B_\varepsilon(e_k) < C\delta_\varepsilon^2, \qquad (3)\ \|\partial_x \mathcal{Q}_\varepsilon\| < C\delta_\varepsilon^2\ ,$$

where we assume additionally that $\delta_\varepsilon < \varepsilon^{9/(2-\kappa)}$ for some small $\kappa > 0$. B_ε is defined as

$$B_\varepsilon(e) = \varepsilon^2\|e_{xx}\|^2 + \|e_x\|^2\ ,$$

while for $g = \sum_{k=1}^{\infty} \gamma_k e_k \in L^2(0,1)$ the linear operator $\partial_x \mathcal{Q}_\varepsilon$ is defined as

$$(\partial_x \mathcal{Q}_\varepsilon) g := \sum_{k=1}^{\infty} \gamma_k \partial_x(\mathcal{Q}_\varepsilon e_k) = \sum_{k=1}^{\infty} \gamma_k a^2_{\varepsilon,k} \partial_x e_k\ .$$

The first assumption on the norm of $\mathcal{Q}_\varepsilon$ as an operator in H implies that the strength of the noise is bounded by $\mathcal{O}(\delta_\varepsilon)$, while the second and third one are additional assumptions on the noise regularity. Note that $B_\varepsilon(\cdot)^{1/2}$ is equivalent to the standard H^2-norm (see (3.15)).

For the calculation of the motion of the interfaces, we will assume that $\xi(t)$ a diffusion process in $\mathbb{R}^N$ (see Section 1.5) is defined for any $k = 1, \dots, N$ by

$$d\xi_k = b_k(\xi)dt + \langle \sigma_k(\xi), dW_\varepsilon\rangle,$$

for some vector field $b : \mathbb{R}^N \to \mathbb{R}^N$ and some variance $\sigma : \mathbb{R}^N \to H^N$. Later in Theorem 3.2 we justify this ansatz.

Following [9] we define the matrix

$$A_{ij}(\xi) = \langle \tilde{u}^\xi_j, E^\xi_i\rangle - \langle \tilde{v}, E^\xi_{ij}\rangle,$$

which is invertible close to the slow manifold. The assumptions on the noise combined with (2.1), gives the following SDE system for the motion of interfaces:

$$\begin{aligned}\sum_j A_{ij}(\xi)d\xi_j =& \langle -\varepsilon^2(\tilde{u}^\xi_{xxxx} + \tilde{v}_{xxxx}) + (f(\tilde{u}^\xi_x + \tilde{v}_x))_x, E^\xi_i\rangle dt \\ &+ \sum_{l,k} \Big[\tfrac{1}{2}\langle \tilde{v}, E^\xi_{ilk}\rangle - \tfrac{1}{2}\langle \tilde{u}^\xi_{kl}, E^\xi_i\rangle - \langle \tilde{u}^\xi_k, E^\xi_{il}\rangle\Big] \langle \mathcal{Q}_\varepsilon \sigma_k(\xi), \sigma_l(\xi)\rangle dt \\ &+ \sum_j \langle \mathcal{Q}_\varepsilon E^\xi_{ij}, \sigma_j(\xi)\rangle dt \\ &+ \langle E^\xi_i, dW_\varepsilon\rangle.\end{aligned} \tag{2.2}$$

(cf. also the equivalent presentation (3.11)). From this we can easily read off b and σ. Moreover, the flow along $\tilde{\mathcal{M}}$ is described by the interface positions. It is now easy to check, by construction, that the difference

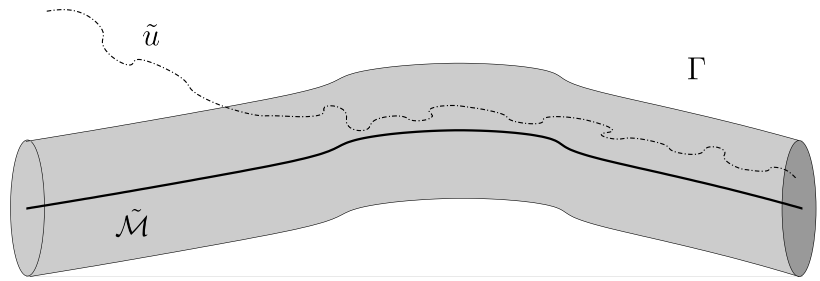

FIGURE 2.1. The stability of the *slow manifold* $\tilde{\mathcal{M}}$ for two interfaces ($N = 1$). A small tubular neighborhood Γ, the *slow channel*, is attracting over long time-scales. Solutions tend to exit at the end of Γ by loosing an interface.

$\tilde{v} = \tilde{u} - \tilde{u}^\xi$ is actually the $\tilde{v}$ of the coordinate system (see Subsec. 1.4). In addition, a solution of (2.2) together with a corresponding equation for $\tilde{v}$ (see (3.14), later) describes a solution $\tilde{u}$ of (ISC-H); see Theorem 3.2.

Further, in Section 3, the variance σ of the multi-dimensional diffusion process ξ of the interfaces is computed first explicitly and then estimated in terms of ε. A main result is the stochastic analysis of the stability of the approximate manifold, which is presented in Theorem 3.6 of Section 3. Over a long time-scale of order $\mathcal{O}(\varepsilon^{-q})$ for any large $q > 0$, we show that, with high probability, the solution of the stochastic Cahn-Hilliard equation stays in a small neighborhood Γ of the integrated manifold $\tilde{\mathcal{M}}$, unless an interface breaks down.

In Section 4, we present first Theorem 4.1 in which we approximate the terms in (2.2) and derive the equations of motion of interfaces. Further, we consider several examples where Theorem 4.1 is simplified. If the noise is exponentially small, then we recover the slow motion results of [8, 9]. There is a slow channel given by a neighborhood of $\tilde{\mathcal{M}}$, in which with high probability the motion of the interfaces is described by the deterministic regime. There is also an interesting intermediate regime of still exponentially small noise, which for simplicity of presentation we do not consider in this article. Here, due to the presence of noise, additional deterministic and stochastic terms appear in the deterministic equation of Bates & Xun [9]. An interesting case from the point of applications is the case where the noise strength is a power of ε. As the general case is quite involved in presentation, we consider only two interfaces (i.e. $N = 1$). Here, obviously the motion of the second interface is determined by the first one which is approximated by the following SDE (cf. (4.11)):

$$d\xi_1 = \frac{1}{32\ell_2^2}\frac{\partial}{\partial\xi_1}\|\mathcal{Q}_\varepsilon^{1/2}E_1^\xi\|^2 dt + \frac{1}{4\ell_2}\langle E_1^\xi, dW_\varepsilon\rangle \,, \tag{2.3}$$

where ℓ_2 is the distance between the two interfaces. We comment later that here ξ_1 is approximately the projection of the Wiener process W_ε onto $\tilde{\mathcal{M}}$.

Finally in this section, we also discuss the case of non-smooth in space space-time white noise ($\mathcal{Q}_\varepsilon = \delta_\varepsilon Id$), which is unfortunately not covered by our assumptions. Here ξ_1 would be close to a Brownian motion with variance $\delta_\varepsilon^2/(4\ell_2)$.

Section 5 provides estimates for the second order derivatives $\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j}$, for the higher order derivatives of E_j^ξ and $\tilde{u}^\xi$, and a bound for the quantity $\langle L^c\tilde{v}, \tilde{u}_{kl}^\xi\rangle$ (needed in the proof of the stability Theorem). Here the operator L^c acting on a general smooth in space function ϕ is given by

$$L^c(\phi) := -\varepsilon^2\phi_{xxxx} + (f'(u^h)\phi_x)_x.$$

The results of this section are quite technical since their proof involves extensive calculations related to properties of solutions of the stationary problem (1.3). The new terms to estimate appear only in the stochastic setting due to the frequent application of Itô-formula, and were therefore not treated in the work of Bates & Xun [8, 9] or Carr & Pego [22, 23].

3. Front motion

In this section, we derive the equations of motions of the fronts and show that the approximate manifold is locally attracting.

3.1. Preliminaries and definitions.

Let us first recall some notation. If u is the solution of (SC-H), then $\tilde{u}(t,x) := \int_0^x u(t,y)dy$ is the solution of the integrated one i.e. of (ISC-H). Let a, ε, ρ, N be given; for some ℓ such that $\varepsilon/\ell < \rho$, we consider the unique solution ϕ of (1.3) which satisfies the properties (a) and (b). Let also $(h_1,\dots,h_{N+1}) \in \Omega_\rho$ be the admissible interface positions and take $h_0 := -h_1$, $h_{N+2} := 2 - h_{N+1}$.

Let $\ell_j = h_j - h_{j-1}$ be the distance between interfaces and $\ell := \min\{\ell_1,\dots,\ell_N\}$ the lower bound on them. Note that by the construction of Ω_ρ the functions ϕ are always well defined. Let

$$r := \varepsilon/\ell, \quad \beta_\pm(r) := 1 \mp \phi(0,\ell,\pm) \quad \text{and } \alpha_\pm(r) := F(\phi(0,\ell,\pm)) \;.$$

In view of (1.5), we also define

$$\phi^j(x) := \phi\left(x - m_j, \ell_j, (-1)^j\right),$$

and $u_j^h := \frac{\partial u^h}{\partial h_j}$ for $j = 1,\dots,N+1$. Considering $r_j := \varepsilon/\ell_j$, let

$$\beta^j(r) := \begin{cases} \beta_+(r_j) & \text{for } j \text{ even} \\ \beta_-(r_j) & \text{for } j \text{ odd,} \end{cases} \quad \text{and} \quad \beta(r) := \max_j \beta^j(r) \;.$$

We recall that in [9], as an application of the implicit function Theorem,

$$\frac{\partial h_{N+1}}{\partial h_j} = (-1)^{N-j} + \mathcal{O}(\varepsilon^{-1}\beta(r)). \tag{3.1}$$

In addition, let

$$\alpha^j(r) := \begin{cases} \alpha_+(r_j) & \text{for } j \text{ even} \\ \alpha_-(r_j) & \text{for } j \text{ odd} \end{cases} \quad \text{and} \quad \alpha(r) := \max_j \alpha^j(r) \;.$$

We see later, that both α and β are exponentially small in ε, if we consider $r_j \le \rho \le \varepsilon^\kappa$ for some small positive κ.

3.2. The SDE for the front motion.

Let $\tilde{u}$ be a solution of (ISC-H). We assume that the N front positions, i.e. the coordinates of $\xi(t) = (\xi_1(t),\dots,\xi_N(t))$, define a multi-dimensional diffusion process which is given by

$$d\xi_k = b_k(\xi)dt + \langle \sigma_k(\xi), dW_\varepsilon \rangle, \quad k = 1,\dots,N, \tag{3.2}$$

for some vector field $b : \mathbb{R}^N \to \mathbb{R}^N$ and some variance $\sigma : \mathbb{R}^N \to H^N$. The main aim of this paragraph is to identify b and σ, which might also depend on $\tilde{v}$, i.e. on the distance from the manifold.

We use Itô-formula, in order to differentiate $\tilde{u}^\xi$ with respect to t, and get

$$d\tilde{u} = \sum_{j=1}^N \tilde{u}_j^\xi d\xi_j + \tfrac{1}{2} \sum_{1\le k,l\le N} \tilde{u}_{kl}^\xi d\xi_k d\xi_l + d\tilde{v} \;, \qquad \text{with } \; \tilde{u}_{kl}^\xi = \frac{\partial^2 \tilde{u}^\xi}{\partial \xi_k \partial \xi_l}. \tag{3.3}$$

We take as in [9], p. 175, the inner product in the space of equation (ISC-H) with E_i^ξ, to get for any $i = 1,\dots,N$

$$\langle E_i^\xi, d\tilde{u} \rangle = \langle \mathcal{L}^c(\tilde{u}), E_i^\xi \rangle dt + \langle E_i^\xi, dW_\varepsilon \rangle \;, \tag{3.4}$$

where we defined the nonlinear ICH-operator as

$$\mathcal{L}^c(u) := -\varepsilon^2 u_{xxxx} + (f(u_x))_x$$

for short-hand notation.

On the other hand, if we take the inner product of (3.3) with E_i^ξ, we derive

$$\langle E_i^\xi, d\tilde{u}\rangle = \sum_{j=1}^{N} \langle \tilde{u}_j^\xi, E_i^\xi\rangle d\xi_j + \tfrac{1}{2} \sum_{1\le k,l\le N} \langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle d\xi_k d\xi_l + \langle E_i^\xi, d\tilde{v}\rangle \ . \tag{3.5}$$

Throughout the rest of this paper, any summation is on $1, 2, \ldots, N$ for any index.

In order to eliminate $d\tilde{v}$, we apply Itô-formula to the orthogonality condition $\langle \tilde{v}, E_i^\xi\rangle = 0$, and arrive at

$$\begin{aligned}\langle E_i^\xi, d\tilde{v}\rangle &= -\langle \tilde{v}, dE_i^\xi\rangle - \langle d\tilde{v}, dE_i^\xi\rangle \\ &= -\sum_j \langle \tilde{v}, E_{ij}^\xi\rangle d\xi_j - \tfrac{1}{2}\sum_{j,k} \langle \tilde{v}, E_{ijk}^\xi\rangle d\xi_j d\xi_k - \sum_j \langle E_{ij}^\xi, d\tilde{v}\rangle d\xi_j.\end{aligned}$$

Now, we use that $d\tilde{v} = d\tilde{u} - d\tilde{u}^\xi$ and the fact that $dtdt = 0$ and $dW_\varepsilon dt = 0$. In detail:

$$\begin{aligned}-\sum_j \langle E_{ij}^\xi, d\tilde{v}\rangle d\xi_j &= -\sum_j \langle E_{ij}^\xi, d\tilde{u}\rangle d\xi_j + \sum_j \langle E_{ij}^\xi, d\tilde{u}^\xi\rangle d\xi_j \\ &= -\sum_j \langle E_{ij}^\xi, \mathcal{L}^c(\tilde{u})\rangle dt d\xi_j - \sum_j \langle E_{ij}^\xi, dW_\varepsilon\rangle d\xi_j + \sum_{j,k} \langle E_{ij}^\xi, \tilde{u}_k^\xi\rangle d\xi_k d\xi_j \\ &= -\sum_j \langle E_{ij}^\xi, dW_\varepsilon\rangle d\xi_j + \sum_{j,k} \langle E_{ij}^\xi, \tilde{u}_k^\xi\rangle d\xi_k d\xi_j,\end{aligned} \tag{3.6}$$

where we took the inner product in space of equation (ISC-H) with E_{ij}^ξ, and used that

$$d\xi_j dt = b_j(\xi) dt dt + \langle \sigma_j(\xi), dW_\varepsilon\rangle dt = 0.$$

Therefore, by (3.6) it follows that

$$\langle E_i^\xi, d\tilde{v}\rangle = -\sum_j \langle \tilde{v}, E_{ij}^\xi\rangle d\xi_j - \tfrac{1}{2}\sum_{j,k} \langle \tilde{v}, E_{ijk}^\xi\rangle d\xi_j d\xi_k - \sum_j \langle dW_\varepsilon, E_{ij}^\xi\rangle d\xi_j + \sum_{j,k} \langle \tilde{u}_k^\xi, E_{ij}^\xi\rangle d\xi_j d\xi_k \ . \tag{3.7}$$

