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MONTE CARLO EULER APPROXIMATIONS
OF HJM TERM STRUCTURE FINANCIAL MODELS

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ABSTRACT. We present Monte Carlo-Euler methods for a weak approximation problem related to the Heath-Jarrow-Morton (HJM) term structure model, based on Itô stochastic differential equations in infinite dimensional spaces, and prove strong and weak error convergence estimates. The weak error estimates are based on stochastic flows and discrete dual backward problems, and they can be used to identify different error contributions arising from time and maturity discretization as well as the classical statistical error due to finite sampling. Explicit formulas for efficient computation of sharp error approximation are included. Due to the structure of the HJM models considered here, the computational effort devoted to the error estimates is low compared to the work to compute Monte Carlo solutions to the HJM model. Numerical examples with known exact solution are included in order to show the behavior of the estimates.

1. THE HJM MODEL

1.1. **Generals.** When valuing derivatives in the bond market it is important to use models that are consistent with the initial term structure observed in the market. The Heath-Jarrow-Morton (HJM) model for the forward rate has this property and in addition offers the freedom to choose the volatility structure, for example to be able to fit other derivative prices quoted in the market (see [6, 7, 15, 19]). This HJM model approach is particularly suitable for Monte Carlo computations, since in general the alternative of tree methods leads, for the multifactor case, to non recombining trees with higher computational cost.

In this work we focus on the numerical approximation of the price of financial instruments in the bond market, using the HJM model of forward rates. We propose Monte Carlo Euler methods fow which we develop a rigorous strong error analysis and provide rigorous weak error expansions, with leading error term in computable a posteriori form, offering computational reliability in the use of more complicated HJM multifactor models, where no explicit formula can be found, or such a formula is just too complicated to use, for the pricing of contingent claims. These weak error expansions can be used in adaptive algorithms to handle *simultaneously* different sources of error, e.g. time discretization, maturity discretization, and finite sampling, see [21]. To develop error estimates we use a Kolmogorov backward equation in an extended domain and carry out further the analysis in [21], from general weak approximation of Itô stochastic differential equations in $\mathbb{R}^n$, to weak approximation of the HJM Itô stochastic differential equations in infinite dimensional spaces. Therefore, the main new ingredient here is to provide error estimates useful for adaptive refinement not only in time t but also in maturity time τ. In addition, using the structure of the HJM model studied here, the application of a simple transformation removes the error caused by the representation of the initial term structure in a finite maturity partition. Finally, the formulas to compute sharp error approximations are simplified by exploiting the structure of the HJM model, reducing the work to compute such error estimates. The use of the error estimates proposed here is compatible with the application of variance reduction techniques, allowing for faster Monte Carlo computations, see [4].

The work at hand is based on a research paper included in the one of the authors PhD Disseration [22].

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1.2. Description of the model. The bond market is assumed to be efficient and without friction, i.e. there is no arbitrage opportunity, and there exists a martingale probability measure, under which bond contracts can be priced as expected values of properly discounted cash flows, see [1, 3, 9]. On what follows, all the equations are assumed to be under such a probability measure.

The HJM model is based on the so called *forward rate*, $f(t, \tau)$, which relates to the price of the most simple type of bond, the zero coupon bond, with contracting time t and maturity time τ, by

$$p(t, \tau) = \exp\left(-\int_t^\tau f(t, \eta)\, d\eta\right).$$

In particular, the non arbitrage assumption in the HJM formulation, see [13, 14], yields an Itô stochastic differential equation, for $\tau \in [0, \tau_{\max}]$,

(1.1)
$$df(t, \tau) = \sum_{j=1}^J \sigma^j(t, \tau)\left(\int_t^\tau \sigma^j(t, s)ds\right) dt + \sum_{j=1}^J \sigma^j(t, \tau)\, dW^j(t), \quad t \in [0, \tau]$$

$$f(0, \tau) = f_o(\tau).$$

Here $(W^j)_{j=1}^J$ are independent Wiener processes, and $(\sigma^j(t, \tau))_{j=1}^J$ are stochastic processes, adapted to the filter structure generated by the Wiener processes. Furthermore, the initial datum for the term structure, $f_o : [0, \tau_{\max}] \to \mathbb{R}$, is a deterministic function in $C^1([0, \tau_{\max}])$. In this setting, the *short rate*, $r(t)$, is defined as $r(t) \equiv f(t, t)$.

On what follows the volatility function $\sigma = (\sigma^1, \ldots, \sigma^J)$ is assumed to be of the form

$$\sigma(t, \tau) = \xi(r(t))\, \lambda(t, \tau)$$
$$= \xi(f(t, t))\, \lambda(t, \tau),$$

where $\xi : \mathbb{R} \to \mathbb{R}$ and $\lambda : [0, t_{\max}] \times [0, \tau_{\max}] \to \mathbb{R}^J$ are given bounded functions on $C^{m_0}(\mathbb{R})$ and $C^{m_0}([0, t_{\max}] \times [0, \tau_{\max}])$, respectively, for m_0 a sufficiently large integer. Then, setting

$$\mathcal{D} \equiv \{(t, \tau) \in [0, t_{\max}] \times [0, \tau_{\max}] : t \leq \tau\}$$

problem (1.1) reads as follows: find $f = f(t, \tau) : \mathcal{D} \to \mathbb{R}$ such that

