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# Vortex lattices for ultracold bosonic atoms in a non-Abelian gauge potential

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The use of coherent optical dressing of atomic levels allows the coupling of ultracold atoms to effective non-dynamical gauge fields. These can be used to generate effective magnetic fields, and have the potential to generate non-Abelian gauge fields. We consider a model of a gas of bosonic atoms coupled to a gauge field with  $U(2)$  symmetry, and with constant effective magnetic field. We include the effects of weak contact interactions by applying Gross-Pitaevskii mean-field theory. We study the effects of a  $U(2)$  non-Abelian gauge field on the vortex lattice phase induced by a uniform effective magnetic field, generated by an Abelian gauge field or, equivalently, by rotation of the gas. We show that, with increasing non-Abelian gauge field, the nature of the groundstate changes dramatically, with structural changes of the vortex lattice. We show that the effect of the non-Abelian gauge field is equivalent to the introduction of effective interactions with non-zero range. We also comment on the consequences of the non-Abelian gauge field for strongly correlated fractional quantum Hall states.

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## I. INTRODUCTION

Atomic Bose-Einstein Condensates (BECs) offer the possibility to study the physics of quantised vortex lines with unprecedented precision and control [1, 2]. Experiments on rapidly rotating gases [3–6] have allowed detailed studies of remarkable states such as large arrays of vortices, forming vortex lattices. Their static features, dynamics and response to periodic lattice potentials have been investigated.

As an alternative to rotation, one can use the dressing by coherent optical fields to create an effective  $U(1)$  gauge potential which simulates the orbital effects of a magnetic field on a charged particle [7]. Using such optically induced gauge fields, the formation of quantized vortices in a rubidium condensate has been demonstrated in pioneering experimental work [8]. Optically induced gauge potentials are not limited to Abelian gauge fields, but can be naturally extended to the non-Abelian case [7]. There exists a variety of proposed ways to generate non-Abelian gauge potentials, both in the continuum [9, 10] and in lattice-based systems [11–13]

In this paper, we study the consequences of a  $U(2)$  non-Abelian gauge field on the groundstate of a weakly interacting atomic BEC. We focus on a gauge-field configuration in which the effective magnetic field is constant in space, and for which there exists a simple exact solution for the single particle wavefunctions [14–17]. In the case of a uniform Abelian magnetic field (or for uniform rotation of the gas) the spectrum has the Landau level structure. In this case, for weak repulsive interparticle interactions the bosonic atoms occupy the lowest energy Landau level and the mean-field ground state is well-known to be a lattice of vortices with triangular symmetry. It is interesting to ask how the addition of a constant non-Abelian magnetic field affects this groundstate.

We show that the effects of the additional non-Abelian

magnetic field can be understood in terms of a change in the effective interatomic interaction potential, and are equivalent to the effects of an interaction potential with non-zero range. Specifically, the effects of the non-Abelian gauge field are completely encoded by the Haldane pseudopotentials that describe the interatomic interactions in the lowest energy single-particle states. We study the consequences on the condensed vortex lattice phases by applying Gross-Pitaevskii mean-field theory to the weakly interacting gas. We show that the nature of the ground state changes dramatically, with structural changes in the symmetry of the vortex lattice brought about with increasing non-Abelian gauge field. We show that these changes are precisely analogous to the introduction of a long-range interaction, as has previously been studied in the context of dipolar interactions [18].

The paper is organised as follows. In Sec. II we introduce the non-Abelian gauge field. In Sec. III we introduce the interaction Hamiltonian and the corresponding Haldane pseudopotentials. In Sec. IV we present the vortex lattices. In Sec. V we comment on fractional quantum Hall states in the system. Sec. VI contains our concluding remarks. Some calculational details are relegated to Appendices.

## II. A NON-ABELIAN GAUGE FIELD

The Hamiltonian for a non-relativistic particle of mass  $m$  and charge  $q$  reads

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 \quad (1)$$

where  $\vec{p}$  is the particle momentum and  $\vec{A}$  is a vector potential. For an Abelian gauge field, the magnetic field is given by the standard relation  $\vec{B} = \vec{\nabla} \times \vec{A}$ . We are

interested in the case of a non-Abelian vector potential which is written in the form

$$\vec{A} = A_x \vec{u}_x + A_y \vec{u}_y + A_z \vec{u}_z \quad (2)$$

and the components  $A_\mu$  are matrices acting on a set of states which, in the present context, correspond to a set of degenerate dressed states[7]. In the simplest case there are two such internal states, which we label by  $\sigma = \uparrow, \downarrow$ , and  $A_\mu$  are  $2 \times 2$  Hermitian matrices. The gauge group is then  $U(2) = U(1) \times SU(2)$  and thus contains the standard  $U(1)$  potential and a  $SU(2)$  part. The corresponding magnetic field is [19]

$$\vec{B} = \vec{\nabla} \times \vec{A} - i \frac{q}{\hbar} \vec{A} \times \vec{A}. \quad (3)$$

It contains an additional cross product term, which is zero for standard Abelian fields. This relation shows that, in the case that the components of  $\vec{A}$  are non-commuting, a nonzero magnetic field is obtained even from a uniform vector potential.