Combining (3.4) with (3.5) and (3.7) we arrive at

$$\begin{aligned}\sum_j \left[\langle \tilde{u}_j^\xi, E_i^\xi\rangle - \langle \tilde{v}, E_{ij}^\xi\rangle\right] d\xi_j =& \langle \mathcal{L}^c(\tilde{u}), E_i^\xi\rangle dt \\ &+ \sum_{l,k} \left[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\right] d\xi_l d\xi_k \\ &+ \sum_j \langle dW_\varepsilon, E_{ij}^\xi\rangle d\xi_j + \langle E_i^\xi, dW_\varepsilon\rangle \ .\end{aligned} \tag{3.8}$$

Lemma 3.1. *For all $1 \le k,\ l \le N$ it holds that*

$$\langle \sigma_k(\xi), dW_\varepsilon\rangle \langle \sigma_l(\xi), dW_\varepsilon\rangle = \langle \mathcal{Q}_\varepsilon \sigma_k(\xi), \sigma_l(\xi)\rangle dt.$$

Proof. Since $d\beta_j d\beta_i = \delta_{ij} dt$ and $W_\varepsilon(t) = \sum_{k=1}^\infty \alpha_{\varepsilon,k}\beta_k(t) e_k$ we obtain, using Parcevals identity,

$$\begin{aligned}\langle \sigma_k(\xi), dW_\varepsilon\rangle \langle \sigma_l(\xi), dW_\varepsilon\rangle &= \sum_{i,j} \alpha_{\varepsilon,i}\alpha_{\varepsilon,j} \langle \sigma_k(\xi), e_i\rangle \langle \sigma_l(\xi), e_j\rangle d\beta_j d\beta_i = \sum_j \alpha_{\varepsilon,j}^2 \langle \sigma_k(\xi), e_j\rangle \langle \sigma_l(\xi), e_j\rangle dt \\ &= \sum_j \langle \mathcal{Q}_\varepsilon \sigma_k(\xi), e_j\rangle \langle \sigma_l(\xi), e_j\rangle dt = \langle \mathcal{Q}_\varepsilon \sigma_k(\xi), \sigma_l(\xi)\rangle dt \ .\end{aligned}$$

□

Analogously to this Lemma we easily obtain (using $dtdW_\varepsilon = 0$)

$$\langle E_{ij}^\xi, dW_\varepsilon\rangle d\xi_j = \langle E_{ij}^\xi, dW_\varepsilon\rangle \langle \sigma_j(\xi), dW_\varepsilon\rangle = \langle \mathcal{Q}_\varepsilon E_{ij}^\xi, \sigma_j(\xi)\rangle dt.$$

Moreover, for short-hand notation, as in [8], we define the matrix $A(\xi) = (A_{ij}(\xi)) \in \mathbb{R}^{N\times N}$ by

$$A_{ij}(\xi) = \langle \tilde{u}_j^\xi, E_i^\xi\rangle - \langle \tilde{v}, E_{ij}^\xi\rangle \ , \tag{3.9}$$

which is invertible, provided that we are near the slow manifold (cf. Lemma 3.4 later). Let us denote the inverse matrix of A by $A^{-1}(\xi) = (A_{ij}^{-1}(\xi)) \in \mathbb{R}^{N\times N}$.

Therefore, for all $i \in \{1,\dots,N\}$ we arrive at

$$\begin{aligned}\sum_j A_{ij}(\xi)d\xi_j = \ & \langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle dt \\ & + \sum_{l,k}\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big]\langle \mathcal{Q}_\varepsilon\sigma_k(\xi), \sigma_l(\xi)\rangle dt \\ & + \sum_j \langle \mathcal{Q}_\varepsilon E_{ij}^\xi, \sigma_j(\xi)\rangle dt + \langle E_i^\xi, dW_\varepsilon\rangle \ .\end{aligned} \tag{3.10}$$

To obtain the equation for $d\xi$ we use that $d\xi = A(\xi)^{-1}A(\xi)d\xi$.

Thus, the final equation for ξ (as long as $\tilde{u}$ is near the manifold) is given for any $r = 1,\dots,N$ by

$$\begin{aligned}d\xi_r = \ & \sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle dt \\ & + \sum_{i,l,k} A_{ri}^{-1}(\xi)\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big]\langle \mathcal{Q}_\varepsilon\sigma_k(\xi), \sigma_l(\xi)\rangle dt \\ & + \sum_i A_{ri}^{-1}(\xi)\sum_j \langle \mathcal{Q}_\varepsilon E_{ij}^\xi, \sigma_j(\xi)\rangle dt + \sum_i A_{ri}^{-1}(\xi)\langle E_i^\xi, dW_\varepsilon\rangle \ .\end{aligned} \tag{3.11}$$

We can now recover σ and b from (3.11). The only term that does involve noise is the last one. Thus, in view of (3.2) we derive

$$\sigma_r(\xi) = \sum_i A_{ri}^{-1}(\xi)E_i^\xi \ . \tag{3.12}$$

After we obtained σ, we can proceed, in order to determine $b(\xi)$ from the remaining terms (cf. (3.2)). So, we get for $r = 1,\dots,N$ that

$$\begin{aligned}b_r(\xi) = & \sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle \\ & + \sum_{i,l,k} A_{ri}^{-1}(\xi)\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big]\langle \mathcal{Q}_\varepsilon\sigma_k(\xi), \sigma_l(\xi)\rangle \\ & + \sum_i A_{ri}^{-1}(\xi)\sum_j \langle \mathcal{Q}_\varepsilon E_{ij}^\xi, \sigma_j(\xi)\rangle.\end{aligned} \tag{3.13}$$

3.3. **Justification of the ansatz.** Let us first give an equation for $\tilde{v} = \tilde{u} - \tilde{u}^\xi$ describing the flow "orthogonal" to $\tilde{\mathcal{M}}$. Following [8] p. 449, we consider equation (3.3)

$$d\tilde{v} = d\tilde{u} - \sum_{j=1}^N \tilde{u}_j^\xi d\xi_j - \frac{1}{2}\sum_{kl}\tilde{u}_{kl}^\xi d\xi_k d\xi_l,$$

and thus the key equation for the distance from the manifold $\tilde{\mathcal{M}}$ is described by

$$d\tilde{v} = \mathcal{L}^c(\tilde{u})dt - \sum_j \tilde{u}_j^\xi b_j(\xi)dt - \sum_j \tilde{u}_j^\xi\langle \sigma_j(\xi), dW_\varepsilon\rangle - \frac{1}{2}\sum_{kl}\tilde{u}_{kl}^\xi\langle \mathcal{Q}_\varepsilon\sigma_k(\xi), \sigma_l(\xi)\rangle dt + dW_\varepsilon \ . \tag{3.14}$$

The following theorem is straightforwardly verified:

Theorem 3.2. *Consider the pair of functions $(\xi, \tilde{v})$ as local solutions of the system given by (3.14) and the ansatz (3.2) where σ and b are given by (3.12) and (3.13).*

As long as $\|\tilde{v}\| = \mathcal{O}(\varepsilon^{3/2})$ and $\xi(t) \in \Omega_\rho$, the function $\tilde{u} = \tilde{u}^\xi + \tilde{v}$ is well defined and solves (ISC-H) with $\langle \tilde{v}, E_j^\xi \rangle = 0$.

The orthogonality condition follows directly from (3.14) as the differential $d\langle v, E_j^\xi \rangle = 0$. The fact that $\tilde{u}$ is a solution follows from a lengthy calculation. Basically, one reverses the calculation of the previous section leading to (3.2).

3.4. **Stability and Attractivity of the manifold.** In this section, we prove the stability and discuss the attractivity of $\tilde{\mathcal{M}}$. Considering the stability, we show that with high probability (over a long time-scale) the solution stays close to $\tilde{\mathcal{M}}$, unless an interface breaks down.

In [8, Theorem B], Bates and Xun show that in the deterministic setting the slow manifold is exponentially attracting in a $\mathcal{O}(\varepsilon^{7/2})$-neighborhood in H^2, until the solution reaches an exponentially small neighborhood, where the motion of the solution along the manifold is exponentially slow. Using large deviation estimates, it is straightforward to verify for small noise, that the stochastic solution follows the deterministic one up to error terms of the order of the noise strength. Hence, the exponential attraction of $\tilde{\mathcal{M}}$ still holds for (ISC-H), until the solution reaches a neighborhood of the manifold that is determined by the strength of the noise.

Here, for simplicity of presentation we will focus only on the stability of $\tilde{\mathcal{M}}$. The proof can be modified, in order to show attraction, too. Once, we are in the slow channel around $\tilde{\mathcal{M}}$, we cannot exit with high probability for a long time-scale T_ϵ, unless one of the interfaces breaks down.

We define A_ε and B_ε as

$$A_\varepsilon(\tilde{v}) = \int_0^1 [\varepsilon^2 \tilde{v}_{xx}^2 + f'(u^\xi)\tilde{v}_x^2]dx \qquad \text{and} \qquad B_\varepsilon(\tilde{v}) = \int_0^1 [\varepsilon^2 \tilde{v}_{xx}^2 + \tilde{v}_x^2]dx. \tag{3.15}$$

Obviously, it holds that

$$\|\partial_x \tilde{v}\|_{L^2}^2 = \int_0^1 \tilde{v}_x^2 dx \leq \int_0^1 [\varepsilon^2 \tilde{v}_{xx}^2 + \tilde{v}_x^2]dx = B_\varepsilon(\tilde{v}) \ .$$

Observe also that even if the function $f'(u^\xi)$ appearing in the definition of A_ε changes sign, then provided ρ is small for $\tilde{v} \in C^2$ satisfying $\tilde{v} = 0$ at $x = 0, 1$ and $\langle \tilde{v}, E_j^\xi \rangle = 0$ for any $j = 1, \cdots, N$, there exists C independent of ε, $\tilde{v}$ such that

$$CA_\varepsilon(\tilde{v}) \geq \varepsilon^2 B_\varepsilon(\tilde{v}).$$

This estimate, which depends heavily on the properties of $\tilde{v}$, is established in [8] after an extensive analysis of the spectrum of the linearized integrated Cahn-Hilliard operator (see pg. 434-446, Lemmas 4.2, 3.2, 3.4). More specifically Bates and Xun proved that for ρ small the spectrum consists of exactly N exponentially small eigenvalues, while all the other eigenvalues are negative and bounded away from 0 uniformly in ε. Under weaker assumptions, such as $\tilde{v} \in H^2$, the same estimate follows, cf. Theorem A.1 of [9] at pg. 209-211. Further, since $f'(u^\xi)$ is bounded we get $A_\varepsilon(\tilde{v}) \leq cB_\varepsilon(\tilde{v})$, while by definition and for $\varepsilon < 1$ it follows that $B_\varepsilon(\tilde{v}) \leq \|\tilde{v}\|_{H^2}^2$. Hence, the next relation holds true

$$\|\partial_x \tilde{v}\|_{L^2}^2 \leq B_\varepsilon(\tilde{v}) \leq C\varepsilon^{-2} A_\varepsilon(\tilde{v}) \leq c\varepsilon^{-2} B_\varepsilon(\tilde{v}) \leq c\varepsilon^{-2} \|\tilde{v}\|_{H^2}^2.$$

In addition, by Lemma 4.1 of [8] at pg. 445, we have

$$\|\tilde{v}\|_\infty^2 \leq B_\varepsilon(\tilde{v}) \ , \qquad \qquad \|\tilde{v}_x\|_\infty^2 \leq \tfrac{1+\varepsilon}{\varepsilon} B_\varepsilon(\tilde{v}) \ . \tag{3.16}$$

Definition 3.3. *(cf.* [8]*, p. 452) Define a neighborhood Γ' of $\tilde{\mathcal{M}}$ by*

$$\Gamma' = \{\tilde{u}^\xi + \tilde{v} \ : \ \xi \in \Omega_\rho, \ B_\varepsilon(\tilde{v}) < \varepsilon^3\},$$

and we define the slow tube Γ by

$$\Gamma := \{\tilde{u}^\xi + \tilde{v} \ : \ \xi \in \Omega_\rho, \ A_\varepsilon(\tilde{v}) < \delta_\varepsilon^{2-\kappa}\} \ ,$$

where $0 < \kappa \ll 1$ presented in the definition of the noise (cf. Assumption 2.3) and δ_ε estimates the noise strength.

The small tube Γ' is a neighborhood of the slow manifold, where the coordinate system (cf. (1.7)) is well defined, while the slow tube Γ is a neighborhood in which with high probability solutions do not exit for long times unless one of the interfaces breaks down. Recall that $\Gamma \subset \Gamma'$ by definition of δ_ε. We even have $B_\varepsilon(\tilde{v}) < C\delta_\varepsilon^{2-\kappa}\varepsilon^{-2} \leq C\varepsilon^7$, which we need in the proof of stability.

As indicated in the introduction, the first term at the right-hand side of the flow given by (3.11) is identical to the right-hand side of the deterministic flow and has been estimated in [8]. In our stochastic case, in order to approximate the flow, we need to bound also the additional higher order terms and estimate the contribution of the noise. Later, in the next Section 4, we will identify the dominant terms in (3.11).

Using (4.27) of [9] and the fact that $\|E_{ij}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$ ([9] p. 187), we obtain in Γ' considering the matrix A the following invertibility result:

Lemma 3.4. *Suppose that $h \in \Omega_\rho$ and $\|\tilde{v}\| = \mathcal{O}(\varepsilon^{3/2})$, then*

$$A_{ij}(\xi) = \mathcal{O}(\varepsilon) + \begin{cases} (-1)^{i+j}4\ell_{j+1} & \text{if} \quad i \geq j \\ 0 & \text{if} \quad i < j \end{cases}$$

and the matrix is invertible, with

$$A_{ij}^{-1}(\xi) = \mathcal{O}(\varepsilon) + \begin{cases} \frac{1}{4\ell_{j+1}} & \text{if} \quad i = j, j-1 \\ 0 & \text{otherwise} \end{cases}$$

where $1 > \ell_i > \varepsilon/\rho$ denotes the length of the i-th interface.

As the equation is deterministically stable, we can show that $\tilde{v}$ stays small for a long time (depending on the noise strength). To be more precise, we show a bound on $A_\varepsilon(\tilde{v})$ for solutions near $\tilde{\mathcal{M}}$. Compare also (86) of [8]).

Fix some large time T_ε and define $\tau^* > 0$ as the first exit time (below the threshold T_ε) of $\tilde{u}$ from Γ'. This is the stopping time

$$\tau^* = T_\varepsilon \wedge \inf\{t > 0 : \xi(t) \notin \Omega_\rho \text{ or } A_\varepsilon(\tilde{v}(t)) \geq \delta_\varepsilon^{2-\kappa}\} .$$

Note that for $t \leq \tau^*$ also $B_\varepsilon(\tilde{v}(t)) \leq C\varepsilon^7$, as discussed above.

Definition 3.5. *We say that a term is $\mathcal{O}(e_\varepsilon)$, if it is asymptotically smaller than any polynomial in ε uniformly for times $t \leq \tau^*$.*

Note that α, β are $\mathcal{O}(e_\varepsilon)$, if $\rho = \varepsilon^\kappa$.