(1.2)
$$df(t, \tau) = \xi^2(f(t, t))\, \widetilde{\lambda}(t, \tau)\, dt + \xi(f(t, t))\, \lambda(t, \tau) \cdot dW(t), \quad t \in [0, \tau],$$

$$f(0, \tau) = f_o(\tau)$$

for $\tau \in [0, \tau_{\max}]$, where

(1.3)
$$\widetilde{\lambda}(t, \tau) \equiv \lambda(t, \tau) \cdot \int_t^\tau \lambda(t, z)dz, \quad \forall\, t \in [0, \tau], \quad \forall\, \tau \in [0, \tau_{\max}].$$

Here the notation $a \cdot b$ denotes the standard inner product in $\mathbb{R}^J$, i.e. $a \cdot b \equiv \sum_{j=1}^J a_j\, b_j$. In many models used in practice, the function λ has the form $\lambda(t, \tau) = \lambda_o(\tau - t)$, and then $\widetilde{\lambda}(t, \tau) = \widetilde{\lambda}_o(\tau - t)$ with

$$\widetilde{\lambda}_o(\tau - t) \equiv \lambda_o(\tau - t) \cdot \int_0^{\tau - t} \lambda_o(z)\, dz.$$

Observe that to solve for f it is enough to have $\lambda_o : \mathbb{R}^+ \to \mathbb{R}$. However, in this work the usual domain of definition $\mathcal{D}$ of λ and f, extends to the set $[0, t_{\max}] \times [0, \tau_{\max}]$, leaving $f|_{\mathcal{D}}$ unchanged. The extension of $\mathcal{D}$ helps to develop a posteriori approximations for the time and maturity discretization errors, depending on a linear backward problem (cf. Theorem 4.1).

A typical contract to price is a call option, with exercise time $t_{\max}$ and strike price K, on a zero coupon bond. Its price can be written in terms of the forward rate as

$$\mathbb{E}\left[e^{-\int_0^{t_{\max}} f(s,s)\, ds}\ \max\left\{e^{-\int_{t_{\max}}^{\tau_{\max}} f(t_{\max}, \tau)d\tau} - K, 0\right\}\right].$$

Another basic contract is a continuous cap, with price

$$\mathbb{E}\left[\int_0^{t_{\max}} e^{-\int_0^t f(s,s)\, ds}\ (f(t, t) - r_c)^+\ dt\right]$$

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where r_c is a given value associated with the contract. With this motivation, and bearing in mind other possible contracts, we consider the approximation of the quantity

$$(1.4) \qquad\qquad \mathbb{E}\left[\mathcal{F}(f)\right]$$

where the functional $\mathcal{F}(f)$ is given by

$$\mathcal{F}(f) \equiv F\left(\int_0^{t_{\max}} f(s,s)\,ds\right) G\left(\int_{\tau_a}^{\tau_{\max}} \Psi(f(t_{\max},\tau))\,d\tau\right) + \int_0^{t_{\max}} F\left(\int_0^s f(s',s')\,ds'\right) U(f(s,s))\,ds$$

with τ_a being a given positive number such that $0 < t_{\max} \leq \tau_a < \tau_{\max}$. Obviously, $\mathcal{F}(f)$ is written equivalently as

$$(1.5) \qquad\qquad \mathcal{F}(f) \equiv F\left(Y(t_{\max})\right) G\left(\Lambda(\Psi(f(t_{\max},\cdot)))\right) + Z(t_{\max}),$$

where

$$(1.6) \qquad\qquad Y(t) \equiv \int_0^t f(s,s)\,ds, \quad Z(t) \equiv \int_0^t F(Y(s))\,U(f(s,s))\,ds,$$

$$\Lambda(w) \equiv \int_{\tau_a}^{\tau_{\max}} w(\tau)\,d\tau, \quad \forall\, w \in L^1(\tau_a, \tau_{\max}).$$

The functions $F : \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$, $\Psi : \mathbb{R} \to \mathbb{R}$, $U : \mathbb{R} \to \mathbb{R}$, and their derivatives up to a sufficiently large order $m_\star$ are assumed to have a polynomial growth. We say that a function $S : \mathbb{R} \to \mathbb{R}$ has a *polynomial growth* if there exist positive constants k' and C' such that: $|S(x)| \leq C'(1 + |x|^{k'})$ for all $x \in \mathbb{R}$.

Let us consider the system of differential equations (1.2)-(1.3) describing the dynamics for the forward rate f along with that for $Y(t)$ and $Z(t)$, i.e.,

$$(1.7) \qquad \begin{aligned} df(t,\tau) &= \xi^2(f(t,t))\,\widetilde{\lambda}(t,\tau)\,dt + \xi(f(t,t))\,\lambda(t,\tau)\cdot dW(t), \\ dY(t) &= f(t,t)\,dt, \\ dZ(t) &= F(Y(t))\,U(f(t,t))\,dt, \end{aligned}$$

for $t \in [0, t_{\max}]$ and $\tau \in [0\tau_{\max}]$, with the initial conditions

$$(1.8) \qquad\qquad f(0,\tau) = f_0(\tau), \quad Y(0) = 0, \quad Z(0) = 0$$

for $\tau \in [0\tau_{\max}]$.