Let us consider a uniform magnetic field perpendicular to the plane  $\vec{B} = B_z \vec{u}_z$ , where  $B_z$  is a  $2 \times 2$  hermitian matrix. The magnetic field can be assumed to be diagonal by an appropriate choice of basis and we write  $B_z = B(\mathcal{I} + 2\beta^2 \sigma_z)$ , where  $\mathcal{I}$  is the identity matrix,  $\sigma_z$  is the diagonal Pauli matrix, and  $\beta$  is a parameter controlling the size of the non-Abelian part of the field. The first Abelian term in such a magnetic field can be produced in the standard way by the rotation term in Eq. (3). The second term of the magnetic field is produced by a vector potential whose components are non-commuting constant matrices, through the second term on the right-hand side of Eq. (3). The complete non-Abelian vector potential may be chosen in the form

$$\vec{A} = -B y \mathcal{I} \vec{u}_x + \beta' (-\sigma_y \vec{u}_x + \sigma_x \vec{u}_y) \quad (4)$$

where  $\sigma_x, \sigma_y$  are Pauli matrices. The constants in the magnetic field and the vector potential are related by  $\beta = \beta' / (l_B B)$ , where we have introduced the magnetic length  $l_B \equiv \sqrt{\hbar / (qB)}$ . We have chosen the Landau gauge for the Abelian part, while the second, non-Abelian term has been chosen to stand in analogy to the symmetric gauge. Up to an overall gauge transformation this gauge potential is equivalent to the gauge potentials studied in Ref. [14, 16] and for the symmetric case  $|a| = |b|$  in Ref. [17]. A method for implementing such a non-Abelian gauge field is described in Ref. [15]. Related gauge fields and their effects have also been studied on the lattice [12, 13, 20].

We introduce the standard creation and annihilation operators  $a^\dagger, a$  for the Landau level problem [16] and the spin ladder operators  $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ , and then the Hamiltonian is written in the form

$$H = \hbar\omega_c \left[ a^\dagger a + \sqrt{2}\beta (a^\dagger \sigma_+ + a \sigma_-) + \frac{1}{2} + \beta^2 \right], \quad (5)$$

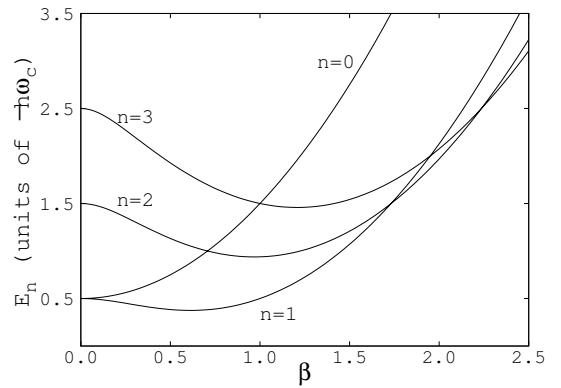


FIG. 1: The energy levels (6) for  $n = 0, 1, 2, 3$  as functions of  $\beta$ . We find that  $E_1$  is the minimum energy for  $0 < \beta < \sqrt{3}$  while  $E_2$  is the minimum for  $\sqrt{3} < \beta < \sqrt{5}$ .

where  $\omega_c = qB/m$  is the cyclotron frequency. This is equivalent to the Jaynes-Cummings Hamiltonian[21] and its spectrum is

$$E_0 = \hbar\omega_c \left( \frac{1}{2} + \beta^2 \right) \\ E_n^\pm = \hbar\omega_c \left( n \pm \frac{\beta_n}{2} + \beta^2 \right), \quad \beta_n \equiv \sqrt{8\beta^2 n + 1}, \quad n \geq 1. \quad (6)$$

We are interested in the ground state as a function of the parameter  $\beta$ . Since  $E_n^+ \geq E_n^-$  we will only discuss  $E_n = E_n^-$ . Fig. 1 shows the first few energy levels as a function of  $\beta$ . One finds that  $E_1$  is the lowest energy for  $0 \leq \beta < \sqrt{3}$ , while the ground state energy is  $E_n$  for  $\sqrt{2n-1} < \beta < \sqrt{2n+1}$  when  $n > 1$ .