Theorem 3.6. *Suppose $\rho = \epsilon^\kappa$ for some small $\kappa > 0$, $\delta_\varepsilon \geq C\varepsilon^q$ for any $q > 0$, and suppose that for all $p > 0$ there exists a constant $c_p > 0$ such that $\mathbb{E}A_\varepsilon(\tilde{v}(0))^p \leq c_p\delta_\varepsilon^{2p}$. Then for all $p > 0$ there exists a constant $C_p > 0$ such that*

$$\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p \leq C_p(T_\epsilon + 1)\delta_\varepsilon^{2p} .$$

Therefore, we can show that the probability that the solution exits from the slow tube before T_ε (i.e. $\tau^* = T_\varepsilon$) or an interface is breaking down (i.e. $\xi(\tau^*) \notin \Omega_\rho$) is bounded above by

$$\mathbb{P}\left(A_\varepsilon(\tilde{v}(\tau^*)) \geq \delta_\varepsilon^{2-\kappa}\right) \leq \mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p\delta_\varepsilon^{-p(2-\kappa)} \leq C_p(T_\varepsilon + 1)\delta_\varepsilon^{\kappa p}$$

for any $p > 0$. Thus the probability that the solution exits from the slow tube before T_ε is of the order of $\mathcal{O}(e_\varepsilon)$ provided $T_\varepsilon \ll \delta_\varepsilon^{-q}$ for some large $q > 0$. The typical case for applications would be to consider a noise strength polynomial in ε, where we can take $T_\varepsilon = \varepsilon^{-q}$ for any $q > 0$.

Remark 3.7. (Exponentially small noise-strength δ_ε) *If we want to have exponentially long times T_ε, then we need to take exponentially small noise strength δ_ε and look closer at the various error terms in the proof of Theorem 3.6. This is straightforward, but for simplicity of presentation, we refrain from stating details here.*

On the other hand, assuming that δ_ε is exponentially small, the probability of the solution leaving the slow tube Γ before time T_ε, without an interface breaking down, is exponentially small, even for some exponentially large time T_ε.

3.5. **Bounds on the SDE.** The following Lemmas replace the bound on $\dot{\xi}$, used in the deterministic setting (cf. Lemma 4.3. in [8]).

Lemma 3.8. *Let $\tilde{u}^\xi + \tilde{v} \in \Gamma'$ and $r = 1, \dots, N$, then (with $E^\xi_{N+1} = 0$ for shorthand notation)*

$$\sigma_r(\xi) = \frac{1}{4\ell_{r+1}}(E^\xi_r + E^\xi_{r+1}) + \mathcal{O}(\varepsilon) \qquad \text{and} \qquad \|\sigma_r(\xi)\| \le C/\ell < C\rho/\varepsilon\ .$$

Proof. Note that $\|\tilde{v}\| \le B_\varepsilon(\tilde{v})^{1/2}$. Thus from the definition of σ (cf. (3.12)), Lemma 3.4, and the bound on E^ξ_i one obtains

$$\|\sigma_r(\xi)\| \le \sum_i |A^{-1}_{ri}(\xi)| \|E^\xi_i\| \le C/\ell\ .$$

Moreover,

$$\sigma_r(\xi) = A^{-1}_{r,r} E^\xi_r + A^{-1}_{r,r+1} E^\xi_{r+1} + \mathcal{O}(\varepsilon),$$

and the claim follows from Lemma 3.4. □

The next Lemma estimates the vector field b of the diffusion process ξ.

Lemma 3.9. *Let $\tilde{u}^\xi + \tilde{v} \in \Gamma'$ and assume that $\rho = \varepsilon^\kappa$ for some small $\kappa > 0$, then there is a constant $c > 0$ such that*

$$|b_r(\xi)| \le c\|\mathcal{Q}_\varepsilon\| \Big\{ \varepsilon^{3\kappa - 7/2} + \varepsilon^{2\kappa - 5/2} \Big\} + \mathcal{O}(e_\varepsilon), \tag{3.17}$$

for any $r = 1, \dots, N$.

Proof. We recall first b_r

$$\begin{aligned} b_r(\xi) = & \sum_i A^{-1}_{ri}(\xi) \langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E^\xi_i \rangle \\ & + \sum_{i,l,k} A^{-1}_{ri}(\xi) \Big[\tfrac{1}{2} \langle \tilde{v}, E^\xi_{ilk} \rangle - \tfrac{1}{2} \langle \tilde{u}^\xi_{kl}, E^\xi_i \rangle - \langle \tilde{u}^\xi_k, E^\xi_{il} \rangle \Big] \langle \mathcal{Q}_\varepsilon \sigma_k(\xi), \sigma_l(\xi) \rangle \\ & + \sum_i A^{-1}_{ri}(\xi) \sum_j \langle \mathcal{Q}_\varepsilon E^\xi_{ij}, \sigma_j(\xi) \rangle. \end{aligned} \tag{3.18}$$

Then we use Lemma 3.4 and the bound on σ. Moreover, in Section 5, after tedious computations the next estimates are derived (cf. (5.44), (5.45), (5.46), (5.41) and (5.42), respectively):

$$\begin{aligned} &|\langle \tilde{u}^\xi_{kl}, E^\xi_i \rangle| \le \mathcal{O}(\varepsilon^{-1/2}) \Big[4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta) \Big], \\ &|\langle \tilde{u}^\xi_k, E^\xi_{il} \rangle| \le \mathcal{O}(\varepsilon^{-1/2} + \varepsilon^{-4} r^{-1} \beta), \\ &|\langle \tilde{v}, E^\xi_{ilk} \rangle| \le \mathcal{O}(\varepsilon^{-3/2} + \varepsilon^{-5} r^{-1} \beta) \|\tilde{v}\| \le c + \mathcal{O}(\varepsilon^{-7/2} r^{-1} \beta)\ , \end{aligned}$$

since in the slow channel $\|\tilde{v}\| \le \|\tilde{v}\|_\infty \le c B_\varepsilon(\tilde{v})^{1/2} \le c\varepsilon^{3/2}$. Moreover,

$$\|E^\xi_i\| \le 4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta), \qquad \|E^\xi_{ij}\| \le \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4} r^{-1} \beta)\ .$$

In addition, we observe that (cf. [9])

$$\Big| \sum_i A^{-1}_{ri}(\xi) \langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E^\xi_i \rangle \Big| = \mathcal{O}(\alpha/\ell) + \mathcal{O}(\varepsilon\alpha) = \mathcal{O}(e_\varepsilon)\ .$$

In this way, since $\sigma = \mathcal{O}(\rho\varepsilon^{-1})$ and $A^{-1}_{ij} = \mathcal{O}(\rho\varepsilon^{-1})$, we obtain

$$|b_r(\xi)| \le c\|\mathcal{Q}_\varepsilon\| \rho^3 \varepsilon^{-3-1/2} + c\|\mathcal{Q}_\varepsilon\| \rho^2 \varepsilon^{-5/2} + \mathcal{O}(e_\varepsilon) \le c\|\mathcal{Q}_\varepsilon\| \Big\{ \varepsilon^{3\kappa - 7/2} + \varepsilon^{2\kappa - 5/2} \Big\} + \mathcal{O}(e_\varepsilon)\ .$$

□

3.6. **Proof of Stability.** Now let us turn to the proof of the Theorem 3.6. Considering the linearized C-H-operator and using Itô-formula we arrive at

$$dA_\varepsilon(\tilde v) = d\langle -L^c\tilde v,\tilde v\rangle = 2\langle -L^c\tilde v, d\tilde v\rangle + \langle -L^c d\tilde v, d\tilde v\rangle + dR\ , \tag{3.19}$$

with

$$dR = \int_0^1 \tilde v_x^2 f''(u^\xi)du^\xi\ dx + \int_0^1 \tilde v_x f''(u^\xi)\ d\tilde v_x\cdot du^\xi\ dx + \int_0^1 \tilde v_x^2 f'''(u^\xi)(du^\xi)^2\ dx\ .$$

All terms in R appear, because A_ε itself depends on ξ through $f'(u^\xi)$. Using Itô-formula and the equations (3.2) and (3.14) for ξ and $\tilde v$, we expand all terms

$$\begin{aligned} dR =& \sum_j \int_0^1 \tilde v_x^2 f''(u^\xi)u_j^\xi\ dx\ b_j\ dt + \sum_{i,j}\int_0^1 \tilde v_x^2 f''(u^\xi)u_{ij}^\xi\ dx\langle \mathcal{Q}_\varepsilon\sigma_j,\sigma_i\rangle dt + \sum_j\int_0^1 \tilde v_x^2 f''(u^\xi)u_j^\xi\ dx\langle\sigma_j, dW_\varepsilon\rangle \\ &+ \tfrac12\sum_{i,j}\int_0^1 \tilde v_x^2 f'''(u^\xi)u_j^\xi u_i^\xi\ dx\langle \mathcal{Q}_\varepsilon\sigma_j,\sigma_i\rangle dt + \sum_{i,j}\int_0^1 \tilde v_x f''(u^\xi)u_j^\xi u_i^\xi\ dx\langle \mathcal{Q}_\varepsilon\sigma_j,\sigma_i\rangle dt \\ &+ \sum_j\int_0^1 \tilde v_x f''(u^\xi)u_j^\xi\partial_x(\mathcal{Q}_\varepsilon\sigma_j)\ dx\ dt\ . \end{aligned}$$

Now we use Theorem 5.8 in ξ variables, to obtain that $\|u_j^\xi\|_\infty = \mathcal{O}(\varepsilon^{-1})$, $\|u_{ij}^\xi\|_\infty = \mathcal{O}(\varepsilon^{-2})$ and $\|u_j^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$. Moreover, by definition it holds that $\|\tilde v_x\|^2 \le B_\varepsilon(\tilde v)$, so using Lemmas 3.9 and 3.8 we have $b_j = \mathcal{O}(\delta_\varepsilon^2\varepsilon^{-7/2})$ and $\sigma_j = \mathcal{O}(\varepsilon^{-1})$. Finally, as u^ξ is uniformly bounded, we can bound the nonlinearity f by a constant and get

$$dR = \mathcal{O}(B_\varepsilon(\tilde v)\varepsilon^{-9/2}\delta_\varepsilon^2)dt + \mathcal{O}(B_\varepsilon(\tilde v)^{1/2}\varepsilon^{-7/2}\delta_\varepsilon^2)dt + \langle I_R, dW_\varepsilon\rangle.$$

with

$$I_R = \sum_j\int_0^1 \tilde v_x^2 f''(u^\xi)u_j^\xi\ dx\ \sigma_j = \mathcal{O}(B_\varepsilon(\tilde v)\varepsilon^{-2})$$

As we are in the slow channel, we obtain

$$dR = \mathcal{O}(\delta_\varepsilon^2)dt + \langle I_R, dW_\varepsilon\rangle. \tag{3.20}$$

This is the crucial and only point where we we need $B_\varepsilon(\tilde v) = \mathcal{O}(\varepsilon^7)$, in order to estimate the 5th term of R.

Now we turn to the other terms in (3.19). Lemma 3.1 gives

$$dA_\varepsilon(\tilde v) - dR = 2\langle -L^c\tilde v, \mathcal{L}^c(\tilde u)\rangle dt \tag{3.21}$$

$$- \sum_j 2\langle -L^c\tilde v, \tilde u_j^\xi\rangle b_j(\xi)dt \tag{3.22}$$

$$- \sum_j 2\langle -L^c\tilde v, \tilde u_j^\xi\rangle\langle\sigma_j(\xi), dW_\varepsilon\rangle$$

$$- \sum_{kl}\langle -L^c\tilde v, \tilde u_{kl}^\xi\rangle\langle \mathcal{Q}_\varepsilon\sigma_k(\xi),\sigma_l(\xi)\rangle dt \tag{3.23}$$

$$+ \sum_{ij}\langle -L^c\tilde u_i^\xi, \tilde u_j^\xi\rangle\langle \mathcal{Q}_\varepsilon\sigma_i(\xi),\sigma_j(\xi)\rangle dt \tag{3.24}$$

$$+ \sum_i\langle -L^c\tilde u_i^\xi, \mathcal{Q}_\varepsilon\sigma_i(\xi)\rangle dt \tag{3.25}$$

$$- 2\langle L^c\tilde v, dW_\varepsilon\rangle$$

$$+ \mathrm{trace}(\mathcal{Q}_\varepsilon^{1/2}L^c\mathcal{Q}_\varepsilon^{1/2})dt. \tag{3.26}$$

For the term in (3.21) we follow [8] pages 449/450, where

$$\mathcal{L}^c(\tilde{u}) = \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}) = L^c\tilde{v} + \mathcal{L}^c(\tilde{u}^\xi) + \partial_x(f_2\partial_x\tilde{v})$$

with

$$\|\partial_x(f_2\partial_x\tilde{v})\| \le C\varepsilon^{-2}B_\varepsilon(\tilde{v}).$$

Moreover, note that by Lemma 5.1 in [8] we have

$$\|\mathcal{L}^c(\tilde{u}^\xi)\|_\infty = \|\partial_x\mathcal{L}^b(u^\xi)\|_\infty \le C\varepsilon^{-1}\alpha(r)\ ,$$

and thus

$$\begin{aligned}\langle -L^c\tilde{v}, \mathcal{L}^c(\tilde{u})\rangle &\le -\|L^c\tilde{v}\|^2 + C(\varepsilon^{-2}B_\varepsilon(\tilde{v}) + \varepsilon^{-1}\alpha(r))\|L^c\tilde{v}\| \\ &\le -\tfrac{2}{3}\|L^c\tilde{v}\|^2 + C\varepsilon^{-2}B_\varepsilon(\tilde{v})\|L^c\tilde{v}\| + C\varepsilon^{-2}\alpha(r)^2 \\ &\le -\tfrac{1}{2}\|L^c\tilde{v}\|^2 + C\varepsilon^{-2}\alpha(r)^2,\end{aligned} \tag{3.27}$$

where we used that for some constant $a > 0$ independent of ε and r (cf. [8], Lemma 3.2 at p. 434, and Lemma 4.2 at p. 446)

$$B_\varepsilon(\tilde{v}) < C\varepsilon^{-2}A_\varepsilon(\tilde{v}) < \frac{C}{2a}\varepsilon^{-2}\|L^c\tilde{v}\|^2\ . \tag{3.28}$$

Using $B_\varepsilon(\tilde{v}) = \mathcal{O}(\varepsilon^{6+\kappa})$ in the slow channel, we obtain

$$2\langle -L^c\tilde{v}, \mathcal{L}^c(\tilde{u})\rangle \le -\tfrac{1}{2}\|L^c\tilde{v}\|^2 - aA_\varepsilon(\tilde{v}) + C\varepsilon^{-2}\alpha(r)^2.$$

Now consider the remaining four deterministic integrals. For the term in (3.22), notice that

$$\langle L^c\tilde{v}, \tilde{u}^\xi_j\rangle = \langle \tilde{v}, L^c\tilde{u}^\xi_j\rangle = \langle \tilde{v}, \partial_x\partial_j\mathcal{L}^b(u^\xi)\rangle\ .$$

Thus integration by parts and Lemma 5.2 of [8] yields

$$|\langle L^c\tilde{v}, \tilde{u}^\xi_j\rangle| \le C\|\partial_x\tilde{v}\|\varepsilon^{-2}\beta(r) = \mathcal{O}(e_\varepsilon). \tag{3.29}$$

We use now (3.29) to arrive at

$$|\sum_j\langle -L^c\tilde{v}, \tilde{u}^\xi_j\rangle b_j(\xi)| \le C\varepsilon^{-5/2}\beta(r)B_\varepsilon(\tilde{v})^{1/2}\sup_j\{|b_j(\xi)|\} = \mathcal{O}(e_\varepsilon), \tag{3.30}$$

which is exponentially small in ε by Lemma 3.9. By Definition 3.5, a term is $\mathcal{O}(e_\varepsilon)$, if it is asymptotically smaller than any polynomial in ε uniformly for times $t \le \tau^*$.