A approximation error for a typical discretization of the problem above will consists of a $t-$discretization error and a $\tau-$discretization error coming from the discretization of the initial condition f_0. Due to the special structure of (1.7)-(1.8), the initial error can be avoided and practically included in the $t-$discretization error by introducing the anzatz

$$g(t,\tau) = f(t,\tau) - f_0(\tau),$$

which implies $f(t,t) = g(t,t) + f_0(t)$. Thus, (1.7)-(1.8) is formulated as follows: find $g = g(t,\tau) : [0, t_{\max}] \times [0, \tau_{\max}] \to \mathbb{R}$ such that

$$(1.9) \qquad \begin{aligned} dg(t,\tau) &= \xi^2(g(t,t) + f_0(t))\,\widetilde{\lambda}(t,\tau)\,dt + \xi(g(t,t) + f_0(t))\,\lambda(t,\tau)\cdot dW(t), \quad \forall\, t \in [0, t_{\max}], \\ dY(t) &= (g(t,t) + f_0(t))\,dt, \\ dZ(t) &= F(Y(t))\,U(g(t,t) + f_0(t))\,dt, \end{aligned}$$

for $t \in [0, t_{\max}]$, with homogeneous initial conditions

$$(1.10) \qquad\qquad g(0,\tau) = 0, \quad Y(0) = 0, \quad Z(0) = 0$$

for all $\tau \in [0, \tau_{\max}]$. Thus, the quantity we want to approximate takes the form

$$(1.11) \qquad\qquad \mathbb{E}\left[\mathcal{G}(g)\right]$$

where

$$(1.12) \qquad\qquad \mathcal{G}(g) := \mathcal{F}(g + f_0).$$

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In the numerical methods, we describe later, the approximations to Y and Z will be always considered to be respectively the last two components of the approximate solution vector.

1.3. **Overview.** Let us give an overiview of the is organized as follows. In Section 2 first we present two Monte Carlo Euler methods for the HJM model (1.9)-(1.10), namely, a stochastic finite difference method, the Euler Finite Difference method (EFD), and a more accurate stochastic finite element method, the Euler Finite Element method (EFE); then, we combine a numerical quadrature rule and the outcome of the (EFD) or the (EFE) methods to construct a numerical approximation of the functional $\mathbb{E}\left[\mathcal{G}(g)\right]$. In Section 3 we provide a stong convergence analysis for the (EFD) and the (EFE) methods. Section 4 states and proves weak error estimates for the (EFD) method, giving explicit formulas for efficient computation of the discrete duals. Finally, Section 5 presents results from numerical experiments.

2. Monte Carlo Euler Methods

In this section first we introduce two time and maturity time discretizations of (1.9)-(1.10): the Euler-Finite Difference (EFD) method and the Euler-Finite Element (EFE) method. Then, we use the (EFD) or the (EFE) approximations along with a quadrature rule to construct approximations of the quantity of interest $\mathbb{E}\left[\mathcal{G}(g)\right]$ defined in (1.11).

2.1. **Time and maturity time discretization.** Given extreme points $0 < t_{\max} \leq \tau_a < \tau_{\max}$ introduced in Section 1, let N and L denote the number of subintervals on $[0, t_{\max}]$ and $[0, \tau_{\max}]$, respectively. Then, consider partitions

$$0 = t_0 < \cdots < t_N = t_{\max} \quad \text{and} \quad 0 = \tau_0 < \cdots < \tau_L = \tau_{\max}$$

of the t-interval $[0, t_{\max}]$ and of the τ-interval $[0, \tau_{\max}]$, respectively. For technical reasons, these partitions are assumed to satisfy the following condition: every τ-node in the interval $[0, t_{\max}]$ is also a t-node, i.e.

(2.1) $\qquad$ there exists an one-to-one index map ρ, such that, $\quad \tau_\ell = t_{\rho(\ell)} \;$ for $\; \tau_\ell \leq t_{\max}$.

In addition, assume that

(2.2) $\qquad\qquad\qquad$ there exists an index $\; \ell_\star \;$ such that $\; t_{\max} = \tau_{\ell_\star}$

and

(2.3) $\qquad\qquad\qquad$ there exists an index $\; \ell_a \;$ such that $\; \tau_a = \tau_{\ell_a}$.

Also, define the auxiliary index function, ℓ_n, by

(2.4) $\qquad\qquad\qquad \ell_n \equiv \max \left\{ \ell \in \mathbb{Z} : \quad 0 \leq \ell \leq L \quad \text{such that} \quad \tau_\ell \leq t_n \right\}$

introduce the notation

$$\Delta t_n \equiv t_{n+1} - t_n, \quad \Delta W_n \equiv W(t_{n+1}) - W(t_n) \quad \text{for} \quad n = 0, \ldots, N-1,$$

$$\Delta \tau_\ell \equiv \tau_{\ell+1} - \tau_\ell \quad \text{for} \quad \ell = 0, \ldots, L-1,$$

and set $\Delta t \equiv \max_{0 \leq n \leq N-1} \Delta t_n$ and $\Delta \tau \equiv \max_{0 \leq \ell \leq L-1} \Delta \tau_\ell$. Finally, introduce the space of piecewise constant and right continuous functions on a τ-partition, $(\tau_\ell)_{\ell=0}^L$, of the interval $[0, \tau_{\max}]$, by

$$S_{\Delta\tau} \equiv \left\{ \chi \in L^\infty(0, \tau_{\max}) : \text{there are constants} \; (c_\ell)_{\ell=0}^{L-1} \; \text{such that} \; \chi|_{[\tau_\ell, \tau_{\ell+1})} = c_\ell, \quad \ell = 0, \ldots, L-1 \right\}.$$