Let us denote the Landau level states by  $\phi_{n,k}$  where  $n$  is the Landau level index and  $k$  is an index for the degenerate states in each Landau level. The two spin states will be denoted by the symbols  $\uparrow$  and  $\downarrow$  so the product states of the Landau level states with the spin states are denoted as  $\phi_{n,k,\uparrow}, \phi_{n,k,\downarrow}$ . The normalized eigenfunctions for the energies  $E_n$  are

$$\psi_{n,k} = \left( \frac{\beta_n + 1}{2\beta_n} \right)^{1/2} \phi_{n-1,k,\downarrow} - \left( \frac{\beta_n - 1}{2\beta_n} \right)^{1/2} \phi_{n,k,\uparrow}. \quad (7)$$

In Appendix A, we give explicit expressions for the spatial dependence of the wavefunctions  $\phi_{n,k}$  in the periodic geometry used below for our numerical calculations.

### III. INTERACTION HAMILTONIAN

We assume that the bosons are interacting. We use a second-quantized description, and write  $\psi_\sigma^\dagger(\vec{r})$  and  $\psi_\sigma(\vec{r})$  for the creation and annihilation operators of a particle with spin  $\sigma$  at position  $\vec{r}$ . The interaction Hamiltonian

for two-body interactions has the form

$$H_I = \frac{1}{2} \sum_{\sigma\sigma'} \int g_{\sigma\sigma'}(|\vec{r}-\vec{r}'|) \psi_{\sigma'}^{\dagger}(\vec{r}') \psi_{\sigma}^{\dagger}(\vec{r}) \psi_{\sigma}(\vec{r}) \psi_{\sigma'}(\vec{r}') d^2\vec{r} d^2\vec{r}'. \quad (8)$$

We have assumed that the interaction potential depends on the distance between the particles and it may also depend on the spins. We have made the additional assumption that the spins of the bosons do not change due to scattering. We suppose a contact interaction potential of the form  $g_{\sigma\sigma'}(|\vec{r}-\vec{r}'|) = g \delta(\vec{r}-\vec{r}')$  and obtain

$$H_I = \frac{g}{2} \sum_{\sigma\sigma'} \int \psi_{\sigma'}^{\dagger} \psi_{\sigma}^{\dagger} \psi_{\sigma} \psi_{\sigma'} d^2\vec{r}. \quad (9)$$

(It is straightforward to retain the spin-dependence to the interactions, which will introduce additional parameters to the model. We restrict attention to the spin-independent case to simplify presentation.)

We consider, first, the effect of interactions within the basis of single-particle eigenstates (7). It is convenient to consider states of definite angular momentum  $m$ . We assume that the interactions are small compared to the energy level spacings between states with different quantum numbers  $n$ , so we focus only on the scattering of pairs of particles with the same value of  $n$ , but between different angular momentum states: that is  $(n, m_1), (n, m_2) \rightarrow (n, m'_1), (n, m'_2)$ . Owing to the rotational symmetry of the (contact) interactions, all two-body scattering processes (9) preserve the total angular momentum of the pair of particles,  $m'_1 + m'_2 = m_1 + m_2$ . As a result, the two-particle states with definite relative angular momentum,  $m$ , are eigenstates of the interaction Hamiltonian; their eigenvalues are the Haldane pseudopotentials[22, 23],  $V_m^{(n)}$ , which encode the entire properties of the interactions within the energy band labelled by  $n$ . For the states (7) these take the form

$$V_m^{(n)} = \frac{g}{4\pi} \int_0^{\infty} \left[ \frac{\beta_n + 1}{2\beta_n} L_{n-1}(q^2/2) + \frac{\beta_n - 1}{2\beta_n} L_n(q^2/2) \right]^2 [L_m(q^2)]^2 e^{-q^2} dq^2. \quad (10)$$

We use here and in the following as a unit of length the magnetic length  $l_B$ .

For bosons, symmetry of the two-particle wave functions means that only even  $m = 0, 2, 4, \dots$  can contribute. For the usual case of rapidly rotating bosons in the lowest Landau level,  $n = 0$ , interacting via contact interactions [2], the only non-zero pseudopotential is  $V_0$ . This represents the situation in which interactions have the shortest possible range. Explicitly introducing a long-range interaction potential, such as the dipolar potential with  $g(r) \propto 1/r^3$  at large distances, leads to non-zero values of pseudopotentials with  $m > 0$ [18]. We find, from an analysis of (10), that when the non-Abelian gauge field is present,  $\beta \neq 0$ , a set of pseudopotentials with  $m > 0$  are also non-zero even for contact interactions  $g(|\vec{r}|) \propto \delta(\vec{r})$ . We therefore argue that *the effect of the*

*non-Abelian gauge field on the properties of the lowest energy state is equivalent to the introduction of effective interactions with non-zero range.*

In the case of dipolar interactions, these changes to the effective interaction were found to lead to changes in the lowest energy vortex lattice phase [18]. In the following we investigate the effects on the vortex lattices due to the changes in effective interaction caused by the non-Abelian gauge field. We first give results of numerical simulations and we then make the connection to the calculated values of the pseudopotentials  $V_m^{(n)}$ .