Now let us turn to (3.24). Similarly, we get

$$|\langle -L^c\tilde{u}^\xi_i, \tilde{u}^\xi_j\rangle| = |\langle \tilde{u}^\xi_i, \partial_x\partial_j\mathcal{L}^b(u^\xi)\rangle| \le \|\tilde{u}^\xi_i\|_{L^1}\|\partial_x\partial_j\mathcal{L}^b(u^\xi)\|_\infty \le C\varepsilon^{-4}\beta(r)\ ,$$

where we used Lemma 5.1 of [8] and the bound $\|\tilde{u}^\xi_i\|_{L^1} = \mathcal{O}(1)$ (cf. (5.38), for β bounded). Thus we obtain for the term in (3.24)

$$|\sum_{ij}\langle -L^c\tilde{u}^\xi_i, \tilde{u}^\xi_j\rangle\langle \mathcal{Q}_\varepsilon\sigma_i(\xi), \sigma_j(\xi)\rangle| \le C\varepsilon^{-4}\beta(r)\|\mathcal{Q}_\varepsilon\|\ell^{-2} = \mathcal{O}(e_\varepsilon)\ . \tag{3.31}$$

For the term in (3.23) we use the bounds on $\langle -L^c\tilde{v}, \tilde{u}^\xi_{kl}\rangle$ provided by Theorem 5.47. Thus, we get

$$|\langle L^c\tilde{v}, \tilde{u}^\xi_{kl}\rangle\langle \mathcal{Q}_\varepsilon\sigma_k(\xi), \sigma_l(\xi)\rangle| \le C\|\mathcal{Q}_\varepsilon\|\varepsilon^{-2}C\varepsilon^{-2}\beta(r)\|\tilde{v}\| = \mathcal{O}(e_\varepsilon)\ .$$

Using similar estimates and Lemma 3.8 the term in (3.25) is also $\mathcal{O}(e_\varepsilon)$.

For the term in (3.26), we use the eigenfunctions e_k of $\mathcal{Q}_\varepsilon$ and the uniform bound on $f'(u^\xi)$, in order to obtain

$$\text{trace}(\mathcal{Q}_\varepsilon^{1/2}L^c\mathcal{Q}_\varepsilon^{1/2}) = \sum_{k=1}^\infty \alpha^2_{\varepsilon,k}\langle L^c e_k, e_k\rangle \le C\sum_{k=1}^\infty \alpha^2_{\varepsilon,k}B_\varepsilon(e_k) \le C\delta_\varepsilon^2\ .$$

This is the largest deterministic term, as the other ones are all $\mathcal{O}(e_\varepsilon)$. This term comes directly from the Itô-correction of the additive noise.

Consider now Equations (3.21) - (3.26), with all deterministic integrals already estimated and include the bound on R from (3.20). For $t \leq \tau^*$

$$dA_\varepsilon(\tilde{v}(t)) \leq C\delta_\varepsilon^2 dt - (\tfrac{1}{2}\|L^c\tilde{v}\|^2 + aA_\varepsilon(\tilde{v}))dt + < I, dW_\varepsilon > , \tag{3.32}$$

where

$$I = \sum_j 2\langle -L^c\tilde{v}, \tilde{u}_j^\xi\rangle \sigma_j(\xi) - 2L^c\tilde{v} + I_R \ ,$$

with $I_R = \mathcal{O}(B_\varepsilon(\tilde{v})\varepsilon^{-3/2})$.

In order to bound I, we use (3.29), and the asymptotic formula for $\sigma_j(\xi)$ of Lemma 3.8 combined with (54)-(55) of [8] to obtain that $\langle L^c\tilde{v}, \tilde{u}_j^\xi\rangle\sigma_j(\xi) = \mathcal{O}(e_\varepsilon)$ and thus

$$|\langle I, \mathcal{Q}_\varepsilon I\rangle| \leq \mathcal{O}(e_\varepsilon) + C\|\mathcal{Q}_\varepsilon\|(\|L^c\tilde{v}\|^2 + B_\varepsilon(\tilde{v})^2\varepsilon^{-4}) \ .$$

Now from (3.28) as in the slow channel at least $B_\varepsilon(\tilde{v}) = \mathcal{O}(\varepsilon^6)$ we obtain $B_\varepsilon(\tilde{v})^2\varepsilon^{-3} \leq C\|L^c\tilde{v}\|^2 B_\varepsilon(\tilde{v})\varepsilon^{-6} \leq C\|L^c\tilde{v}\|^2$ and thus

$$|\langle I, \mathcal{Q}_\varepsilon I\rangle| \leq \mathcal{O}(e_\varepsilon) + C\|\mathcal{Q}_\varepsilon\|\|L^c\tilde{v}\|^2 \ .$$

Now we can bound powers of A_ε for $t \leq \tau^*$

$$\begin{aligned} \tfrac{1}{p} dA_\varepsilon(\tilde{v})^p &= A_\varepsilon(\tilde{v})^{p-1} dA_\varepsilon(\tilde{v}) + \tfrac{p-1}{2} A_\varepsilon(\tilde{v})^{p-2}(dA_\varepsilon(\tilde{v}))^2 \\ &\leq C\varepsilon^{2\delta_s} A_\varepsilon(\tilde{v})^{p-1}dt - (\tfrac{1}{2}\|L^c\tilde{v}\|^2 + aA_\varepsilon(\tilde{v}))A_\varepsilon(\tilde{v})^{p-1}dt \\ &\quad + A_\varepsilon(\tilde{v})^{p-1}\langle I, dW_\varepsilon\rangle + \tfrac{p-1}{2} A_\varepsilon(\tilde{v})^{p-2}\langle I, \mathcal{Q}_\varepsilon I\rangle dt \ . \end{aligned} \tag{3.33}$$

Taking integrals up to τ^* and expectation, we easily obtain from (3.32) and (3.33) (using that the expectation of a stochastic integral is 0)

$$\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*)) + \tfrac{1}{2}\mathbb{E}\int_0^{\tau^*} \|L^c\tilde{v}\|^2 dt + a\mathbb{E}\int_0^{\tau^*} A_\varepsilon(\tilde{v})dt \leq A_\varepsilon(\tilde{v}(0)) + CT_\varepsilon\delta_\varepsilon^2 \ ,$$

and for $p \geq 2$

$$\begin{aligned} &\tfrac{1}{p}\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p + \tfrac{1}{2}\mathbb{E}\int_0^{\tau^*} \|L^c\tilde{v}\|^2 A_\varepsilon(\tilde{v})^{p-1}dt + a\mathbb{E}\int_0^{\tau^*} A_\varepsilon(\tilde{v})^p dt \\ &\leq \tfrac{1}{p}\mathbb{E}A_\varepsilon(\tilde{v}(0))^p + C\delta_\varepsilon^2\mathbb{E}\int_0^{\tau^*} A_\varepsilon(\tilde{v})^{p-1}dt + \mathcal{O}(e_\varepsilon)\cdot\mathbb{E}\int_0^{\tau^*} A_\varepsilon(\tilde{v})^{p-2}dt + C\|\mathcal{Q}_\varepsilon\|\cdot\mathbb{E}\int_0^{\tau^*} A_\varepsilon(\tilde{v})^{p-2}\|L^c\tilde{v}\|^2 dt \ . \end{aligned}$$

Now (using $\delta_\varepsilon \geq C\varepsilon^q$) it is easy to verify by induction on p that

$$\tfrac{1}{p}\mathbb{E}A_\varepsilon(\tilde{v}(\tau^*))^p + \tfrac{1}{2}\mathbb{E}\int_0^{\tau^*} \|L^c\tilde{v}\|^2 A_\varepsilon(\tilde{v})^{p-1}dt + a\mathbb{E}\int_0^{\tau^*} A_\varepsilon(\tilde{v})^p dt \leq C(T_\varepsilon + 1)\delta_\varepsilon^{2p} \ .$$

This implies the claim.

4. Motion of the interfaces

In this section, we investigate some important special cases in detail to see what the SDE (2.2) for ξ actually implies for the motion of the interfaces. We assume first that the noise is exponentially small. Then considering the two interfaces problem (i.e. when $N = 1$) we discuss the case of noise strength being polynomial in ε. Although not covered by our theorems, we present some comments on how the equation would look like for non-smooth space-time white noise, which means that $\mathcal{Q}_\varepsilon$ is proportional to the identity. Finally, we present the approximate SDE system for the front motion considering the general precise system (2.1).

Let us first state the result we achieved so far. The motion of the interfaces for the stochastic model is given by the following theorem.

Theorem 4.1. *Let $\tilde{u}^\xi + \tilde{v} \in \Gamma$ and assume that ρ is small, then the equations dominating the flow of the Stochastic Cahn-Hilliard equation within the slow channel are given by*

$$
\begin{aligned}
d\xi_1 &= \frac{1}{4\ell_2}(\alpha^3 - \alpha^1)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(1)} \\
d\xi_2 &= \frac{1}{4\ell_2}(\alpha^3 - \alpha^1)dt + \frac{1}{4\ell_3}(\alpha^4 - \alpha^2)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(2)} \\
d\xi_3 &= \frac{1}{4\ell_3}(\alpha^4 - \alpha^2)dt + \frac{1}{4\ell_4}(\alpha^5 - \alpha^3)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(3)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
d\xi_N &= \frac{1}{4\ell_N}(\alpha^{N+1} - \alpha^{N-1})dt + \frac{1}{4\ell_{N+1}}(\alpha^{N+2} - \alpha^N)dt + \mathcal{O}(\varepsilon\alpha)dt + d\mathcal{A}_s^{(N)},
\end{aligned}
\tag{4.1}
$$

where

$$
\alpha^j = \frac{1}{2}K_\pm^2 A_\pm^2 \exp(-A_\pm \ell_j/\varepsilon)\Big[1 + \mathcal{O}\Big(\frac{\ell_j}{\varepsilon}\exp\Big(\frac{-A_\pm \ell_j}{2\varepsilon}\Big)\Big)\Big] \quad j = 1, 2, \ldots, N+2,
\tag{4.2}
$$

for

$$
A_\pm := f'(\pm 1) \quad \text{and} \quad K_\pm := 2\exp\Big[\int_0^1 \Big[\frac{A_\pm}{2F(\pm t)^{1/2}} - \frac{1}{1-t}\Big]dt\Big].
\tag{4.3}
$$

Here, the stochastic processes $\mathcal{A}_s^{(r)}$, $r = 1, \ldots, N$ are related to the noise; they depend on the symmetric operator $\mathcal{Q}_\varepsilon$ and the variance σ, and are given by the formula

$$
\begin{aligned}
d\mathcal{A}_s^{(r)} :=& \sum_{i,l,k} A_{ri}^{-1}(\xi)\Big[\tfrac{1}{2}\langle \tilde{v}, E_{ilk}^\xi\rangle - \tfrac{1}{2}\langle \tilde{u}_{kl}^\xi, E_i^\xi\rangle - \langle \tilde{u}_k^\xi, E_{il}^\xi\rangle\Big]\langle \mathcal{Q}_\varepsilon \sigma_k(\xi), \sigma_l(\xi)\rangle dt \\
&+ \sum_i A_{ri}^{-1}(\xi)\sum_j \langle \mathcal{Q}_\varepsilon E_{ij}^\xi, \sigma_j(\xi)\rangle dt + \sum_i A_{ri}^{-1}(\xi)\langle E_i^\xi, dW_\varepsilon\rangle\ .
\end{aligned}
\tag{4.4}
$$

The quantities $K_\pm$ are constants introduced by Carr and Pego in [22].

Proof. Recall that as long as $\tilde{u}$ is near the manifold, then by (3.11) we obtained for any $r = 1, \ldots, N$

$$
d\xi_r = \sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle dt + d\mathcal{A}_s^{(r)}.
$$

Lemma 3.4 gives that the matrix A^{-1} and therefore the terms $\sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}), E_i^\xi\rangle$ are identical to those presented in [8, 9] for the deterministic case (i.e. when $d\mathcal{A}_s^{(r)} = 0$ for any r). Hence, using (4.32) of [9] we obtain the result. □

Remark 4.2. *Note that using the relation $\ell_j = h_j - h_{j-1}$ and the asymptotic formula for $\frac{\partial h_{N+1}}{\partial h_j}$ we can derive an analogous system in h_j or in ℓ_j (cf.* [9]*).*

Remark 4.3. *In view of the assumptions of Theorem 4.1, and as mentioned throughout our analysis, it is sufficient that $\rho = \varepsilon^\kappa$ for some small $\kappa > 0$. In this case α^j are exponentially small. So, by stability, for a sufficiently bounded noise strength, the distance $\|\tilde{v}\|$ will remain small and thus the matrix A^{-1} will remain well defined.*

We observe

$$
d\mathcal{A}_s^{(r)} := \mathcal{A}_{\mathcal{Q}}^{(r)}dt + \sum_i A_{ri}^{-1}(\xi)\langle E_i^\xi, dW_\varepsilon\rangle\ ,
\tag{4.5}
$$

for

$$\mathcal{A}_{\mathcal{Q}}^{(r)} := \sum_{i,l,k} A_{ri}^{-1}(\xi)\Big[\tfrac{1}{2}\langle\tilde{v}, E_{ilk}^{\xi}\rangle - \tfrac{1}{2}\langle\tilde{u}_{kl}^{\xi}, E_{i}^{\xi}\rangle - \langle\tilde{u}_{k}^{\xi}, E_{il}^{\xi}\rangle\Big]\langle\mathcal{Q}_{\varepsilon}\sigma_k(\xi), \sigma_l(\xi)\rangle + \sum_{i} A_{ri}^{-1}(\xi)\sum_{j}\langle\mathcal{Q}_{\varepsilon}E_{ij}^{\xi}, \sigma_j(\xi)\rangle \tag{4.6}$$

Following Lemma 3.9 we obtain in the slow channel

$$|\mathcal{A}_{\mathcal{Q}}^{(r)}| \le c\|\mathcal{Q}_{\varepsilon}\|\rho^2(\rho\varepsilon^{-3-1/2} + \varepsilon^{-5/2}), \quad \text{for all } r = 1, \dots, N\ . \tag{4.7}$$

Thus, in case of $\|\mathcal{Q}_{\varepsilon}\| = \mathcal{O}(\varepsilon^{4+1/2}\alpha)$, since ρ is at least bounded, we can show that $\mathcal{A}_{\mathcal{Q}}^{(r)} = \mathcal{O}(\varepsilon\alpha)$. It is not hard to show that we can also neglect the stochastic term from (4.1), in order to recover the result of Bates & Xun on metastable slow motion, at least with high probability.

An interesting case arises, when the additional terms in $\mathcal{A}_{s}^{(r)}$ are of the order of $\mathcal{O}(\alpha)$. Then we obtain additional terms in (4.1). Nevertheless, for simplicity of presentation, we refrain from stating details here. Obviously, for a polynomial noise strength the extra drift $\mathcal{A}_{\mathcal{Q}}^{(r)}dt$ coming from stochastic dynamics would dominate the exponentially small terms involving α^j and α.