Define the standard L^2–projection $\Pi : L^2(0, \tau_{\max}) \to S_{\Delta\tau}$ by

$$\int_0^{\tau_{\max}} \Pi v \, \chi \, d\tau = \int_0^{\tau_{\max}} v \, \chi \, d\tau, \quad \forall \chi \in S_{\Delta\tau}, \quad \forall v \in L^2(0, \tau_{\max}),$$

which satisfies

$$\Pi v \big|_{[\tau_\ell, \tau_{\ell+1})} = \tfrac{1}{\Delta \tau_\ell} \int_{\tau_\ell}^{\tau_{\ell+1}} v(\tau) \; d\tau, \quad \ell = 0, \ldots, L-1, \quad \forall v \in L^2(0, \tau_{\max}).$$

For $\chi \in S_{\Delta\tau}$ and $\ell = 0, \ldots, L-1$, denote by χ_ℓ the constant value of χ in $[\tau_\ell, \tau_{\ell+1})$. When considering a function, $w = w(t, \tau)$, depending on two variables, the L^2 projection is always with respect to τ, i.e. for $\ell = 0, \ldots, L-1$ and $\tau \in [\tau_\ell, \tau_{\ell+1})$, we have $\Pi w(t; \tau) \equiv \Pi(w(t, \cdot)) \big|_{[\tau_\ell, \tau_{\ell+1})} = \tfrac{1}{\Delta \tau_\ell} \int_{\tau_\ell}^{\tau_{\ell+1}} w(t, s) \, ds$.

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2.2. The Euler-Finite Difference (EFD) method.

For each time level the (EFD) method approximates $g(t_n,.)$ by a piecewise constant function, $\overline{\overline{g}}(t_n,.) \in S_{\Delta\tau}$. In particular, it finds the approximate values $\overline{\overline{g}}_{n,\ell} \approx g(t_n, \tau_\ell)$ for $\ell = 0, \ldots, L-1$, $\overline{\overline{g}}_{n,L} \approx Y(t_n)$, $\overline{\overline{g}}_{n,L+1} \approx Z(t_n)$ by setting first

$$(2.5) \qquad \overline{\overline{g}}_{0,\ell} \equiv 0, \quad \ell = 0, \ldots, L+1,$$

and, then recursively, for $n = 0, \ldots, N-1$, define

$$
\begin{aligned}
(2.6) \qquad \overline{\overline{g}}_{n+1,\ell} &= \overline{\overline{g}}_{n,\ell} + \Delta t_n \, \xi^2 (\overline{\overline{g}}_{n,\ell_n} + f_0(t_n)) \, \widetilde{\lambda}(t_n, \tau_\ell) \\
&\quad + \xi \left(\overline{\overline{g}}_{n,\ell_n} + f_0(t_n) \right) \lambda(t_n, \tau_\ell) \cdot \Delta W_n, \quad \ell = 0, \ldots, L-1, \\
\overline{\overline{g}}_{n+1,L} &= \overline{\overline{g}}_{n,L} + \Delta t_n \left(\overline{\overline{g}}_{n,\ell_n} + f_0(t_n) \right), \\
\overline{\overline{g}}_{n+1,L+1} &= \overline{\overline{g}}_{n,L+1} + \Delta t_n \, F(\overline{\overline{g}}_{n,L}) \, U \left(\overline{\overline{g}}_{n,\ell_n} + f_0(t_n) \right)
\end{aligned}
$$

where the index ℓ_n has been defined in (2.4).

2.3. The Euler-Finite Element (EFE) method.

The (EFE) method also approximates the τ-function $g(t_n, \cdot)$, by a piecewise constant function $\overline{\overline{g}}(t_n, \cdot) \in S_{\Delta\tau}$, but is based in a variational formulation of (1.9)-(1.10) with $S_{\Delta\tau}$ being the space of trial and test functions. In particular, the (EFE) is defined by the initial datum

$$(2.7) \qquad \overline{\overline{g}}_{0,\ell} \equiv 0, \quad \ell = 0, \ldots, L+1,$$

and, for $n = 0, \ldots, N-1$, the recursion

$$
\begin{aligned}
(2.8) \qquad \overline{\overline{g}}_{n+1,\ell} &= \overline{\overline{g}}_{n,\ell} + \Delta t_n \, \xi^2 \left(\overline{\overline{g}}_{n,\ell_n} + f_0(t_n) \right) \Pi\widetilde{\lambda}(t_n; \tau_\ell) \\
&\quad + \xi \left(\overline{\overline{g}}_{n,\ell_n} + f_0(t_n) \right) \Pi\lambda(t_n; \tau_\ell) \cdot \Delta W_n, \quad \ell = 0, \ldots, L-1, \\
\overline{\overline{g}}_{n+1,L} &= \overline{\overline{g}}_{n,L} + \Delta t_n \left(\overline{\overline{g}}_{n,\ell_n} + f_0(t_n) \right), \\
\overline{\overline{g}}_{n+1,L+1} &= \overline{\overline{g}}_{n,L+1} + \Delta t_n \, F(\overline{\overline{g}}_{n,L}) \, U \left(\overline{\overline{g}}_{n,\ell_n} + f_0(t_n) \right)
\end{aligned}
$$

where the index ℓ_n has been defined in (2.4).