#### IV. VORTEX LATTICE PHASES

To study vortex lattices of an infinite system, it is convenient to work in a rectangular geometry with periodic boundary conditions in the two spatial co-ordinates, that is with the topology of a torus. We consider a rectangular cell of size  $L_x$  and  $L_y$  in the  $x$  and  $y$  directions respectively. This spatial periodicity imposes the condition

$$L_x L_y = 2\pi l_B^2 N_V, \quad (11)$$

where  $N_V$  is a positive integer equal to the number of states in the cell for each Landau level. This can be interpreted as the number of vortices within the cell. Depending on the symmetry of the vortex lattice phase (described within mean field theory below), the aspect ratio of the cell  $L_y/L_x$  must be chosen to match the natural periods of the vortex lattice.

For small enough values of the interaction strength  $g\bar{\rho} \ll \hbar\omega_c$ , where  $\bar{\rho} \equiv N/L_x L_y$  is the mean particle density, we can assume that the interaction potential does not mix excited states of the non-interacting Hamiltonian. Therefore the ground state of the interacting system is found by minimizing the interaction energy within the space of the wave functions (7) for a specific  $n$ .

We make the ansatz

$$\Psi_n = \sum_k c_k \psi_{n,k} \quad (12)$$

where  $c_k$  are complex coefficients, and  $\psi_{n,k}$  are the single particle wavefunctions (7) for the periodic geometry (see Appendix A). The choice for the Landau level index  $n$  depends on the value of  $\beta$ . We minimize the energy

$$E_I \equiv \frac{g}{2} \sum_{k_1, k_2, k_3, k_4} V_{k_1, k_2, k_3, k_4} c_{k_1}^* c_{k_2}^* c_{k_3} c_{k_4} \quad (13)$$

(the form of the interaction potentials  $V_{k_1, k_2, k_3, k_4}$  is given in Appendix B) in the subspace of wave functions (12) for a given number of particles

$$N = \sum_k c_k^* c_k. \quad (14)$$

The mean density is then  $\bar{\rho} = \frac{N}{N_V} \frac{1}{2\pi l_B^2}$ , and the condition that mean-field theory provides an accurate description is that  $N/N_V \gg 1$  [2].

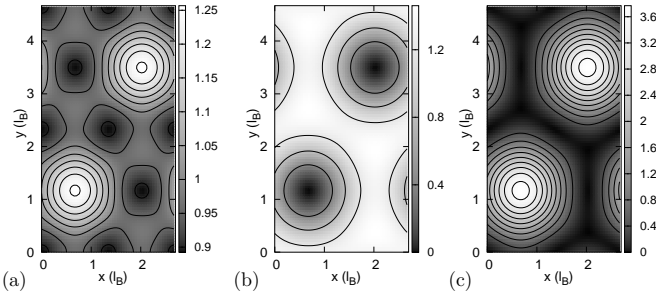


FIG. 2: Contour plots for the particle density of a triangular vortex lattice for  $\beta = 1.0$  (aspect ratio  $a_r = \sqrt{3}$ ). We present one cell which contains  $N_V = 2$  vortices. (a) The total particle density (in units of the mean particle density). The particle density is nowhere zero, since the single particle wavefunctions involve contributions from more than one Landau level, as shown in Eqs. (7), (12). (b) The density for particles in the lowest Landau level (in units of the mean particle density in the lowest Landau level). (c) The density for particles in the first Landau level (in units of the mean particle density in the first Landau level).

We find minima of the energy (13) numerically using the Fletcher-Reeves-Polak-Ribiere method [25] which uses derivatives of the function to be minimized. The number of basis wave functions in the ansatz (12) is  $N_V$  and we used values up to  $N_V = 64$  for which the method rapidly converges to a minimum. Setting  $N_V$  to a certain value we fix the number of Landau level states which are introduced in the area  $[L_x, 0] \times [0, L_y]$ , and the cell area is fixed through Eq. (11). As a second parameter we choose the aspect ratio  $a_r = L_y/L_x$  of the rectangular cell. We have explored the minima of the interaction energy in the space of the two parameters  $N_V, a_r$ . For every state under consideration we evaluate the quantity

$$b \equiv \frac{E_I}{g N^2} (4\pi N_V), \quad (15)$$

where  $E_I$  is the interaction energy and  $N$  is the number of particles in a cell with  $N_V$  vortices, from which the chemical potential is  $\mu = \frac{dE}{dN} = bg\bar{\rho}$ .