4.1. **Polynomial noise strength.** For the remainder of this section we fix $N = 1$, which is the case of two interfaces, and a noise strength $\delta_{\varepsilon} = \varepsilon^{\delta}$ for some $\delta > 9/2$. To be more precise suppose $\mathcal{Q}_{\varepsilon} = \mathcal{Q}_0\varepsilon^{\delta}$ with $\mathcal{Q}_0 = \mathcal{O}(1)$.

Using (4.1), we notice that the equation of motion for the first interface is given by

$$d\xi_1 = \mathcal{O}(\alpha)dt + d\mathcal{A}_{s}^{(1)}\ ,$$

and the motion of the second interface is fixed due to mass conservation.

Recall that ℓ_2 is the distance between the two interfaces, and fix $\rho = \varepsilon^{\kappa}$, which means that the lower bound on ℓ_2 is $\varepsilon^{1-\kappa}$. Let us now first look at (3.12)

$$\sigma_1(\xi) = A_{11}^{-1}E_1^{\xi}.$$

Since $\tilde{u}_1^{\xi} = \tilde{u}_2^{h}\frac{\partial h_2}{\partial h_1} + \tilde{u}_1^{h}$ while $\frac{\partial h_2}{\partial h_1} = 1 + \mathcal{O}(e_{\varepsilon})$ and $E_1^{\xi} = \tilde{u}_1^{h} + \tilde{u}_2^{h} + \mathcal{O}(e_{\varepsilon})$, it follows that

$$E_1^{\xi} = \tilde{u}_1^{\xi} + \mathcal{O}(e_{\varepsilon})\ ,$$

and again the error term remains of the same order under differentiation w.r.t. ξ_1. Secondly, from (4.24) in [9] there is a constant $c_{\star}$ such that $\|\tilde{u}_1^{\xi}\|^2 = 4\ell_2 + c_{\star}\varepsilon + \mathcal{O}(e_{\varepsilon})$, and the error term remains $\mathcal{O}(e_{\varepsilon})$ under differentiation. (In our case $N = 1$ we have that $\tilde{w}_1$ used in [9] is up to errors of order $\mathcal{O}(e_{\varepsilon})$ equal to $\tilde{u}_1^{\xi}$.). Moreover, by definition

$$A_{11} = \langle\tilde{u}_1^{\xi}, E_1^{\xi}\rangle - \langle\tilde{v}, E_{11}^{\xi}\rangle = \|\tilde{u}_1^{\xi}\|^2 + \|\tilde{v}\|_{\infty}\mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(e_{\varepsilon})\ , \tag{4.8}$$

where we used (5.42) (cf. also [9], where the same estimate is used, though never presented analytically) for $E_{11}^{\xi} = \mathcal{O}(\varepsilon^{-1/2})$. Recall that in the slow channel Γ

$$\|v\|_{\infty} \le (B_{\varepsilon}(v))^{1/2} \le C\varepsilon^{-1}(A_{\varepsilon}(v))^{1/2} \le C\varepsilon^{-1}(\delta_{\varepsilon}^{2-\kappa})^{1/2} \le C\varepsilon^{-1+\delta(1-\kappa/2)}\ . \tag{4.9}$$

Thus we proved

$$A_{11} = 4\ell_2 + c_{\star}\varepsilon + \mathcal{O}(\varepsilon^{\delta(1-\kappa/2)-\frac{3}{2}}) \quad \text{and} \quad \sigma_1(\xi) = \frac{1}{4\ell_2 + c_{\star}\varepsilon + \mathcal{O}(\varepsilon^{\delta(1-\kappa/2)-\frac{3}{2}})}E_1^{\xi} + \mathcal{O}(e_{\varepsilon})\ . \tag{4.10}$$

Now we can consider the deterministic drift

$$\begin{aligned}
\mathcal{A}_{\mathcal{Q}}^{(1)} &= A_{11}^{-1}(\xi)\Big[\tfrac{1}{2}\langle\tilde{v}, E_{111}^{\xi}\rangle - \tfrac{1}{2}\langle\tilde{u}_{11}^{\xi}, E_1^{\xi}\rangle - \langle\tilde{u}_1^{\xi}, E_{11}^{\xi}\rangle\Big]\langle\mathcal{Q}_{\varepsilon}\sigma_1(\xi), \sigma_1(\xi)\rangle + A_{11}^{-1}(\xi)\langle\mathcal{Q}_{\varepsilon}E_{11}^{\xi}, \sigma_j(\xi)\rangle \\
&= A_{11}^{-3}\Big[\mathcal{O}(\varepsilon^{-3/2})\|\tilde{v}\| - \tfrac{3}{4}\tfrac{\partial}{\partial\xi_1}\|E_1^{\xi}\|^2\Big]\|\mathcal{Q}_{\varepsilon}^{1/2}E_1^{\xi}\|^2 + A_{11}^{-2}\tfrac{1}{2}\tfrac{\partial}{\partial\xi_1}\|\mathcal{Q}_{\varepsilon}^{1/2}E_1^{\xi}\|^2 + \mathcal{O}(e_{\varepsilon}).
\end{aligned}$$

Thus, in the slow channel Γ (cf. (4.9)) the equation of motion for the interface is reduced to

$$\begin{aligned} d\xi_1 =& A_{11}^{-3}\mathcal{O}(\varepsilon^{\delta(1-\kappa/2)-5/2})\|\mathcal{Q}_\varepsilon^{1/2}E_1^\xi\|^2 dt - \tfrac{3}{4}A_{11}^{-3}\left(\tfrac{\partial}{\partial\xi_1}\|E_1^\xi\|^2\right)\|\mathcal{Q}_\varepsilon^{1/2}E_1^\xi\|^2 dt \\ &+ A_{11}^{-2}\tfrac{1}{2}\tfrac{\partial}{\partial\xi_1}\|\mathcal{Q}_\varepsilon^{1/2}E_1^\xi\|^2 dt + A_{11}^{-1}\langle E_1^\xi, dW_\varepsilon\rangle + \mathcal{O}(e_\varepsilon)dt\,. \end{aligned}$$

By (45) of [8] we know that

$$\tilde{u}_1^\xi = 1 - u^\xi + \mathcal{O}(e_\varepsilon) \quad \text{and} \quad u_1^\xi = -u_x^\xi + \mathcal{O}(e_\varepsilon)\,,$$

(as $[0,1] = I_1 \cup I_2$ and $u^\xi(m_1) = u^\xi(0) = -1 + \mathcal{O}(e_\varepsilon)$). Furthermore, the error terms remain $\mathcal{O}(e_\varepsilon)$, under differentiation with respect to ξ. Thus, we obtain

$$\|\tilde{u}_1^\xi\|^2 = \|1 - u^\xi\|^2 + \mathcal{O}(e_\varepsilon) = 1 - 2M + \|u^\xi\|^2 + \mathcal{O}(e_\varepsilon).$$

Differentiation yields

$$\tfrac{\partial}{\partial\xi_1}\|\tilde{u}_1^\xi\|^2 = 2\langle u_1^\xi, u^\xi\rangle + \mathcal{O}(e_\varepsilon) = -2\langle u_x^\xi, u^\xi\rangle + \mathcal{O}(e_\varepsilon) = u^\xi(0)^2 - u^\xi(1)^2 + \mathcal{O}(e_\varepsilon) = \mathcal{O}(e_\varepsilon)\,.$$

Thus we verified that

$$\tfrac{\partial}{\partial\xi_1}\|E_1^\xi\|^2 = \mathcal{O}(e_\varepsilon).$$

Therefore, the equation of motion for ξ simplifies to:

$$d\xi_1 = \mathcal{O}(\varepsilon^{\delta(3-\kappa/2)-11/2})dt + A_{11}^{-2}\tfrac{1}{2}\tfrac{\partial}{\partial\xi_1}\|\mathcal{Q}_\varepsilon^{1/2}E_1^\xi\|^2 dt + A_{11}^{-1}\langle E_1^\xi, dW_\varepsilon\rangle\,. \tag{4.11}$$

Remark 4.4. *Let us comment in more detail, what this formula implies for the motion of the interface. First, A_{11} is approximately the constant $4\ell_2$ with very small derivatives. Moreover, from (4.8) we see that $A_{11}^{-1}E_1^\xi$ is a normalized tangent vector at $\tilde{\mathcal{M}}$. So the deterministic drift in (4.11) is an Itô-Stratonovic correction and the motion of ξ is approximately the Wiener-process W_ε projected onto $\tilde{\mathcal{M}}$.*

Although this is not covered by our assumptions, as a final example we consider a space-time white noise with $\mathcal{Q}_\varepsilon = \varepsilon^\delta Id$. In this case

$$d\xi = \mathcal{O}(\varepsilon^{3\delta-7/2})dt + \varepsilon^\delta A_{11}^{-1}\langle E_1^\xi, d\hat{W}_\varepsilon\rangle,$$

which is a rescaled equation valid on the timescale $\mathcal{O}(\varepsilon^{-\delta})$. Up to the small deterministic error terms, ξ is a stochastic process with mean zero and linear quadratic variation. More specifically,

$$\begin{aligned} \int_0^t \varepsilon^{2\delta}A_{11}^{-2}\langle E_1^\xi, E_1^\xi\rangle dt &= \varepsilon^{2\delta}\int_0^t A_{11}^{-2}\|\tilde{u}_1^\xi\|^2 dt + \mathcal{O}(e_\varepsilon)t \\ &= \varepsilon^{2\delta}\int_0^t A_{11}^{-1}dt + \mathcal{O}(\varepsilon^{\delta-3/2+\kappa})\frac{t}{\ell_2^2} = \frac{\varepsilon^{2\delta}}{4\ell_2}t + \mathcal{O}(\varepsilon^{2\delta+1})t + \mathcal{O}(\varepsilon^{3\delta-7/2+\kappa})t\,. \end{aligned}$$

Recalling Levy's characterization of Brownian motion, in first approximation for times not too large the interface behaves similar to a Brownian motion with variance $\varepsilon^{2\delta}/(4\ell_2)$.

4.2. Conclusions.

Let us summarize the results of our analysis:

(1) There exists a slow tube Γ (around the slow manifold Γ') where the coordinate system (cf. (1.7)) is well defined and from which solutions with high probability do not exit for long times T_ε unless one of the interfaces breaks down (stochastic stability).

More specifically, according to Theorem 3.6 this probability is bounded below by

$$1 - C_p(T_\varepsilon + 1)\delta_\varepsilon^{\kappa p}$$

for any $p > 0$ where δ_ε measures the noise strength (less than $\varepsilon^{9/2}$). So if the noise is exponentially small then this probability is large for exponentially long time, while in case the noise being polynomially small the probability is large for any polynomially long time.

(2) In Γ the approximate SDE of front motion for the stochastic C-H is given by (4.1). Further, the extra stochastic terms from co-relations of the interfaces motions are important since the deterministic dynamics are exponentially small.

5. Higher order estimates

5.1. **Preliminaries.** This section deals with the estimation of all the following higher order terms that appear due to stochastic integration when deriving the equations of motion in the slow channel:

$$\langle \tilde{v}, E^{\xi}_{ilk} \rangle \,, \qquad \langle \tilde{u}^{\xi}_{kl}, E^{\xi}_{i} \rangle \qquad \text{and} \quad \langle \tilde{u}^{\xi}_{k}, E^{\xi}_{il} \rangle.$$

In addition, we bound the quantity $\langle L^c\tilde{v}, \tilde{u}^{\xi}_{kl}\rangle$, where for a general smooth in space function ϕ the operator L^c is defined by

$$L^c\phi := -\varepsilon^2\phi_{xxxx} + (f'(u^h)\phi_x)_x.$$

In order to achieve rigorous estimates for all these terms, we investigate the properties of the stationary problem (1.3). Our analysis admits extensive calculations and is based on the ideas and techniques presented in [8, 9, 22, 23] for the deterministic case where analogous terms of lower order have been estimated already.

Note first, that for the construction of the approximate manifold of solutions for the stochastic Cahn-Hilliard equation we use a local coordinate system when presenting the admissible interface positions. The h_{N+1} variable depends on $h_i = \xi_i$, $i = 1, \dots, N$, therefore, when differentiating two times in ξ variables and applying the chain rule the second order term $\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j}$ appears. More specifically, for a general function f smooth in space and any $i, j = 1, \dots, N$, we obtain

$$\begin{aligned}
&\frac{\partial f}{\partial \xi_i} = \frac{\partial f}{\partial h_i} + \frac{\partial f}{\partial h_{N+1}}\frac{\partial h_{N+1}}{\partial h_i}, \quad \text{and}\\
&\frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 f}{\partial h_i \partial h_j} + \Big(\frac{\partial^2 f}{\partial h_{N+1}\partial h_j} + \frac{\partial^2 f}{\partial h^2_{N+1}}\frac{\partial h_{N+1}}{\partial h_j}\Big)\frac{\partial h_{N+1}}{\partial h_i}\\
&+ \frac{\partial f}{\partial h_{N+1}}\Big(\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j} + \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_{N+1}}\frac{\partial h_{N+1}}{\partial h_j}\Big).
\end{aligned} \tag{5.1}$$

By the next lemma considering $\rho = \varepsilon^\kappa$ for some small $\kappa > 0$ and thus α, β are exponentially small, we estimate $\left|\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j}\right|$. As in [8], where the analogous first order estimate has been derived, we use an implicit function theorem argument combined with the mass conservation constraint. If u^h is in the second approximate manifold $\mathcal{M}$ then, by definition, mass conservation holds i.e.

$$M = M(h) = \int_0^1 u^h(x)dx.$$

Differentiating twice with respect to h variables, we get

$$\frac{d^2}{dh_i dh_j}M(h) = \int_0^1 u^h_{ij}dx,$$

where $u^h_{ij} := \frac{\partial^2 u^h}{\partial h_i \partial h_j} = \frac{\partial u^h_i}{\partial h_j}$.