2.4. Approximation of the quantity of interest $\mathbb{E}[\mathcal{G}(g)]$.

The numerical approximation of $\mathcal{G}(g)$ defined in (1.11) involves both an approximation of the processes g, Y, Z, by computable quantities, and an approximation of the τ-integral in (1.6).

To construct an approximation of $\Lambda(\Psi(g(t_{\max}, \cdot) + f_0(\cdot)))$ we apply a composite quadrature formula, over the partition of $[0, \tau_{\max}]$, based on a quadrature rule $Q : C[0,1] \to \mathbb{R}$ with N_Q nodes $s_Q = (s_{Q,i})_{i=1}^{N_Q}$ and weights $w_Q = (w_{Q,i})_{i=1}^{N_Q}$, i.e., for $v \in C([0,1]; \mathbb{R})$ the quantity $Q(v) = \sum_{i=1}^{N_Q} w_{Q,i} \, v(s_{Q,i})$ approximates the integral $\int_0^1 v(x) \, dx$. Also, we assume that the quadrature rule Q is of order p_Q, i.e., it is exact for polynomials of order less or equal to $p_Q - 1$. For example, the Simpson rule has $N_Q = 3$, $s_Q = (0, \frac{1}{2}, 1)$ and $w_Q = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$, with $p_Q = 4$. Another example is the Gaussian quadrature with $N_Q = 2$, $s_Q = (\frac{1}{2} - \frac{1}{2\sqrt{3}}, \frac{1}{2} + \frac{1}{2\sqrt{3}})$, $w_Q = (\frac{1}{2}, \frac{1}{2})$ and $p_Q = 4$. We note that it is well known from the mathematical analysis of numerical quadrature that in general we have $p_Q \leq 2N_Q$, and the maximum value $p_Q = 2N_Q$ is achieved only by the Gaussian quadrature.

Thus, for a fixed realization of $\overline{\overline{g}}$ obtained by the (EFD) or the (EFE) method, first we approximate $\Lambda_\Psi(g) := \Lambda(\Psi(g(t_{\max}, \cdot) + f_0(\cdot)))$ by $\Lambda_\Psi(\overline{\overline{g}}) = \Lambda(\Psi(\overline{\overline{g}}(t_{\max}, \cdot) + f_0(\cdot)))$ and then we apply the composite quadrature formula to construct an approximation $\overline{\Lambda}_{\Psi,Q}(\overline{\overline{g}})$ of $\Lambda_\Psi(\overline{\overline{g}})$ as follows

$$
\begin{aligned}
(2.9) \qquad \overline{\Lambda}_{\Psi,Q}(\overline{\overline{g}}) &= \sum_{\ell=\ell_a}^{L-1} \Delta\tau_\ell \, Q \left(\Psi \left(\overline{\overline{g}}(t_{\max}, \tau_\ell + \cdot \, \Delta\tau_\ell) + f_0(\tau_\ell + \cdot \, \Delta\tau_\ell) \right) \right) \\
&= \sum_{\ell=\ell_a}^{L-1} \Delta\tau_\ell \left[\sum_{i=1}^{N_Q} w_{Q,i} \, \Psi \left(\overline{\overline{g}}_{N,\ell} + f_0(\tau_\ell + s_{Q,i} \, \Delta\tau_\ell) \right) \right].
\end{aligned}
$$

Note that $\overline{\overline{g}}(t_{\max}, \cdot)$ is piecewise constant over the partition of $[0, \tau_{\max}]$ and numerical quadrature error in (2.9) is caused only from the presence of the initial datum f_0. In particular, if the initial datum for

the term structure, f_0, is a piecewise constant function on the maturity time partition, then there is no quadrature error. Finally, an approximation $\overline{\overline{\mathcal{G}}}(\overline{g})$ of $\mathcal{G}(g)$ is computed by

$$(2.10) \qquad \overline{\overline{\mathcal{G}}}(\overline{g}) \equiv F\left(\overline{g}_{N,L}\right) G\left(\overline{\Lambda}_{\Psi,Q}(\overline{g})\right) + \overline{\overline{g}}_{N,L+1}.$$

The Monte Carlo method, [16], approximates the expectation of a given random variable X by a sample average of M independent realizations of X, i.e. $\mathbb{E}[X] \approx \mathcal{A}(M;X) \equiv \frac{1}{M}\sum_{j=1}^{M} X(\omega_j)$. In particular, here we approximate $\mathbb{E}[\mathcal{G}(g)]$ by a sample average of $\overline{\overline{\mathcal{G}}}(\overline{g})$,

$$(2.11) \qquad \mathcal{A}\left(M;\overline{\overline{\mathcal{G}}}(\overline{g})\right) \equiv \frac{1}{M}\sum_{j=1}^{M}\left[F\left(\overline{g}_{N,L}(\omega_j)\right) G\left(\overline{\Lambda}_{\Psi,Q}(\overline{g}(\omega_j))\right) + \overline{\overline{g}}_{N,L+1}(\omega_j)\right].$$