For  $\beta = 0$  the ground state wave functions are the lowest Landau level states  $\phi_{0,k,\downarrow}$ , and we can verify that a triangular vortex lattice (identical to Fig. 1a in Ref. [26]) is the lowest energy state with  $b = 1.596$ , as expected by the equivalent problem in the Ginsburg-Landau model for type-II superconductors [27, 28].

For  $0 < \beta < \sqrt{3}$  the ground state wave functions are obtained by setting  $n = 1$  in Eq. (7), and they are superpositions of the lowest and first Landau level states. Minimizing (13) we find a triangular lattice as the lowest energy wave function for the range  $0 \leq \beta \leq 1.13$ . Note, however, that the groundstate wave function depends on  $\beta$  explicitly through the coefficients of  $\phi_{0,k,\downarrow}$  and  $\phi_{1,k,\uparrow}$  in Eq. (7). In Fig. 2 we present plots of the particle density for  $\beta = 1.0$ . We plot the total density as well as the densities for the two Landau levels of the wavefunction. As  $\beta$  increases the contribution of  $\phi_{1,k,\uparrow}$  in the wave function

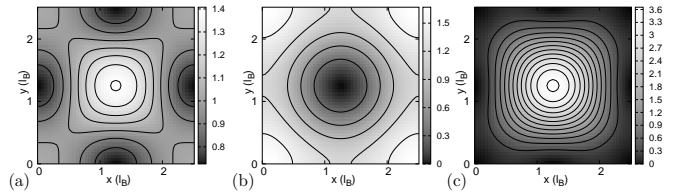


FIG. 3: Contour plots for the particle density of a square vortex lattice for  $\beta = 1.5$  (aspect ratio  $a_r = 1$ ). We present one cell which contains  $N_V = 1$  vortex. (a) The total particle density. The particle density is nowhere zero, since the single particle wavefunctions involve contributions from more than one Landau level. (b) The density for particles in the lowest Landau level. (c) The density for particles in the first Landau level.

increases as indicated in Eq. (7). The contribution to the particle density from the first Landau level wave function  $\phi_{1,k,\uparrow}$  has maximum values where the vortex centers for  $\phi_{0,k,\uparrow}$  are located. These remarks suffice in order to follow the change of the particle density in the lattice as  $\beta$  increases. Note that we have density minima, but no zeros of the density, in Fig. 2a.

For  $1.13 < \beta < \sqrt{3}$  our numerical method converges to a square lattice which has lower energy than any other state we investigated. Fig. 3 shows the result of the energy minimization in a single unit cell for a square lattice for  $\beta = 1.5$ .

A transition from a triangular to square and other vortex lattices has been investigated for the case of long-range dipolar interactions in the LLL [18]. In order to quantify the contribution of non-local interactions we calculate the values of pseudopotentials  $V_m^{(n)}$ . It is convenient to define as a control parameter the ratio of the two first nonzero pseudopotentials at a certain Landau level [26]

$$\alpha \equiv \frac{V_2}{V_0}. \quad (16)$$

We have simplified the notation setting  $V_m = V_m^{(1)}$ . If  $V_m = 0$  for  $m > 2$  a transition from triangular to square lattice occurs at  $\alpha = \alpha_1 \equiv 0.0865$  [26]. For  $n = 1$  in the present model we find that  $V_m = 0$  for  $m > 2$ , and the value  $\alpha = \alpha_1$  is obtained for  $\beta = 1.13$ . Therefore, the transition to a square lattice for  $\beta > 1.13$  reported here is in perfect agreement with the results of Ref. [26]. The parameter  $\alpha$  increases with  $\beta$ , and for  $\beta = \sqrt{3}$  we have  $\alpha = 1.118$ . In further agreement of the present results to the results of Ref. [26], no transition to a vortex phase different than the square lattice is found for these parameter values.