Lemma 5.1. *For any $i, j = 1, \dots, N$ the next inequality follows*

$$\Big|\frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j}\Big| \leq \mathcal{O}(e_\varepsilon).$$

Proof. Let ℓ be a generic positive variable. According to the analysis presented in [22], when comparing the x and ℓ derivatives of the solution ϕ of the stationary problem (1.3), we obtain a residual function w given by the following relation

$$2\phi_\ell(x, \ell, \pm1) = -(\operatorname{sgn} x)\phi_x(x, \ell, \pm1) + 2w(x, \ell, \pm1). \tag{5.2}$$

Let us define $I_j := [m_j, m_{j+1}]$, $\chi^j(x) := \chi\Big(\frac{x-h_j}{\varepsilon}\Big)$. If $w^j(x) := w(x - m_j, h_j - h_{j-1}, (-1)^j)$, then the interval $[h_{j-1} - \varepsilon, h_{j+1} + \varepsilon]$ contains the support of u^h_j and

$$u^h_j(x) = \begin{cases} \chi^{j-1} w^j & \text{for } x \in I_{j-1} \\ (1-\chi^j)(-\phi^j_x + w^j) + \chi^j(-\phi^{j+1}_x - w^{j+1}) + \chi^j_x(\phi^j - \phi^{j+1}) & \text{for } x \in I_j \\ -(1-\chi^{j+1})w^{j+1} & \text{for } x \in I_{j+1} \end{cases} \tag{5.3}$$

where $\chi^j_x = \partial_x\Big(\chi\Big(\frac{x-h_j}{\varepsilon}\Big)\Big)$ and $\phi^j_x = \phi_x(x - m_j, l_j - l_{j-1}, (-1)^j)$ (cf. [22], p. 561). We note that in I_j (cf. [8] p. 430)

$$u^h_j = -u^h_x + (1-\chi^j)w^j - \chi^j w^{j+1}$$

and thus

$$\begin{aligned} u^h_{ji} = &- \frac{\partial u^h_x}{\partial h_i} + (-\delta_{j,i}\chi^j_x)w^j + (1-\chi^j)(A_{j,i}w^j_x + B_{j,i}w^j_\ell) \\ &- \delta_{j,i}\chi^j_x w^{j+1} - \chi^j(A_{j+1,i}w^{j+1}_x + B_{j+1,i}w^{j+1}_\ell), \text{ in } I_j \end{aligned} \tag{5.4}$$

where $w^j_x = w_x(x - m_j, l_j - l_{j-1}, (-1)^j)$ and $w^j_\ell = w_l(x - m_j, l_j - l_{j-1}, (-1)^j)$, with $\delta_{j,i}$ being the Kronecker delta. Moreover,

$$A_{j,i} := \frac{\partial(x - m_j)}{\partial h_i} = \begin{cases} 0 & \text{for } i \neq j, j-1 \\ -1/2 & \text{for } i = j, j-1 \end{cases}$$

and

$$B_{j,i} := \frac{\partial(h_j - h_{j-1})}{\partial h_i} = \begin{cases} 0 & \text{for} i \neq j, j-1 \\ 1 & \text{for } i = j \\ -1 & \text{for } i = j-1\,. \end{cases}$$

In a similar way we obtain

$$u^h_{ji} = \delta_{j-1,i}\chi^{j-1}_x w^j + \chi^{j-1}(A_{j,i}w^j_x + B_{j,i}w^j_\ell), \text{ in } I_{j-1}, \tag{5.5}$$

and

$$u^h_{ji} = \delta_{j+1,i}\chi^{j+1}_x w^{j+1} - (1-\chi^{j+1})(A_{j+1,i}w^{j+1}_x + B_{j+1,i}w^{j+1}_\ell), \text{ in } I_{j+1}. \tag{5.6}$$

Using now the bounds on w^j, w^j_x, and w^j_ℓ (cf. [22], or [8] at p.172), we obtain for $r > 0$ sufficiently small

$$\Big|\int_{I_{j-1}\cup I_{j+1}} u^h_{ji}(x)dx\Big| \leq C\varepsilon^{-2}(r^{-1}+1)\beta(r)\mathcal{K}_{j,i} + \mathcal{O}(e_\varepsilon)(\delta_{j-1,i} + \delta_{j+1,i})\,,$$

with $\mathcal{K}_{j,i} = |A_{j,i}| + |A_{j+1,i}| + |B_{j,i}| + |B_{j+1,i}|$ and

$$\begin{aligned} \Big|\int_{I_j} \Big[(-\delta_{j,i}\chi^j_x)w^j + (1-\chi^j)(A_{j,i}w^j_x + B_{j,i}w^j_\ell) - \delta_{j,i}\chi^j_x w^{j+1} - \chi^j(A_{j+1,i}w^{j+1}_x + B_{j+1,i}w^{j+1}_\ell)\Big]dx\Big| \\ \leq C\varepsilon^{-2}(r^{-1}+1)\beta(r)\mathcal{K}_{j,i} + \mathcal{O}(e_\varepsilon)\delta_{j,i}. \end{aligned}$$

Therefore, using the estimates for w^i it follows that

$$\begin{aligned}\frac{d^2}{dh_j dh_i}M(h) &= \int_0^1 u^h_{ji}dx \\ &= \int_{I_j} -\frac{\partial^2 u^h}{\partial x \partial h_i}dx + \mathcal{O}(\varepsilon^{-2}(r^{-1}+1)\beta(r))\mathcal{K}_{j,i} + \mathcal{O}(e_\varepsilon)(\delta_{j-1,i}+\delta_{j,i}+\delta_{j+1,i}) \\ &= \int_{I_j} (-\frac{\partial u^h_i}{\partial x})dx + \mathcal{O}(\varepsilon^{-2}(r^{-1}+1)\beta(r))\mathcal{K}_{j,i} + \mathcal{O}(e_\varepsilon)(\delta_{j-1,i}+\delta_{j,i}+\delta_{j+1,i}) \\ &= -(u^h_i(m_{j+1}) - u^h_i(m_j)) + \mathcal{O}(\varepsilon^{-2}(r^{-1}+1)\beta(r))\mathcal{K}_{j,i} \\ &+ \mathcal{O}(e_\varepsilon)(\delta_{j-1,i}+\delta_{j,i}+\delta_{j+1,i}).\end{aligned}$$

Since the support of u^h_i is $I_{i-1}\cup I_i \cup I_{i+1} \ni m_{i-1}, m_i, m_{i+1}, m_{i+2}$ we get that $\frac{d^2}{dh_i dh_j}M = 0$ if $j \neq i-1, i, i+1, i+2$, while for example $u^h_i(m_i) = \chi^{i-1}w^i|_{m_i} = \chi^{i-1}|_{m_i}w(0,l_i,\pm 1)$ and $u^h_i(m_{i+1}) = -(1-\chi^{i+1})w^{i+1}|_{m_{i+1}} = -(1-\chi^{i+1})|_{m_{i+1}}w(0,l_{i+1},\pm 1)$. But $w(0) = \mathcal{O}(\varepsilon^{-1})\alpha'_\pm(r)$, [22] p. 558, since $\phi_{xx}(0)^{-1} = \varepsilon^2/f(\phi(0))$ and ε/l is uniformly bounded, while χ is C^∞.

Let us now for simplicity consider $N = 1$ then $M(h_1, y) =$ constant, when $y = h_2$ where h_2 is a function of h_1, so

$$\frac{\partial M}{\partial h_1} + \frac{\partial M}{\partial y}\frac{\partial y}{h_1} = 0$$

and thus

$$\frac{\partial^2 M}{\partial h_1 \partial h_1} + \frac{\partial^2 M}{\partial y^2}\Big(\frac{\partial y}{\partial h_1}\Big)^2 + \frac{\partial M}{\partial h_1}\frac{\partial^2 y}{\partial h_1^2} = 0\ .$$

We set $y = h_2$ to get, using the estimate $\frac{\partial h_{N+1}}{\partial h_j} = \mathcal{O}(1)$,

$$\mathcal{O}(e_\varepsilon) + \mathcal{O}(e_\varepsilon)\mathcal{O}(1) + \mathcal{O}(1)\frac{\partial^2 h_2}{\partial h_1^2} = 0$$

and thus

$$\frac{\partial^2 h_2}{\partial h_1^2} = O(e_\varepsilon)\ .$$

The case $N > 1$ follows in a similar way. □

5.2. **The estimates.** We define $I_s := [-\ell/2 - \varepsilon, \ell/2 + \varepsilon]$, then for any $x \in I_s$ it holds that ([9, 22, 23])

$$\begin{aligned}&|w| \le c\varepsilon^{-1}\beta_\pm(r), \\ &|w_x| \le c\varepsilon^{-2}r^{-1}\beta_\pm(r), \\ &|w_\ell| \le c\varepsilon^{-2}\beta_\pm(r), \\ &|w_{x\ell}| \le c\varepsilon^{-3}r^{-1}\beta_\pm(r), \\ &|w_{xx}| \le c\varepsilon^{-3}\beta_\pm(r).\end{aligned} \tag{5.7}$$

For the purposes of our proof we will need estimates for the terms

$$|w_{\ell\ell}|,\ \ |w_{xxx}|,\ \ |w_{xx\ell}|,\ \ |w_{x\ell\ell}|,\ \ |w_{xxxxx}|,\ \ |w_{xxx\ell}|,\ \text{and}\ |w_{xx\ell\ell}|\ .$$

It is sufficient to estimate these terms in $I := [0, \ell/2 + \varepsilon]$ or in $(0, \ell/2 + \varepsilon]$. We write $I = [0, \ell/2 - \varepsilon H] \cup [\ell/2 - \varepsilon H, \ell/2 + \varepsilon]$, for a positive H to be defined in the sequel. We set

$$I_H := [0, \ell/2 - \varepsilon H], \quad \text{and} \quad J := [\ell/2 - \varepsilon H, \ell/2 + \varepsilon],$$

and prove the following lemma bounding the second derivative of w in ℓ on I_s.

Lemma 5.2. *For any $x \in I_s$ it holds*

$$|w_{\ell\ell}(x)| \leq c\varepsilon^{-3}\beta_\pm(r). \tag{5.8}$$

Proof. Motivated by the proof of [23] for the estimate of $|w_\ell|$, we use that

$$\varepsilon^2 w_{xx} = f'(\phi(x))w \ \text{ in } \ (0, \ell/2+\varepsilon) \supset I_H^\circ,$$

and differentiate twice with respect to ℓ to obtain

$$\varepsilon^2 (w_{\ell\ell})_{xx} - f'(\phi)w_{\ell\ell} = \mathcal{F}$$

for $\mathcal{F} := f'''(\phi)\phi_\ell^2 w + f''(\phi)\phi_{\ell\ell}w + 2f''(\phi)\phi_\ell w_\ell$. By maximum principle it follows that

$$|w_{\ell\ell}(x)| \leq \max\Big\{|w_{\ell\ell}(0)|,\ \ |w_{\ell\ell}(\ell/2-\varepsilon H)|,\ \ \sup_{x\in I_H}\Big|\mathcal{F}/f'(\phi)\Big|\Big\} \ \text{ for any } x \in I_H. \tag{5.9}$$

Following Carr and Pego (cf. [22] p. 560), we choose H such that $f'(\phi(x)) \geq c_0 > 0$ for $0 < x < \ell/2 - \varepsilon H$. Since $\varepsilon^2\phi_x^2 = 2(F(\phi)-\alpha)$, there exists a constant $C > 0$ such that $\frac{1}{|\phi_x|} \leq \frac{\varepsilon}{C}$ for any $x \in J = [\ell/2-\varepsilon H, \ell/2+\varepsilon]$ (cf. [22] p. 560, and p. 557).

First we estimate $|w_{\ell\ell}(x,\ell,-1)|$ in J. It holds that (cf. [22] p. 558)

$$w(x,\ell,-1) = \varepsilon^{-1}\ell^{-2}\alpha_-'(r)\phi_x(|x|,\ell,-1)\int_{\ell/2}^{|x|}\frac{ds}{\phi_x(s,\ell,-1)^2}. \tag{5.10}$$

Let us define $\mathcal{A} := \int_{\ell/2}^{|x|}\frac{ds}{\phi_x(s,\ell,-1)^2}$. With a slight abuse of notation and for simplicity of notation, we neglect the index '$-$' in α_- by using α. Differentiation of (5.10) yields

$$\begin{aligned} w_{\ell\ell} = \varepsilon^{-1}\Big\{&(\ell^{-2}\alpha'(r))_{\ell\ell}\phi_x\mathcal{A} + 2(\ell^{-2}\alpha'(r))_\ell\phi_{x\ell}\mathcal{A} + 2(\ell^{-2}\alpha'(r))_\ell\phi_x\mathcal{A}_\ell \\ &+ (\ell^{-2}\alpha'(r))\phi_{x\ell\ell}\mathcal{A} + 2(\ell^{-2}\alpha'(r))\phi_{x\ell}\mathcal{A}_\ell + (\ell^{-2}\alpha'(r))\phi_x\mathcal{A}_{\ell\ell}\Big\}. \end{aligned} \tag{5.11}$$

According to [22, 23] it follows that

$$|\alpha'(r)| \leq cr^{-2}\alpha, \qquad \text{and} \qquad |\alpha''(r)| \leq cr^{-4}\alpha\ .$$

Analogously we obtain

$$|\alpha'''(r)| \leq cr^{-6}\alpha\ .$$

Observing that $r = \varepsilon/\ell$ is bounded, i.e. $\ell^{-1} \leq c\varepsilon^{-1}$, we derive

$$|\ell^{-2}\alpha'(r)| \leq c\varepsilon^{-2}\alpha(r), \qquad |(\ell^{-2}\alpha'(r))_\ell| \leq c\varepsilon^{-3}\alpha(r), \qquad \text{and} \qquad |(\ell^{-2}\alpha'(r))_{\ell\ell}| \leq c\varepsilon^{-4}\alpha(r)\ . \tag{5.12}$$

Obviously since $x \in J$ one has $|\mathcal{A}| \leq c\varepsilon^{2+1}$. It holds that (cf. [22] p. 552)

$$\varepsilon^2\phi_x^2 = 2(F(\phi)-\alpha)\ , \tag{5.13}$$

while

$$\varepsilon^2\phi_{xx} = f(\phi). \tag{5.14}$$

Since $\int_{-\ell/2}^{\ell/2}|\phi_x|dx \leq 2$ (cf. [22] p. 558), and ϕ satisfies a Dirichlet problem we get by trace inequality that ϕ is uniformly bounded. Therefore, we obtain

$$|\phi_x| \leq c\varepsilon^{-1}, \qquad |\phi_{xx}| \leq c\varepsilon^{-2}, \qquad \text{and} \qquad |\phi_{xxx}| \leq c\varepsilon^{-3}\ .$$

Using now the definition (5.2) of w, and the fact that $|w| + |\phi_x| \leq c\varepsilon^{-1}$, we arrive at

$$|\phi_\ell| \leq c\varepsilon^{-1},$$

while $|\phi_{xl}| \leq c|\phi_{xx}| + c|w_x|$. So, using $|w_x| \leq c\varepsilon^{-2}$, (cf. [9]), we get

$$|\phi_{x\ell}| \leq c\varepsilon^{-2}\ .$$

By (5.14) it follows that

$$|\phi_{xx\ell}| \leq c\varepsilon^{-3}.$$

Finally, we also need an estimate for the term $\phi_{x\ell\ell}$. We differentiate (5.13) twice with respect to ℓ, in order to obtain

$$|\varepsilon^2\phi_x\phi_{x\ell\ell}| \leq c\varepsilon^{-2} \ .$$

Hence using the bound $\frac{1}{|\phi_x|} \leq c\varepsilon$ valid in J, it holds that

$$|\phi_{x\ell\ell}| \leq c\varepsilon^{-3} \ \text{ in } \ J$$

In order to compute the derivatives of $\mathcal{A}$ in (5.11), we apply the formulas

$$\frac{d}{d\ell}\int_{s(\ell)}^{b} g(s,\ell)ds = \int_{s(\ell)}^{b} g_\ell(s,\ell)ds - s'(\ell)g(s(\ell),\ell)$$

and

$$\frac{d^2}{d\ell^2}\int_{s(\ell)}^{b} g(s,\ell)ds = \int_{s(\ell)}^{b} g_{\ell\ell}(s(\ell),\ell)ds - s'(\ell)g_\ell(s(\ell),\ell) - s''(\ell)g(s(\ell),\ell) - s'(\ell)^2 g_x(s(\ell),l) - s'(\ell)g_\ell(s(\ell),\ell).$$

After tedious calculations, using the estimates above and the fact that the length of the interval is of order $\mathcal{O}(\varepsilon)$, we arrive at

$$|\mathcal{A}_\ell| \leq c\varepsilon^2, \ \ |\mathcal{A}_{\ell\ell}| \leq c\varepsilon \ .$$