Therefore, the exact computational weak error

$$(2.12) \qquad \mathcal{E}_c \equiv \mathbb{E}[\mathcal{G}(g)] - \mathcal{A}\left(M;\overline{\overline{\mathcal{G}}}(\overline{g})\right)$$

naturally separates into three error contributions as follows:

$$(2.13) \qquad \mathcal{E}_c = E_D + E_Q + E_S$$

with

$$(2.14) \qquad \begin{array}{c} E_D \equiv \mathbb{E}\left[\mathcal{G}(g)\right] - \mathbb{E}\left[\mathcal{G}(\overline{g})\right], \quad E_Q \equiv \mathbb{E}\left[\mathcal{G}(\overline{g})\right] - \mathbb{E}\left[\overline{\overline{\mathcal{G}}}(\overline{g})\right], \\ E_S \equiv \mathbb{E}\left[\overline{\overline{\mathcal{G}}}(\overline{g})\right] - \mathcal{A}\left(M;\overline{\overline{\mathcal{G}}}(\overline{g})\right) \end{array}$$

where E_D is the error contribution from t- and τ- discretization, E_Q is the quadrature error in (2.9), and E_S is the statistical error.

3. Strong Convergence

To carry out an error analysis for the numerical methods proposed in Section 2, we assume that there exists nonnegative constants $C_{\xi,1}$ and $C_{\xi,2}$ such that

$$(3.1) \qquad |\xi^2(x)| \leq C_{\xi,1}\left(1 + |x|\right) \quad \forall\, x \in \mathbb{R},$$

and

$$(3.2) \qquad |\xi^2(x) - \xi^2(z)| + |\xi(x) - \xi(z)| \leq C_{\xi,2}\, |x - z|, \quad \forall\, x,\, z \in \mathbb{R}.$$

3.1. **Bounds for moments.** In Lemmas 3.1 and 3.2, we show, respectively, boundness for the moments of the $\tau-$derivatives of the solution g to the problem (1.9)–(1.10), and for the functional value $\mathcal{G}(g)$.

Lemma 3.1. *Let $D_\star \equiv [0,t_{\max}] \times [0,\tau_{\max}]$, g be the solution of (1.9)–(1.10) and $\nu \in \mathbb{N}_0$. Also, we assume that the derivatives $\partial_\tau^\ell \widetilde{\lambda}$ and $(\partial_\tau^\ell \lambda_j)_{j=1}^J$ are well defined and continuous on $D_\star$, for $\ell = 0,\ldots,\nu$. Then, for $\ell = 0,\ldots,\nu$ and $\kappa \in \mathbb{N}$, there exists a positive constant $C_{\kappa,\ell}^M$, depending on κ, ℓ, $(\partial_\tau^\ell \lambda_j)_{j=1}^J$, $\partial_\tau^\ell \widetilde{\lambda}$, f_0, $C_{\xi,1}$, $\tau_{\max}$ and $t_{\max}$, such that*

$$(3.3) \qquad \max_{(t,\tau)\in D_\star} \mathbb{E}\left[\left|\partial_\tau^\ell g(t,\tau)\right|^{2\kappa}\right] \leq C_{\kappa,\ell}^M,$$

where $C_{\xi,1}$ is the constant in (3.1).

Proof. Let $\kappa \in \mathbb{N}$, $\ell \in \{0,\ldots,\nu\}$ and $(t,\tau) \in D_\star$. Also, in order to simplify the notation, we set $t_\star := t_{\max}$ and $\tau_\star := \tau_{\max}$. Our first step is to use (1.9) to get

$$(3.4) \qquad \mathbb{E}\left[\left|\partial_\tau^\ell g(t,\tau)\right|^{2\kappa}\right] \leq (J+1)^{2\kappa-1}\left[T_{1,\kappa}^\ell(t,\tau) + T_{2,\kappa}^\ell(t,\tau)\right],$$

where

$$T_{1,\kappa}^\ell(t,\tau) \equiv \mathbb{E}\left[\left(\int_0^t \partial_\tau^\ell \widetilde{\lambda}(s,\tau)\,\xi^2(g(s,s) + f_0(s))\,ds\right)^{2\kappa}\right],$$

$$T_{2,\kappa}^\ell(t,\tau) \equiv \sum_{j=1}^J \mathbb{E}\left[\left(\int_0^t \partial_\tau^\ell \lambda_j(s,\tau)\,\xi(g(s,s) + f_0(s))\,dW^j(s)\right)^{2\kappa}\right].$$

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Using (3.1) and applying the Hölder inequality we have

$$
T^{\ell}_{1,\kappa}(t,\tau) \le (C_{\xi,1})^{2\kappa}\, \mathbb{E}\left[\left(\int_0^t |\partial_\tau^\ell \widetilde{\lambda}(s,\tau)|\, (1 + |f_{\scriptscriptstyle 0}(s)| + |g(s,s)|)\ ds \right)^{2\kappa} \right]
$$

$$
\le 2^{2\kappa-1}\,(C_{\xi,1})^{2\kappa}\, \mathbb{E}\left[\left(\int_0^t |\partial_\tau^\ell \widetilde{\lambda}(s,\tau)|\,(1 + |f_{\scriptscriptstyle 0}(s)|)\, ds \right)^{2\kappa} \right.
$$