When  $\beta$  increases to values  $\sqrt{3} < \beta < \sqrt{5}$  the ground state of the non-interacting problem is a superposition of the first and second Landau levels obtained from wave function (7) with  $n = 2$ . Fig. 4 shows the particle density of the ground state for  $\beta = 2.0$ . The picture is similar for the whole range of values of  $\beta$ . The particle density has

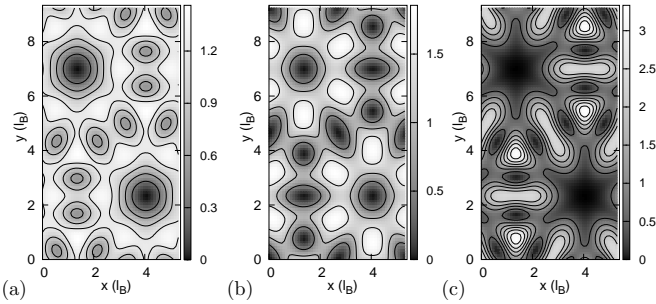


FIG. 4: Contour plots for the particle density of a  $q = 4$  bubble crystal lattice for  $\beta = 2.0$  (aspect ratio  $a_r = \sqrt{3}$ ). We present a cell which contains  $N_V = 8$  vortices. (a) The total particle density. The particle density is zero on the sites of a triangular lattice. (b) The density for particles in the first Landau level. (c) The density for particles in the second Landau level.

vortex minima arranged in a triangular lattice and these are surrounded by density dips. The clusters of vortex and density dips may be called *bubbles* and we thus have a bubble crystal phase in Fig. 4.

Following Refs. [18, 26] we may characterize the vortex lattice in Fig. 4 as a  $q = 4$  bubble as each unit cell contains 4 vortices. The particular bubble phase of Fig. 4 is similar to that in Fig. 1e in Ref. [26]. The Haldane pseudopotentials for  $\beta = \sqrt{3}$  are  $\alpha = V_2/V_0 = 0.65$  and  $V_4/V_0 = 0.16$ , where we use the simplified notation  $V_m = V_m^{(2)}$ . The vortex lattice for this value of  $\alpha$  gives a  $q = 4$  bubble state in the case of dipolar interactions [18] as well as in the model of Ref. [26]. The present results are thus in agreement with the preceding reports showing that the non-Abelian gauge potential has an effect analogous to long-range interactions on the vortex lattice phases.

Note that we find no evidence of the appearance of “stripe crystal” phases, which appear for  $0.12 \leq \alpha \leq 0.70$  for the model of Ref. [26]. For the present model we have shown that  $0 < \alpha < 0.12$  for  $0 < \beta < \sqrt{3}$  and  $\alpha > 0.70$  for  $\beta > \sqrt{3}$  as a result of the transition for the groundstate in the noninteracting model at  $\beta = \sqrt{3}$ . A direct transition from square lattice to bubble crystal at  $\beta = \sqrt{3}$  is therefore consistent with the results of Ref. [26].

## V. CONSEQUENCES FOR FRACTIONAL QUANTUM HALL STATES

The above mean-field studies are valid in the regime of high filling factor  $\nu \equiv N/N_V \gg 1$  [2]. For small values of the filling factor, the vortex lattice phases can be replaced by strongly correlated phases, which are bosonic analogues of fractional quantum Hall (FQH) states. For atoms in a uniform Abelian gauge field interacting with contact interactions, the groundstate at  $\nu = 1/2$  is the Laughlin state[29], and those at  $\nu = 1, 3/2, 2$  are well de-

scribed by the Moore-Read and Read-Rezayi states[30]. Variations in the (ratios of the) Haldane pseudopotentials can lead to changes in the nature of the groundstate.

Our result, Eqn (10), provides the values of the pseudopotentials for the non-Abelian gauge potential studied here. By combining these results with existing numerical studies of the effects of variations of the pseudopotentials on strongly correlated phases[18, 26, 31, 32] we can deduce consequences of the non-Abelian gauge field for the FQH states.

For filling factor  $\nu = 1/2$  the effect of changing the Haldane pseudopotentials was studied in Refs. [18, 26]. For a model with only the lowest two pseudopotentials, with ratio  $\alpha = V_2/V_0$ , it was found that the groundstate is described by the bosonic Laughlin state for  $\alpha \lesssim 0.4$  [26]. For  $\alpha \gtrsim 0.4$  the groundstate is a compressible crystalline phase related to the bubble crystal phase of mean-field theory described above. For larger filling factors,  $\nu > 1/2$  the crystalline phase becomes increasingly stable. Thus, for  $\alpha \gtrsim 0.4$  the groundstates at all filling factors  $\nu \geq 1/2$  are compressible crystalline states. This has been confirmed in numerical studies for filling factors  $\nu = 1, 3/2, 2$ [31–33].

