ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ - ΤΜΗΜΑ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
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On a uniform estimate for positive solutions of semilinear elliptic equations

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ON A UNIFORM ESTIMATE FOR POSITIVE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider the semilinear elliptic equation $\Delta u = W'(u)$ with Dirichlet boundary conditions in a Lipschitz, possibly unbounded, domain $\Omega \subset \mathbb{R}^n$. Under suitable assumptions on the potential W, we deduce a condition on the size of the domain that implies the existence of a positive solution satisfying a uniform pointwise estimate. Here uniform means that the estimate is independent of Ω. Besides of its simplicity, the main advantage of our approach is that we can remove a restrictive monotonicity assumption on W that was imposed in the recent paper [12]. Moreover, we can remove a non-degeneracy condition on the global minimum of W that was assumed in the latter reference.

1. Introduction and statement of the main result

Recently, the authors of [12] considered the semilinear elliptic problem

$$\begin{cases} \Delta u = W'(u), & x \in \Omega, \\ \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a domain with nonempty Lipschitz boundary (see for instance [10]), under the following assumptions on the C^2 function $W : \mathbb{R} \to \mathbb{R}$:

(a): There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t), \ t \in [0, \infty), \ t \neq \mu,$$
$$W(-t) \geq W(t), \ t \in [0, \infty);$$

(b): $W'(t) \leq 0, \ t \in (0, \mu)$;

(c): $W''(\mu) > 0$.

For a typical example of such a potential, see (1.8) below. We stress that, in the case where the domain is unbounded, the boundary conditions in (1.1) *do not* refer to $u(x) \to 0$ as $|x| \to \infty$ with $x \in \Omega$.

For $x \in \mathbb{R}^n$, $\rho > 0$, we let

$$B_\rho(x) = \{y \in \mathbb{R}^n \ : \ |y - x| < \rho\}, \quad B_\rho = B_\rho(0),$$
$$A + B = \{x + y \ : \ x \in A, \ y \in B\}, \quad A, B \subset \mathbb{R}^n,$$

and denote $d(x, E)$ the Euclidean distance of the point $x \in \mathbb{R}^n$ from the set $E \subset \mathbb{R}^n$, and $|E|$ the n-dimensional Lebesgue measure of E (see [10]).

The main result of [12] was the following:

Theorem 1.1. *Assume Ω and W as above. There are positive constants R^*, k, K, depending only on W and n, such that if Ω contains a closed ball of radius R^*, then problem (1.1) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ verifying*

$$0 < u(x) < \mu, \quad x \in \Omega, \tag{1.2}$$

$$\mu - u(x) \leq K e^{-kd(x,\partial\Omega)}, \quad x \in \Omega. \tag{1.3}$$

In addition, there are $r^* \in (0, R^*)$ and $a^* \in (0, \mu)$, depending only on W and n, such that

$$\mu - a^* < u(x), \quad x \in \Omega_{R^*} + B_{r^*}, \tag{1.4}$$

where

$$\Omega_{R^*} = \{x \in \Omega \ : \ d(x, \partial\Omega) > R^*\}.$$

The approach of [12] to the proof of Theorem 1.1 is variational, involving the construction of various judicious radial comparison functions on B_{R^*}, see also [1]. We note that, once (1.4) is established, the exponential decay estimate (1.3) can also be deduced as in Lemma 4.2 in [11], making use of the non-degeneracy condition **(c)** (it holds that $W''(u) > 0, \ u \in [\mu - a^*, \mu]$). Moreover, an examination of the proof of Lemma 2.1 in [12] shows that assumption **(a)** above can be relaxed to

(a'): There exists a constant $\mu > 0$ such that

$$0 = W(\mu) < W(t), \ t \in [0, \mu), \quad W(t) \geq 0, \ t \geq \mu,$$
$$W(-t) \geq W(t), \ t \in [0, \infty).$$

The main purpose of this note is to show that relation (1.4) can be established in a simple manner *without assuming* the monotonicity condition **(b)**. We will accomplish this by making use of some variational arguments that can be traced back to [7], in the context of semilinear elliptic singular perturbation problems. In passing, we remark that a similar monotonicity assumption to **(b)** also appears in [1], in the context of variational systems of the form (1.1), where $W : \mathbb{R}^n \to \mathbb{R}^n$ (see also Remark 2.4 below). Moreover, we remove the non-degeneracy condition **(c)**.