(3.5)

$$
\left. + \left(\int_0^t |\partial_\tau^\ell \widetilde{\lambda}(s,\tau)|\, |g(s,s)|\, ds \right)^{2\kappa} \right]
$$

$$
\le C_1^{\kappa,\ell} + C_2^{\kappa,\ell} \int_0^t \mathbb{E}\left[(g(s,s))^{2\kappa} \right]\, ds,
$$

where $C_1^{\kappa,\ell} \equiv 2^{2\kappa-1}\,(C_{\xi,1})^{2\kappa} \max\limits_{\tau \in [0,\tau_\star]} \left(\int_0^{t_\star} |\partial_\tau^\ell \widetilde{\lambda}(s,\tau)|\,(1 + |f_{\scriptscriptstyle 0}(s)|)\, ds \right)^{2\kappa}$ and

$$
C_2^{\kappa,\ell} \equiv 2^{2\kappa-1}\,(C_{\xi,1})^{2\kappa} \max\limits_{\tau \in [0,\tau_\star]} \left(\int_0^{t_\star} |\partial_\tau^\ell \widetilde{\lambda}(s,\tau)|^{\frac{2\kappa}{2\kappa-1}}\, ds \right)^{2\kappa-1}.
$$

Next, using the properties of the Itô integral and (3.1), we obtain

$$
T^{\ell}_{2,\kappa}(t,\tau) \le (2\kappa-1)!! \sum_{j=1}^{J} \left(\int_0^t (\partial_\tau^\ell \lambda_j(s,\tau))^2\, \mathbb{E}\left[\xi^2(g(s,s) + f_{\scriptscriptstyle 0}(s)) \right]\, ds \right)^\kappa
$$

$$
\le (2\kappa-1)!!\,(C_{\xi,1})^\kappa \sum_{j=1}^{J} \left(\int_0^t (\partial_\tau^\ell \lambda_j(s,\tau))^2\,(1 + |f_{\scriptscriptstyle 0}(s)| + \mathbb{E}\left[|g(s,s)| \right])\, ds \right)^\kappa
$$

(3.6)

$$
\le (2\kappa-1)!!\,(C_{\xi,1})^\kappa \sum_{j=1}^{J} \left(\int_0^t (\partial_\tau^\ell \lambda_j(s,\tau))^2\,(2 + |f_{\scriptscriptstyle 0}(s)| + \mathbb{E}\left[|g(s,s)|^2 \right])\, ds \right)^\kappa
$$

$$
\le C_4^{\kappa,\ell} + C_3^{\kappa,\ell} \left(\int_0^t \mathbb{E}\left[|g(s,s)|^2 \right]\, ds \right)^\kappa,
$$

where $C_3^{\kappa,\ell} \equiv (2\kappa-1)!!\,2^{\kappa-1}\,(C_{\xi,1})^\kappa \left(\sum_{j=1}^{J} \max\limits_{D_\star} |\partial_\tau^\ell \lambda_j|^{2\kappa} \right)$ and $C_4^{\kappa,\ell} \equiv C_3^{\kappa,\ell} \left(\int_0^{t_{\max}} (2 + |f_{\scriptscriptstyle 0}(s)|)\, ds \right)^\kappa$.
Now, combine (3.4), (3.5) and (3.6), to arrive at

(3.7) $$\mathbb{E}\left[(\partial_\tau^\ell g(t,\tau))^{2\kappa} \right] \le C_I^{\kappa,\ell} + C_{II}^{\kappa,\ell} \left(\int_0^t \mathbb{E}\left[|g(s,s)|^2 \right]\, ds \right)^\kappa + C_{III}^{\kappa,\ell} \int_0^t \mathbb{E}\left[|g(s,s)|^{2\kappa} \right]\, ds,$$

where $C_I^{\kappa,\ell} = (J+1)^{2\kappa-1}\,(C_1^{\kappa,\ell} + C_4^{\kappa,\ell})$, $C_{II}^{\kappa,\ell} = (J+1)^{2\kappa-1}\,C_2^{\kappa,\ell}$ and $C_{III}^{\kappa,\ell} = (J+1)^{2\kappa-1}\,C_3^{\kappa,\ell}$.
Consider the case $\kappa = 1$ and $\ell = 0$, and set $\tau = t$ in (3.7), to obtain

$$
\mathbb{E}\left[|g(t,t)|^2 \right] \le C_I^{1,0} + \left(C_{II}^{1,0} + C_{III}^{1,0} \right) \int_0^t \mathbb{E}\left[|g(s,s)|^2 \right]\, ds, \quad \forall t \in [0,t_\star],
$$

which, after the application of the Grönwall lemma, yields

(3.8) $$\mathbb{E}\left[|g(t,t)|^2 \right] \le C_I^{1,0}\, e^{(C_{II}^{1,0}+C_{III}^{1,0})\, t}, \quad \forall t \in [0,t_\star].$$