In the present model, with a non-Abelian gauge field, we have shown that for  $0 \leq \beta \leq \sqrt{3}$  the lowest energy single particle state has  $n = 1$  and only the pseudopotentials  $V_0$  and  $V_2$  are nonzero. The ratio  $\alpha = V_2/V_0$  is in the range  $0 \leq \alpha \leq 0.118$ . Thus, over this range, one expects the groundstate at  $\nu = 1/2$  to be an incompressible liquid that is well described by the Laughlin state[26]. Similarly, over this range,  $\alpha$  is sufficiently small that the groundstates at  $\nu = 1, 3/2$  and  $2$  are well described by the Moore-Read state, the  $k = 3$  Read-Rezayi state, and the  $k = 4$  Read-Rezayi state respectively[30]. Indeed, a small non-zero value of  $\alpha$  has been shown to tune the system into a regime where these very interesting non-Abelian quantum Hall phases describe the groundstate accurately[31–33].

For  $\sqrt{3} < \beta < \sqrt{5}$ , the lowest energy state has  $n = 2$ . The non-zero pseudopotentials are  $V_0, V_2$  and  $V_4$ . Over this range of  $\beta$ , the ratio  $\alpha = V_2/V_0 \simeq 0.65$ , while  $V_4/V_0 \lesssim 0.2$ . Neglecting the effect of this small value of  $V_4$ , we can again make use of the results for the pure  $V_0$ - $V_2$  model. (Neglecting  $V_4$  was accurate for interpreting the mean-field groundstate described above.) Now, the ratio  $\alpha = 0.65$  is so large ( $> 0.4$ ) that the groundstate at all filling factors  $\nu \geq 1/2$  is expected to be a compressible crystalline state[26, 31–33].

Thus we can conclude that, for the filling factors  $\nu = 1/2, 1, 3/2, 2$ , there is a transition from incompressible quantum liquid states (Laughlin and Read-Rezayi states) to a compressible crystalline state as the non-Abelian field is increased through  $\beta = \sqrt{3}$ . This conclusion is in agreement with independent exact diagonalization calculations reported in Ref.[17] in those parameter regimes for which there is overlap.

## VI. CONCLUSIONS

We have studied vortex phases in a model for charged particles in a non-Abelian gauge potential pertaining to a  $U(2)$  symmetry. Applying Gross-Pitaevskii mean field theory we have considered the effect of contact interactions between particles. These lead to the formation of triangular, square, and bubble crystal lattices for increasing values of the parameter for the non-Abelian term in the gauge potential. We calculated the Haldane pseudopotentials for the energy states of the non-Abelian model and find that they indicate effective interactions with non-zero range in the system. We find a general agreement with results on the effect of long-range interactions [18, 26]. We conclude that the effect of the non-Abelian gauge field on the properties of the lowest energy state is equivalent to the introduction of effective interactions with non-zero range.

We suppose throughout that the interactions are small compared to Landau level spacing. However, at values of the parameter  $\beta \sim \sqrt{2n+1}$  for which the successive energy levels are the non-Abelian model cross, our approximation is not valid, since Landau level mixing [34] is then expected even for small interactions. We have also considered only the case of a uniform non-Abelian gauge field. For a non-uniform gauge field one can consider two limiting cases, in which the lengthscale on which the gauge field varies is either large or small compared to the vortex spacing, as set by the magnetic length  $l_B$ . For a gauge field that varies slowly on the scale of  $l_B$ , one can apply our results within a local density approximation: spatial variations in the gauge field can then induce spatial variations in the vortex lattice structure. For a gauge field that varies more rapidly in space – for example with periodic variations on a lengthscale of order  $l_B$  – separate considerations are required. The single particle energy eigenstates will not be of the form described here, and the degenerate lowest Landau level will be broadened into a band with nonzero bandwidth. The nature of the condensed groundstate will then depend on whether the chemical potential (set by the mean interaction energy) is small or large compared to this bandwidth.

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### Appendix A: Spatially periodic wave functions

The single-particle energy eigenstates of the non-Abelian gauge field are given by Eqn. (7), in which  $\phi_{n,k}$

are the normalized Landau level wavefunctions for uniform Abelian magnetic field. For the periodic system we study, with a rectangular cell of size  $L_x \times L_y$ , these functions must have spatial periods of  $L_x$  and  $L_y$  in the  $x$  and  $y$  directions.