Our result is

Theorem 1.2. *Assume that Ω is as above and $W \in C^2$ satisfies **(a')**. There exist positive constants R' and $a \in (0, \mu)$, depending only on W and n, such that if Ω contains some ball $B_{R'}(x_0)$, then problem (1.1) has a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ verifying (1.2), and*

$$\mu - a < u(x), \quad x \in B_{\frac{R'}{2}}(x_0). \tag{1.5}$$

In our opinion, Theorems 1.1 and 1.2 are important for the following reasons. If we additionally assume that W is even, namely

$$W(-t) = W(t), \quad t \in \mathbb{R}, \tag{1.6}$$

by means of these theorems, we can derive the existence of various sign-changing entire solutions for the problem

$$\Delta u = W'(u), \quad x \in \mathbb{R}^n, \tag{1.7}$$

by first proving existence of a positive solution is a suitable "fundamental" domain Ω_F, as above, and consecutive odd reflections to cover the entire space. We refer the interested reader to the introduction of [12]. In the case where

$$W(t) = \frac{1}{4}(t^2 - 1)^2, \quad t \in \mathbb{R}, \tag{1.8}$$

then (1.7) becomes the well known Allen-Cahn equation (see for instance [18]). Assuming (1.6), then (1.1) has always the trivial solution. In this regard, the purpose of estimate (1.5) is twofold: In the case where Ω_F is bounded, it ensures that the solution of (1.1), provided by Theorem 1.2, is nontrivial. The situation of unbounded domains Ω_F can be treated by exhausting them by an infinite sequence

$\{\Omega_n\}$ of bounded ones, each containing the same ball $B_{R'}(x_0)$, and a standard compactness argument, making use of (1.2) together with elliptic estimates. The fact that estimate (1.5) is independent of the domain is needed to rule out the possibility of subsequences of the (chosen) solutions u_n of $(1.1)_n$ on Ω_n converging, uniformly in compact subsets of Ω, to the trivial solution of (1.1) on Ω_F. Another approach can be found in [6].

2. Proof of the main result

We will need the following lemma, which is motivated from Lemma 2 in [15], and can be traced back to [7].

Lemma 2.1. *Assume that $W \in C^2$ satisfies condition **(a')**. There exist positive constants R' and $a \in (0, \mu)$, depending only on W and n, such that any global minimizer of the energy functional*

$$J(v; B_R) = \int_{B_R} \left\{ \frac{1}{2}|\nabla v|^2 + W(v) \right\} dx \quad \textit{in } W_0^{1,2}(B_R),$$

satisfies

$$0 < u(x) < \mu, \;\; x \in B_R, \tag{2.1}$$

and

$$\mu - a < u(x), \;\; x \in B_{\frac{R}{2}}, \tag{2.2}$$

provided that $R \geq R'$.

Proof. Under our assumptions on W, it is standard to show the existence of a global minimizer $u \in W_0^{1,2}(B_R)$ satisfying $0 \leq u(x) \leq \mu$ a.e. in B_R, see [12]. (The second bound in the latter inequality can also be derived from Lemma 2.3 in [7] or Lemma 1 in [15]). Moreover, this minimizer is a smooth solution, in $C^2(B_R) \cap C(\bar{B_R})$, of

$$\Delta u = W'(u) \;\; \text{in} \;\; B_R; \; u = 0 \;\; \text{on} \;\; \partial B_R. \tag{2.3}$$

By the strong maximum principle, see for example Lemma 3.4 in [14], we deduce that $u(x) < \mu, \; x \in B_R$, and that either u is identically equal to zero or $u(x) > 0, \; x \in B_R$ (recall that assumption **(a')** implies that $W'(0) \leq 0$ and $W'(\mu) = 0$). Observe that u depends on R but we have chosen not to make this apparent in the notations.

Next, adapting an argument in Section 4 in [18], we will show that u is nontrivial, provided R is sufficiently large. It is easy to cook up a test function, and use it as a competitor, to show that there exists a constant C_1, depending only on n and W, such that

$$J(u; B_R) \leq C_1 R^{n-1}, \;\; \text{say for} \;\; R \geq 2. \tag{2.4}$$

(Plainly construct a function which interpolates smoothly from μ to 0 in a layer of size 1 around the boundary of B_R and which is identically equal to μ elsewhere). Notice that the energy of the trivial solution is

$$J(0, B_R) = \int_{B_R} W(0) dx = C_2 R^n,$$

where $C_2 > 0$ depends only on $n, \; W$. From the relation

$$C_2 R^n = J(0; B_R) \leq C_1 R^{n-1}, \;\; R \geq 2,$$

we infer that u is certainly not identically equal to zero for

$$R \geq C_1 C_2^{-1} + 2.$$

We thus conclude that (2.1) holds. (In the above calculation, we relied on the fact that **(a')** implies that $W(0) > 0$; in this regard, see Remark 2.2 below).