We note that ε/ℓ is bounded i.e. $\ell^{-1} \leq c\varepsilon^{-1}$. Thus by (5.11) and (5.12) we obtain

$$|w_{\ell\ell}| \leq c\varepsilon^{-3}\alpha \ \text{ in } \ J. \tag{5.15}$$

So by (5.15), since $\ell/2 - \varepsilon H \in J$, it follows that

$$|w_{\ell\ell}(\ell/2 - \varepsilon H)| \leq c\varepsilon^{-3}\alpha. \tag{5.16}$$

By the definition of $\mathcal{F}$, the fact that $f' \geq c_0 > 0$ in I_H, and the first and third estimate of (5.7) we get (using $\beta := \beta_-$) that

$$\sup_{I_H}\Big|\mathcal{F}/f'(\phi)\Big| \leq c\Big[|\phi_\ell|^2|w| + |\phi_{\ell\ell}||w| + |\phi_\ell||w_\ell|\Big] \leq c\varepsilon^{-1}\beta\Big[|\phi_\ell|^2 + |\phi_{\ell\ell}| + \varepsilon^{-1}|\phi_\ell|\Big].$$

In addition, since $|w_\ell| + |\phi_{x\ell}| \leq c\varepsilon^{-2}$ [22, 9], it follows that

$$|\phi_{\ell\ell}| \leq c\varepsilon^{-2} \ .$$

Thus as we already proved $|\phi_\ell| \leq c\varepsilon^{-1}$, we derive

$$\sup_{I_H}\Big|\mathcal{F}/f'(\phi)\Big| \leq c\varepsilon^{-3}\beta. \tag{5.17}$$

Let us now turn to the missing estimate on $|w_{\ell\ell}(0)|$. In [23] by using $w(0) = -\frac{\partial\beta}{\partial\ell}(\varepsilon/\ell)$, it was demonstrated that $|w_\ell(0)| \leq c\varepsilon^{-2}\beta$. Analogously by differentiation in ℓ, it follows that

$$|w_{\ell\ell}(0)| \leq c\varepsilon^{-3}\beta. \tag{5.18}$$

Using now (5.9), (5.15), (5.16), (5.17) and (5.18) we obtain that $|w_{\ell\ell}(x)| \leq c\varepsilon^{-3}\beta$ for any x in $I = I_H \cup J$. By symmetry we prove finally that $|w_{\ell\ell}| \leq c\varepsilon^{-3}\beta_\pm(r)$ in I_s. □

The next three lemmas present bounds for the third or higher order terms

Lemma 5.3. *For any $x \in I_s^\circ - \{0\}$ it holds that*

$$|w_{xxx}(x)| \leq c\varepsilon^{-4}r^{-1}\beta_\pm(r) \tag{5.19}$$

and

$$|w_{xx\ell}(x)| \leq c\varepsilon^{-4}\beta_\pm(r). \tag{5.20}$$

Proof. We consider $x \in (0, \ell/2+\varepsilon)$ and $\varepsilon^2 w_{xx} = f'(\phi)w$. By differentiation in x, using (5.7), and the bound on $|\phi_x|$, or by differentiating in ℓ, using (5.7), and the bound on $|\phi_\ell|$ we get the following

$$\begin{aligned}|w_{xxx}| &\le c\varepsilon^{-2}\Big[|f'(\phi)||w_x| + |f''(\phi)||\phi_x||w|\Big]\\ &\le c\varepsilon^{-2}\Big[c\varepsilon^{-2}r^{-1}\beta + c\varepsilon^{-1}\varepsilon^{-1}\beta\Big] \le c\varepsilon^{-4}r^{-1}\beta\end{aligned}$$

and

$$\begin{aligned}|w_{xx\ell}| &\le c\varepsilon^{-2}\Big[|f'(\phi)||w_\ell| + |f''(\phi)||\phi_\ell||w|\Big]\\ &\le c\varepsilon^{-2}\Big[c\varepsilon^{-2}\beta + c\varepsilon^{-1}\varepsilon^{-1}\beta\Big] \le c\varepsilon^{-4}\beta,\end{aligned}$$

with $\beta = \beta_-$. Again by symmetry, we obtain the bounds for all x in $I_s^\circ - \{0\}$. ☐

Lemma 5.4. *For any $x \in I_s - \{0\}$ it holds that*

$$|w_{x\ell\ell}(x)| \le c\varepsilon^{-4}r^{-1}\beta_\pm(r). \tag{5.21}$$

Proof. Consider $x \in (0, \ell/2+\varepsilon]$ and write $w_{x\ell\ell}(\ell/2) - w_{x\ell\ell}(x) = \int_x^{\ell/2} w_{xx\ell\ell}(s)ds$, in order to obtain

$$|w_{x\ell\ell}(x)| \le |w_{x\ell\ell}(\ell/2)| + \int_x^{\ell/2} |w_{xx\ell\ell}(s)|ds. \tag{5.22}$$

We use the definition of w given in (5.10), set $p = \varepsilon^{-1}\ell^2\alpha'$, and recall that $\mathcal{A} = \int_{\ell/2}^{|x|}\phi_x^{-2}ds$. Differentiating first with respect to x and then twice w.r.t. ℓ yields

$$\begin{aligned}w_{x\ell\ell} =& p_{\ell\ell}\phi_{xx}\mathcal{A} + p_\ell\phi_{xx\ell}\mathcal{A} + 2p_\ell\phi_{xx}\mathcal{A}_\ell + p_\ell\phi_{xx\ell}\mathcal{A} + p\phi_{xx\ell\ell}\mathcal{A}\\ &+ 2p\phi_{xx\ell}\mathcal{A}_\ell + p\phi_{xx}\mathcal{A}_{\ell\ell} - \frac{p_\ell\phi_{x\ell}}{\phi_x^2} - p\frac{(\phi_{x\ell\ell}\phi_x^2 - 2\phi_{x\ell}^2\phi_x)}{\phi_x^4} + \frac{p_{\ell\ell}}{\phi_x} - \frac{p_\ell\phi_{x\ell}}{\phi_x^2}.\end{aligned}$$

Observe that $\mathcal{A} = 0$ at $x = \ell/2$, while

$$\mathcal{A}_\ell(\ell/2) = -\frac{1}{2}\phi_x(\ell/2)^{-2}, \qquad \text{and} \qquad \mathcal{A}_{\ell\ell}(\ell/2) = \phi_x(\ell/2)^{-3}\phi_{x\ell}(\ell/2) + \phi_{x\ell}(\ell/2)\phi_x(\ell/2)^{-3}\ .$$

We also note $\ell/2 \in J$, thus by the bounds of Lemma 5.2 we obtain $|\phi_{x\ell}(\ell/2)| \le c\varepsilon^{-2}$ and $|\phi_x(\ell/2)|^{-1} \le c\varepsilon$. Hence, as in Lemma 5.2 for a general $x \in J$, we get $\varepsilon^{-1}|\mathcal{A}_\ell(\ell/2)| + |\mathcal{A}_{\ell\ell}(\ell/2)| \le c\varepsilon$.

In addition using the third estimate from (5.12) yields $|p_{\ell\ell}(\ell/2)| \le c\varepsilon^{-5}\alpha$. Furthermore, as $\ell/2 \in J$ by the proof of Lemma 5.2 we have $|\phi_{xx}(\ell/2)| \le c\varepsilon^{-2}$ and $|\phi_{x\ell\ell}(\ell/2)| \le c\varepsilon^{-3}$. Therefore, we obtain finally (with $\alpha = \alpha_-$)

$$|w_{x\ell\ell}(\ell/2)| \le c\varepsilon^{-4}\alpha\ . \tag{5.23}$$

Recall $\varepsilon^2 w_{xx} = f'(\phi)w$ in $(0, \ell/2+\varepsilon)$. By taking twice the ℓ-derivative yields ($\beta = \beta_-$)

$$|w_{xx\ell\ell}| \le c\varepsilon^{-2}\Big[|\phi_\ell|^2|w| + |\phi_\ell||w_\ell| + |w_{\ell\ell}|\Big] \le c\varepsilon^{-5}\beta\ . \tag{5.24}$$

Here, we used the bound $|\phi_\ell| \le c\varepsilon^{-1}$ from the proof of Lemma 5.2, the first and third estimate of (5.7), the fact that $|w| \le c\varepsilon^{-1}\beta$ and $|w_\ell| \le c\varepsilon^{-2}\beta$, and the bound $|w_{\ell\ell}| \le c\varepsilon^{-3}\beta$ of Lemma 5.2.

Using $r = \varepsilon/\ell$ and $x \in (0, \ell/2)$ we get $|x - \ell/2| \le c(\ell/2+\varepsilon) \le c\varepsilon r^{-1}$. Therefore, (5.22), (5.23), and (5.24) imply

$$|w_{x\ell\ell}(x)| \le c\varepsilon^{-4}r^{-1}\beta, \quad x \in (0, \ell/2+\varepsilon]\ .$$

By symmetry the analogous result holds for any $x \in [-\ell/2-\varepsilon, 0)$. ☐

Analogously the following lemma follows:

Lemma 5.5. *For any $x \in I_s - \{0\}$ it holds that*

$$|w_{xxxxx}(x)| + |w_{xxx\ell}(x)| + |w_{xx\ell\ell}(x)| \le c\varepsilon^{-5}r^{-1}\beta_\pm(r). \tag{5.25}$$

In order to estimate E_i^ξ, E_{ij}^ξ and E_{ijk}^ξ we fist need the following lemma for the correction terms Q_j.

Lemma 5.6. *For any i, j, k it follows that*

$$
\begin{aligned}
&|Q_j| \le c\varepsilon^{-3}\beta,\\
&|Q_{ij}| \le c\varepsilon^{-4}r^{-1}\beta, \qquad (5.26)\\
&|Q_{ijk}| \le c\varepsilon^{-5}r^{-1}\beta.
\end{aligned}
$$

Proof. We recall that

$$
u_j^h = \begin{cases} \chi^{j-1}w^j & \text{on } I_{j-1}\\ (1-\chi^j)(-\phi_x^j + w^j) + \chi^j(-\phi_x^{j+1} - w^{j+1}) + \chi_x^j(\phi^j - \phi^{j+1}) & \text{on } I_j\\ -(1-\chi^{j+1})w^{j+1} & \text{on } I_{j+1}. \end{cases}
$$

Consider the functions on $x = 0, 1$ in the first and last set of their support. Using the bounds on $|w|, |w_{xx}|$, we arrive at

$$
\begin{aligned}
&|\tilde{u}_j^h| \le c\varepsilon^{-1}\beta \text{ and thus } |\tilde{w}_j| \le c\varepsilon^{-1}\beta,\\
&|\tilde{u}_{jxx}^h| \le c\varepsilon^{-3}\beta \text{ and thus } |\tilde{w}_{jxx}| \le c\varepsilon^{-3}\beta.
\end{aligned}
$$

The estimates of $|w_x|$ and $|w_\ell|$ and of $|w_{xxx}|$ and $|w_{\ell xx}|$, respectively, give

$$
\begin{aligned}
&|\tilde{u}_{ji}^h| \le c\varepsilon^{-2}r^{-1}\beta \text{ and thus } |\tilde{w}_{ji}| \le c\varepsilon^{-2}r^{-1}\beta,\\
&|\tilde{u}_{jixx}^h| \le c\varepsilon^{-4}r^{-1}\beta \text{ and thus } |\tilde{w}_{jixx}| \le c\varepsilon^{-4}r^{-1}\beta.
\end{aligned}
$$

Finally, using the estimates of $|w_{xx}|, |w_{x\ell}|$, and $|w_{\ell\ell}|$ and of $|w_{xxxx}|, |w_{xxx\ell}|$, and $|w_{xx\ell\ell}|$, respectively, we obtain

$$
\begin{aligned}
&|\tilde{u}_{jik}^h| \le c\varepsilon^{-3}r^{-1}\beta \text{ and thus } |\tilde{w}_{jik}| \le c\varepsilon^{-3}r^{-1}\beta,\\
&|\tilde{u}_{jikxx}^h| \le c\varepsilon^{-5}r^{-1}\beta \text{ and thus } |\tilde{w}_{jikxx}| \le c\varepsilon^{-5}r^{-1}\beta.
\end{aligned}
$$

Recall also

$$
\begin{aligned}
&\tilde{w}_j := \tilde{u}_j^h + \tilde{u}_{j+1}^h,\\
&Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\tilde{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\tilde{w}_{jxx}(1) + x\tilde{w}_j(1), \;\; j = 1, \dots, N.
\end{aligned}
$$

This definition of Q_j combined with the previously obtained bounds on $\tilde{w}_j$ imply the result. ☐

Remark 5.7. *By [22] p. 557-556, the following estimates hold true. First,*

$$
\int_{-\ell/2}^{0} \phi_x(x, \ell, -1)^2 + \int_0^{\ell/2} \phi_x(x, \ell, +1)^2 \le \varepsilon^{-1}S_\infty + E(r), \tag{5.27}
$$

where $|E| \le c\varepsilon^{-1}\beta$ and $S_\infty = \int_{-1}^1 \sqrt{2F(u)}du$. Moreover,

$$
\int_{-\ell/2}^{\ell/2} |\phi_x| dx \le 2 \tag{5.28}
$$

and

$$
\int_{-\ell/2}^{\ell/2} |\phi_{xx}|^2 dx \le c\varepsilon^{-3}. \tag{5.29}
$$

In addition, there exists a constant $c > 0$ such that for all $x \in [h_j - \varepsilon, h_j + \varepsilon], j = 0, \dots, N+1$ we have

$$
|\phi^j(x) - \phi^{j+1}(x)| \le c|\alpha^j - \alpha^{j+1}|, \tag{5.30}
$$

$$
|\phi_x^j(x) - \phi_x^{j+1}(x)| \le c\varepsilon^{-1}|\alpha^j - \alpha^{j+1}|, \tag{5.31}
$$

and

$$|\phi^j_{xx}(x) - \phi^{j+1}_{xx}(x)| \le c\varepsilon^{-2}|\alpha^j - \alpha^{j+1}|, \tag{5.32}$$

provided $\varepsilon/\ell_j, \varepsilon/\ell_{j+1} < r_0$ *for some sufficiently small* $r_0 > 0$ *(cf.* [8]*).*

Now, we are able to bound the terms $\tilde{u}^h$ and u^h.