The lowest Landau level wave functions with these periodicities are [24]

$$\phi_{0,k} = \frac{1}{(L_x \pi^{1/2})^{1/2}} \sum_{p=-\infty}^{\infty} e^{i(Y_k + pL_y)x} e^{-\frac{1}{2}(Y_k + pL_y - y)^2}, \quad (\text{A1})$$

$$Y_k \equiv \frac{2\pi}{L_x} k,$$

where  $k$  takes the integer values  $k = 0, \dots, N_V - 1$ , with  $N_V$  given by Eqn. (11). Similarly, the spatially periodic wave functions in the first and second Landau levels are

$$\phi_{1,k} = \frac{1}{(L_x 2\pi^{1/2})^{1/2}} \sum_{p=-\infty}^{\infty} 2(Y_k + pL_y - y) e^{i(Y_k + pL_y)x} e^{-\frac{1}{2}(Y_k + pL_y - y)^2}, \quad (\text{A2})$$

$$\phi_{2,k} = \frac{1}{(L_x 8\pi^{1/2})^{1/2}} \sum_{p=-\infty}^{\infty} [4(Y_k + pL_y - y)^2 - 2] e^{i(Y_k + pL_y)x} e^{-\frac{1}{2}(Y_k + pL_y - y)^2}. \quad (\text{A3})$$

These wavefunctions are orthonormal within the periodic cell. Consider, for example, the wave functions (A1). We have

$$\int_0^{L_x} \int_0^{L_y} \phi_{0,k_1}^* \phi_{0,k_2} dx dy = \sum_{p_1, p_2} \frac{\delta_{k_1 + p_1 N_V, k_2 + p_2 N_V}}{\pi^{1/2}} \int_0^{L_y} e^{-(Y_{k_1} + p_1 L_y - y)^2} dy, \quad (\text{A4})$$

where all summations in the  $p$  symbols (here and in the following) extend from  $-\infty$  to  $\infty$ . The  $\delta$  symbol in the latter equation gives zero for every  $p_1 \neq 0 \neq p_2$  since the  $k$ 's take values in the range  $0 \leq k_1, k_2 < N_V$ . Therefore it is equal to the product  $\delta_{k_1, k_2} \delta_{p_1, p_2}$ . Using the result

$$\sum_p \int_0^{L_y} dy e^{-(Y_m + pL_y - y)^2} = \int_{-\infty}^{\infty} dy e^{-(y - Y_m)^2} = \sqrt{\pi}, \quad (\text{A5})$$

which we substitute in Eq. (A4), we finally find that  $\phi_{0,k}$  are orthonormal. A similar procedure for the first ( $n = 1$ ) and second ( $n = 2$ ) Landau level wave functions proves the orthonormality condition

$$\int_0^{L_x} \int_0^{L_y} \phi_{n,k_1}^* \phi_{n,k_2} dy dx = \delta_{k_1, k_2}. \quad (\text{A6})$$

## Appendix B: Interaction potentials

The interaction potentials entering in the calculation of the interaction energy (13) are of the form

$$\begin{aligned}
 V_{k_1, k_2, k_3, k_4} &= \left( \frac{\beta_n + 1}{2\beta_n} \right)^2 \left[ V_{k_1, k_2, k_3, k_4}^{\downarrow \downarrow \downarrow \downarrow} \right. \\
 &\quad + \frac{\beta_n - 1}{\beta_n + 1} (V_{k_1, k_2, k_3, k_4}^{\uparrow \downarrow \downarrow \uparrow} + V_{k_1, k_2, k_3, k_4}^{\downarrow \uparrow \uparrow \downarrow}) \\
 &\quad \left. + \left( \frac{\beta_n - 1}{\beta_n + 1} \right)^2 V_{k_1, k_2, k_3, k_4}^{\uparrow \uparrow \uparrow \uparrow} \right], \quad (\text{B1})
 \end{aligned}$$

where

$$\begin{aligned}
 V_{k_1, k_2, k_3, k_4}^{\downarrow \downarrow \downarrow \downarrow} &\equiv \int \phi_{n-1, k_1}^* \phi_{n-1, k_2}^* \phi_{n-1, k_3} \phi_{n-1, k_4} d^2 \vec{r}, \\
 V_{k_1, k_2, k_3, k_4}^{\uparrow \uparrow \uparrow \uparrow} &\equiv \int \phi_{n, k_1}^* \phi_{n, k_2}^* \phi_{n, k_3} \phi_{n, k_4} d^2 \vec{r}, \\
 V_{k_1, k_2, k_3, k_4}^{\uparrow \downarrow \downarrow \uparrow} &\equiv \int \phi_{n, k_1}^* \phi_{n-1, k_2}^* \phi_{n-1, k_3} \phi_{n, k_4} d^2 \vec{r}, \\
 V_{k_1, k_2, k_3, k_4}^{\downarrow \uparrow \uparrow \downarrow} &\equiv \int \phi_{n-1, k_1}^* \phi_{n, k_2}^* \phi_{n, k_3} \phi_{n-1, k_4} d^2 \vec{r},
 \end{aligned}$$

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