Since u is strictly positive in the ball B_R, by (2.3) and the method of moving planes [5, 13], we infer that u is radially symmetric and

$$u_r(r) < 0, \quad r \in (0, R), \tag{2.5}$$

(with the obvious notation). We note that, since u is a global minimizer, the same conclusion can be asserted as in [17]. Relation (2.4) and the positivity of W clearly imply that

$$\int_{\bar{B_R} \backslash B_{\frac{R}{2}}} W(u)dx \le C_1 R^{n-1}, \quad R \ge C_1 C_2^{-1} + 2.$$

So, by the mean value theorem, there exists a $\xi \in \left(\frac{R}{2}, R\right)$ such that

$$W\left(u(\xi)\right) \left|\bar{B_R} \backslash B_{\frac{R}{2}}\right| \le C_1 R^{n-1},$$

i.e.,

$$W\left(u(\xi)\right) \le C_3 R^{-1},$$

where the positive constat $C_3 > 0$ depends only on n and W. Now, observe that assumption **(a')** implies that

$$\text{there exist } \delta, \ a > 0 \text{ such that } \ W^{-1}(0, \delta) \cap (0, \mu) = (\mu - a, \mu). \tag{2.6}$$

Thus, choosing

$$R' > \max\left\{\frac{C_3}{\delta}, \ C_1 C_2^{-1} + 2\right\},$$

then assumption **(c')**, (2.1), and the above relation, yield that

$$u(\xi) > \mu - a.$$

Recalling that $\xi \in \left(\frac{R}{2}, R\right)$, by relation (2.5), we infer that the desired estimate (2.2) holds.

The proof of the lemma is complete. □

Remark 2.1. *If we additionally assume that $W''(0) < 0$, then there exists an explicitly computable (in terms of the principal eigenvalue of the Dirichlet Laplacian in B_R), critical radius $R_c \in (0, R')$ such that (2.3) admits a nontrivial positive solution which is a global minimizer of $J(\cdot; B_R)$ in $W_0^{1,2}(B_R)$, as long as $R > R_c$. If we further assume that*

$$W'(t) \ge W''(0)t, \quad t \ge 0,$$

then (2.3) for $R \in (0, R_c)$ has no positive solution. These assertions can be proven by adapting Lemma 2.1 in [9].

Remark 2.2. *In the case where $W(0) = 0$ and Ω is Lipschitz, bounded, and star-shaped with respect to some point, then Pohozaev's identity easily implies that there does not exist a nontrivial solution of (1.1) such that $W(u(x)) \ge 0, \ x \in \Omega$ (see for instance relation (11) in* [3]*).*

We can now proceed to the

`Proof of Theorem 1.2:` Once Lemma 2.1 is established, the proof of Theorem 1.2 proceeds in a rather standard way. We will adapt an argument from the proof of

Theorem 11 in [8], and prove existence of the desired solution to (1.1) by the method of upper solutions. We use $\bar{u} \equiv \mu$ as an upper solution (recall that $W'(\mu) = 0$), and

$$\underline{u} = \begin{cases} u_{R'}(x), & x \in B_{R'}(x_0), \\ \\ 0, & x \in \Omega \backslash B_{R'}(x_0), \end{cases}$$

where $u_{R'}$ is as in Lemma 2.1 but centered at x_0, as a lower solution (here we used that $W'(0) \leq 0$ and Proposition 1 in [4] to make sure that $\underline{u}$ is a lower solution). Note that $\underline{u}(x) < \bar{u}(x),\ x \in \Omega$. In the case where Ω is bounded, it follows immediately from the method of monotone iterations, see Theorem 2.3.1 in [19], that there exists a solution u of (1.1) such that

$$\underline{u}(x) < u(x) < \bar{u}(x), \quad x \in \Omega.$$

The same property also holds in the case where Ω is unbounded, by exhausting it with a sequence of bounded domains, see Theorem 2.10 in [16] (also recall our discussion following the statement of Theorem 1.2). The validity of estimates (1.2) and (1.5) follows at once, in view of (2.1), (2.2).

The proof of the theorem is complete. □

Remark 2.3. *The same assertions of Theorem 1.2 continue to hold if assumption **(a')** is replaced by the weaker one*

(a"): *There exists a constant $\mu > 0$ such that*

$$0 = W(\mu) \leq W(t),\ t \in [0, \infty),$$

$$\textit{there exist } \delta,\ a > 0 \textit{ such that } W^{-1}(0, \delta) \cap [0, \mu) \subseteq (\mu - a, \mu),$$

$$W(-t) \geq W(t),\ t \in [0, \infty),$$

which allows W to have zeros in any interval containing μ.

Remark 2.4. *We recently found the paper* [2]*, where it is stated that G. Fusco, in work in progress, has been able to remove the corresponding monotonicity assumption to **(b)** from the vector-valued Allen-Cahn type equation that was treated in* [1]*. After the first version of the current paper was completed, I was informed by G. Fusco that himself, F. Leonetti and C. Pignotti are working in a paper where, using the same technique developed for the vector case, they also extend the result in* [12] *to general potentials.*

Acknowledgment. The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7-REGPOT-2009-1) under grant agreement n^{o} 245749.

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