Theorem 5.8. *For any* i, j, k *it holds that*

$$\begin{aligned} &\|\tilde{u}^h_j\|_\infty \le \mathcal{O}(1) + \mathcal{O}(\|w\|_\infty),\\ &\|\tilde{u}^h_{ji}\| \le c\varepsilon^{-1/2}(1 + S_\infty^{1/2} + \max(r_j\alpha^j, r_{j+1}\alpha^{j+1})^{1/2}) + c\|w_x\| + c\|w_\ell\|,\\ &\|\tilde{u}^h_{jik}\| \le c\varepsilon^{-3/2} + c\|w_x\| + c\|w_\ell\| + c\|w_{xx}\| + c\|w_{x\ell}\| + c\|w_{\ell\ell}\|,\\ &\|u^h_j\|_\infty \le \mathcal{O}(\varepsilon^{-1}), \qquad \|u^h_{ij}\|_\infty \le \mathcal{O}(\varepsilon^{-2}), \qquad \|u^h_j\| \le \mathcal{O}(\varepsilon^{-\frac{1}{2}}). \end{aligned} \tag{5.33}$$

Proof. We use the definition of u^h_j and get by (5.28) that

$$|\tilde{u}^h_j| \le c\int_0^x |\phi_x| dx + c\|w\|_\infty \le c + c\|w\|_\infty \ .$$

Also, it follows that $|u^h_j| = \mathcal{O}(|\phi_x|) = \mathcal{O}(\varepsilon^{-1})$, and thus

$$\|u^h_j\|_\infty \le \mathcal{O}(\varepsilon^{-1}) \ .$$

By [8] p. 38,

$$u^h_j = -u^h_x + (1-\chi^j)w^j - \chi^j w^{j+1} \qquad \text{on } I_j \ . \tag{5.34}$$

Combining this with (5.3) we obtain

$$u^h_{ji}(x) = \begin{cases} \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j-1}\\ -u^h_{xi}(x) + \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_j\\ \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j+1} \ . \end{cases}$$

Therefore, we arrive at

$$\tilde{u}^h_{ji}(x) = \int_0^x u^h_{ji}(y)dy = \begin{cases} \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j-1}\\ \mathcal{O}(u^h_i + w_x + w_\ell) & \text{for } x \in I_j\\ \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j+1}. \end{cases}$$

By [22] (cf. p. 563, (8.6)),

$$\|u^h_i\| \le \varepsilon^{-1/2}(S_\infty^{1/2} + \max(r_j\alpha^j, r_{j+1}\alpha^{j+1})^{1/2}) \ . \tag{5.35}$$

Using this estimate we obtain

$$\|\tilde{u}^h_{ji}\| \le c\varepsilon^{-1/2}(1 + S_\infty^{1/2} + \max(r_j\alpha^j, r_{j+1}\alpha^{j+1})^{1/2}) + c\|w_x\| + c\|w_\ell\| \ .$$

For higher derivatives, observe now that

$$\tilde{u}^h_{jik}(x) = \int_0^x u^h_{jik}(y)dy = \begin{cases} \mathcal{O}(w_{xx} + w_{x\ell} + w_{\ell\ell}) & \text{for } x \in I_{j-1}\\ \mathcal{O}(u^h_{xi} + w_{xx} + w_{x\ell} + w_{\ell\ell}) & \text{for } x \in I_j\\ \mathcal{O}(w_{xx} + w_{x\ell} + w_{\ell\ell}) & \text{for } x \in I_{j+1}. \end{cases}$$

In addition, since $u^h_j = -u^h_x + (1-\chi^j)w^j - \chi^j w^{j+1}$ on I_j, we obtain

$$\|u^h_{xi}\| \le \|u^h_{xx}\| + c\|w_x\| \ .$$

The argument of Lemma 8.3 of [22] p. 562 applied to u^h on I_j using the support of $|\phi^j_x - \phi^{j+1}_x|$ combined with (5.31), (5.28), and

$$u^h_x = \mathcal{O}(|\phi_x|) + \mathcal{O}(|\phi^j_x - \phi^{j+1}_x|) \ ,$$

finally yields

$$\|u^h_x\| \le \|\phi_x\| + \sqrt{\mathcal{O}(\varepsilon^{-2}\varepsilon)} \le c\varepsilon^{-\frac{1}{2}}. \tag{5.36}$$

Analogously, differentiating u^h twice with respect to x, using the bounds of (5.32) and (5.29), and the support of $|\phi^j_{xx} - \phi^{j+1}_{xx}|$ yields

$$u^h_{xx} = \mathcal{O}(|\phi_{xx}|) + \mathcal{O}(|\phi^j_{xx} - \phi^{j+1}_{xx}|)\ .$$

Thus

$$\|u^h_{xx}\| \le \|\phi_{xx}\| + \sqrt{\mathcal{O}(\varepsilon^{-4}\varepsilon)} \le c\varepsilon^{-3/2}.$$

So, it follows that

$$\|u^h_{xi}\| \le c\varepsilon^{-3/2} + c\|w_x\|. \tag{5.37}$$

Combining the previous estimates yields

$$\|\tilde{u}^h_{jik}\| \le c\varepsilon^{-3/2} + c\|w_x\| + c\|w_\ell\| + c\|w_{xx}\| + c\|w_{x\ell}\| + c\|w_{\ell\ell}\|\ .$$

Using again (5.34) we obtain

$$|u^h_{ij}| \le \mathcal{O}(u^h_{xj}) = \mathcal{O}(u^h_{xx}) = \mathcal{O}(\phi_{xx}) = \mathcal{O}(\varepsilon^{-2})\ .$$

Therefore

$$\|u^h_{ij}\|_\infty \le \mathcal{O}(\varepsilon^{-2})\ .$$

Further, by (5.34) and (5.36) it follows that

$$\|u^h_j\| \le \mathcal{O}(\|u^h_x\|) = \mathcal{O}(\varepsilon^{-\frac{1}{2}})\ .$$

□

Combining now the bound $|\tilde{u}^h_j| \le \mathcal{O}(1) + \mathcal{O}(\|w\|)$ with the implicit function theorem, for the change of variables for h to ξ, we obtain

$$|\tilde{u}^\xi_j| \le (\mathcal{O}(1) + \mathcal{O}(\|w\|))[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)]\ . \tag{5.38}$$

Moreover, for the second derivative in ξ variables

$$\tilde{u}^\xi_{jk} \le |\tilde{u}^h_{jk}|[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)]^2 + |\tilde{u}^h_{jk}|[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)] + |\tilde{u}^h_j|\mathcal{O}(e_\varepsilon)\ .$$

So we verified the following lemma:

Lemma 5.9. *For all j,k it holds:*

$$\|\tilde{u}^\xi_j\| \le (\mathcal{O}(1) + \mathcal{O}(\|w\|))[\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-1}\beta)] \tag{5.39}$$

and

$$\|\tilde{u}^\xi_{jk}\| \le [\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-2}\beta^2) + \mathcal{O}(\varepsilon^{-1}\beta)][\mathcal{O}(w_x + w_\ell) + \varepsilon^{-1/2} + \varepsilon^{-1/2}A] + \mathcal{O}(e_\varepsilon)[\mathcal{O}(1) + \mathcal{O}(\|w\|)] \tag{5.40}$$

with defined as $A = S^{1/2}_\infty + \max_j(r_j\alpha^j, r_{j+1}\alpha^{j+1})^{1/2}$.

The following theorem gives the bounds on E^ξ_i in the L^2-norm.

Theorem 5.10. *For all i,j,k the following inequalities hold:*

$$\|E^\xi_i\| \le 4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta), \tag{5.41}$$

$$\|E^\xi_{ij}\| \le \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4}r^{-1}\beta), \tag{5.42}$$

$$\|E^\xi_{ijk}\| \le \mathcal{O}(\varepsilon^{-3/2}) + \mathcal{O}(\varepsilon^{-5}r^{-1}\beta). \tag{5.43}$$

Proof. Using the bound $\|E^\xi_j\| \le \|\tilde{w}_j\| + \|Q_j\|$, the estimate of $\|\tilde{w}_j\|$ presented in (4.24) on p. 186 of [9], and our Lemma 5.6, we obtain (5.41). Also, observe

$$E^\xi_{ji} = \tilde{w}_{ji} + \mathcal{O}(Q_{ji}) + \mathcal{O}(Q_{ijx}) = \mathcal{O}(w_x + w_\ell) + \int_0^x (-u^h_{xi})dy + \mathcal{O}(Q_{ji}) + \mathcal{O}(Q_{ijx})$$
$$\le \mathcal{O}(w_x + w_\ell) + \mathcal{O}(u^h_i) + \mathcal{O}(Q_{ji}) \ .$$

Hence, by (5.35) and Lemma 5.6, we conclude

$$\|E^\xi_{ji}\| \le \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4} r^{-1}\beta) \ .$$

Furthermore, using (5.34) we obtain

$$E^\xi_{jik} = \tilde{w}_{jik} + \mathcal{O}(Q_{jik}) + \mathcal{O}(Q_{jikx}) = \mathcal{O}(w_{xx} + w_{\ell\ell} + w_{x\ell}) + \int_0^x (-u^h_{xxk})dy + \mathcal{O}(Q_{jik}) + \mathcal{O}(Q_{jikx})$$
$$\le \mathcal{O}(w_{xx} + w_{\ell\ell} + w_{x\ell}) + \mathcal{O}(u^h_{xk}) + \mathcal{O}(Q_{jik}) \ .$$

Thus, by (5.37) and Lemma 5.6 this implies

$$\|E^\xi_{ijk}\| \le \mathcal{O}(\varepsilon^{-3/2}) + \mathcal{O}(\varepsilon^{-5} r^{-1}\beta) \ .$$

□

Remark 5.11. *We note that the bound on $\|E^\xi_{ij}\|$ presented in theorem 5.10 coincides in the main order term with the estimate that was used but not presented analytically in* [9].

Using all the results of the previous analysis we are now ready to derive by Cauchy-Schwarz inequality all the desired estimates for the higher order derivatives. They are presented in the following main theorem of this section.

Theorem 5.12. *These inequalities hold for all i, l, k:*

$$|\langle \tilde{u}^\xi_{kl}, E^\xi_i \rangle| \le \mathcal{O}(\varepsilon^{-1/2})\Big[4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta)\Big], \tag{5.44}$$

$$|\langle \tilde{u}^\xi_{k}, E^\xi_{il} \rangle| \le \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4} r^{-1}\beta), \tag{5.45}$$

$$|\langle \tilde{v}, E^\xi_{ilk} \rangle| \le \big[\mathcal{O}(\varepsilon^{-3/2}) + \mathcal{O}(\varepsilon^{-5} r^{-1}\beta)\big] \cdot \|\tilde{v}\|. \tag{5.46}$$

It remains to analyze $\langle L^c\tilde{v}, \tilde{u}^\xi_{kl}\rangle$. Here we provide the following main result.

Theorem 5.13. *For all k and l it holds that*

$$|\langle L^c\tilde{v}, \tilde{u}^\xi_{kl}\rangle| \le \varepsilon^{-5}\beta(r)\Big(\mathcal{O}(1) + \mathcal{O}(\varepsilon^{-2}\beta(r)^2)\Big)\|\tilde{v}\|. \tag{5.47}$$

Proof. Note that by symmetry and definition

$$\langle L^c\tilde{v}, \tilde{u}^\xi_{kl}\rangle = -\langle \tilde{v}, \partial_x \partial_{\xi_k}\partial_{\xi_l}\mathcal{L}^b(u^\xi)\rangle \ .$$

Recall that we defined $\mathcal{L}^b(\phi) = \varepsilon^2\phi_{xx} - f(\phi)$. As in [8] (cf. p. 452-453) for $x \in [h_j - \varepsilon, h_j + \varepsilon]$, $j = 1, 2, \ldots, N+1$ we write

$$\mathcal{L}^b(u^h) = f_1 + f_2 + G, \tag{5.48}$$

where we defined

$$f_1 := \varepsilon^2\chi^j_{xx}(\phi^{j+1} - \phi^j), \qquad f_2 := 2\varepsilon^2\chi^j_x(\phi^{j+1}_x - \phi^j_x),$$
$$G := (\phi^{j+1} - \phi^j)^2\Big\{(1 - \chi^j)\int_0^{\chi^j} s f''(\theta)ds + \chi^j \int_{\chi^j}^1 (1-s) f''(\theta)ds\Big\},$$

with $\theta = \theta(s) := (1-s)\phi^j(x) + s\phi^{j+1}(x)$. For all other x, we have no contribution of $\mathcal{L}^b(u^h)$.

In Lemma 5.2 of [8] at p. 454, after differentiating f_1, f_2, G with respect to h_j it is derived that

$$\Big|\frac{\partial}{\partial h_j}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-2}\beta(r)\ .$$

Applying the analogous argument to the second differential with respect to h_j and h_i yields after some calculations

$$\Big|\frac{\partial^2}{\partial h_j \partial h_i}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-3}\beta(r). \tag{5.49}$$

Note that in this argument the worst term is $|\phi^j_{xxx}(x)-\phi^{j+1}_{xxx}(x)|$. But as $\varepsilon^2\phi_{xxx} = f'(\phi)\phi_x$ with $f(\phi)=\phi^3-\phi$ and $f'(\phi)=3\phi^2-1$, using the estimates of ϕ, ϕ_x and the results for the differences presented at p. 453 of [8], we get

$$\begin{aligned}
|\phi^j_{xxx}(x)-\phi^{j+1}_{xxx}(x)| &= \varepsilon^{-2}|f'(\phi^j)\phi^j_x(x) - f'(\phi^{j+1})\phi^{j+1}_x(x)| \\
&= \varepsilon^{-2}|f'(\phi^j)\phi^j_x(x) - f'(\phi^{j+1})\phi^{j+1}_x(x) - f'(\phi^j)\phi^{j+1}_x(x) + f'(\phi^j)\phi^{j+1}_x(x)| \\
&\leq \varepsilon^{-2}|f'(\phi^j)||\phi^j_x(x) - \phi^{j+1}_x(x)| + \varepsilon^{-2}|\phi^{j+1}_x(x)||f'(\phi^j) - f'(\phi^{j+1})| \\
&\leq c\varepsilon^{-2}|\phi^j_x(x) - \phi^{j+1}_x(x)| + c\varepsilon^{-2}\varepsilon^{-1}|f'(\phi^j) - f'(\phi^{j+1})| \\
&\leq c\varepsilon^{-3}|\alpha^j - \alpha^{j+1}| + c\varepsilon^{-3}|3\phi^j(x)^2 - 1 - 3\phi^{j+1}(x)^2 + 1| \\
&\leq c\varepsilon^{-3}|\alpha^j - \alpha^{j+1}| + c\varepsilon^{-3}|\phi^j(x) + \phi^{j+1}||\phi^j(x) - \phi^{j+1}| \\
&\leq c\varepsilon^{-3}|\alpha^j - \alpha^{j+1}| + c\varepsilon^{-3}|\alpha^j - \alpha^{j+1}| \\
&\leq c\varepsilon^{-3}|\alpha^j - \alpha^{j+1}|.
\end{aligned}$$

Again as in [8] (cf. p. 456), by using that $\varepsilon^2 w_{xx} = f'(\phi(x))w$ and differentiation in x, we obtain

$$\Big|\frac{\partial^2}{\partial h_j \partial h_i}\frac{\partial}{\partial x}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-5}\beta(r). \tag{5.50}$$

Changing now to ξ variables, using that the second derivative appears by applying the formula (5.1) to (5.50), and since (cf. [8] p. 454) it holds that

$$\Big|\frac{\partial}{\partial h_j}\frac{\partial}{\partial x}\mathcal{L}^b u^h\Big| \leq c\varepsilon^{-4}\beta(r), \tag{5.51}$$

we finally obtain

$$\begin{aligned}
\Big|\frac{\partial^2}{\partial \xi_k \partial \xi_l}\frac{\partial}{\partial x}\mathcal{L}^b u^h\Big| &\leq \varepsilon^{-5}\beta(r)\Big\{(\mathcal{O}(1)+\varepsilon^{-1}\beta(r))^2 + (\mathcal{O}(1)+\varepsilon^{-1}\beta(r))\Big\} + \varepsilon^{-4}\beta(r)\mathcal{O}(e_\varepsilon) \\
&\leq \varepsilon^{-5}\beta(r)\Big(\mathcal{O}(1) + \varepsilon^{-2}\beta(r)^2\Big).
\end{aligned} \tag{5.52}$$

So, the result follows. □

Acknowledgment. The third author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme, MIRG-CT-2007-200526, and partially supported by the FP7-REGPOT-2009-1 project "Archimedes Center for Modeling, Analysis and Computation". The first and third author are partially supported by Thales "AMOSICSS" research project.

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