Now, combine (3.8) and (3.7) (with $\kappa = 1$), to get

(3.9) $$\mathbb{E}\left[|\partial_\tau^\ell g(t,\tau)|^2 \right] \le C_I^{1,\ell} + \frac{(C_{II}^{1,\ell}+C_{III}^{1,\ell})\,C_I^{1,0}}{C_{II}^{1,0}+C_{III}^{1,0}} \left[e^{(C_{II}^{1,0}+C_{III}^{1,0})\, t} - 1 \right], \quad \forall (t,\tau) \in D_\star,$$

for $\ell = 0,\ldots,\nu$, which establishes (3.3) for $\kappa = 1$.
Now, consider the case $\kappa \ge 2$. Then, use (3.8) and (3.7), to obtain

(3.10) $$\mathbb{E}\left[|\partial_\tau^\ell g(t,\tau)|^{2\kappa} \right] \le C_{IV}^{\kappa,\ell} + C_{III}^{\kappa,\ell} \int_0^t \mathbb{E}\left[|g(s,s)|^{2\kappa} \right]\, ds, \quad \forall (t,\tau) \in D_\star, \quad \ell = 0,\ldots,\nu,$$

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where $C_{IV}^{\kappa,\ell} = C_I^{\kappa,\ell} + C_{II}^{\kappa,\ell} \left[\frac{C_I^{1,0}}{C_{II}^{1,0} + C_{III}^{1,0}} \left(e^{\left(C_{II}^{1,0} + C_{III}^{1,0} \right) t_{\max}} - 1 \right) \right]^{\kappa}$. Take $\ell = 0$ and set $\tau = t$ in (3.10), to obtain

$$\mathbb{E}\left[|g(t,t)|^{2\kappa} \right] \leq C_{IV}^{\kappa,0} + C_{III}^{\kappa,0} \int_0^t \mathbb{E}\left[|g(s,s)|^{2\kappa} \right] \, ds, \quad \forall\, t \in [0, t_\star].$$

Apply again the Grönwall lemma, to conclude that

$$(3.11) \qquad\qquad \mathbb{E}\left[|g(t,t)|^{2\kappa} \right] \leq C_{IV}^{\kappa,0} \, e^{C_{III}^{\kappa,0} t}, \quad \forall\, t \in [0, t_\star].$$

Finally, combine (3.11) and (3.10) to have

$$\mathbb{E}\left[|\partial_\tau^\ell g(t,\tau)|^{2\kappa} \right] \leq C_{IV}^{\kappa,\ell} + \frac{C_{III}^{\kappa,\ell} C_{IV}^{\kappa,0}}{C_{III}^{\kappa,0}} \left(e^{C_{III}^{\kappa,0} t} - 1 \right), \quad \forall\, (t,\tau) \in D_\star, \quad \ell = 0, \ldots, \nu,$$

which yields the desired bound (3.3) for $\kappa \geq 2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.2. *Let (g, Y, Z) be the solution of the system* (1.9)–(1.10). *Also, we assume that the functions F, G, Ψ, $U : \mathbb{R} \to \mathbb{R}$ have polynomial growth p_F, p_G, p_Ψ and p_U with constants C_F, C_G, C_Ψ and C_U, respectively. Then, for $\kappa \in \mathbb{N}$, there exists a positive constant C_κ, depending on κ and the data of the problem, such that*

$$(3.12) \qquad \mathbb{E}\left[|F(Y(t_{\max}))|^{2\kappa} \right] + \mathbb{E}\left[|G(\Lambda(\Psi(g(t_{\max}, \cdot) + f_0)))|^{2\kappa} \right] + \mathbb{E}\left[|Z(t_{\max})|^{2\kappa} \right] \leq C_\kappa.$$

Proof. Let $\kappa \in \mathbb{N}$. To simplify the notation, we set $\tau_\star := \tau_{\max}$, $t_\star := t_{\max}$ and $\Upsilon(\tau) := g(t_{\max}, \tau) + f_0(\tau) = f(t_{\max}, \tau)$ for $\tau \in [0, \tau_{\max}]$. Since F, U and G have polynomial growth, using the Hölder inequality and (3.3) for $\ell = 0$, we obtain

$$(3.13) \qquad\qquad \mathbb{E}\left[|F(Y(t))|^{2\kappa} \right] \leq (C_F)^{2\kappa} \, 2^{2\kappa - 1} \left(1 + \mathbb{E}\left[|Y(t)|^{2\kappa p_F} \right] \right), \quad \forall\, t \in [0, t_\star],$$

$$(3.14) \qquad \begin{aligned} \mathbb{E}\left[|Y(t)|^{2m} \right] &\leq (2\, t_\star)^{2m-1} \int_0^t \left(\mathbb{E}\left[|g(s,s)|^{2m} \right] + |f_0(s)|^{2m} \right) \, ds \\ &\leq (2\, t_\star)^{2m-1} \int_0^{t_\star} \left(C_{2m,0}^M + |f_0(s)|^{2m} \right) \, ds, \quad \forall\, t \in [0, t_\star], \quad \forall\, m \in \mathbb{N}, \end{aligned}$$

$$(3.15) \qquad \mathbb{E}\left[|Z(t_\star)|^{2\kappa} \right] \leq (t_\star)^{2\kappa - 1} \left\{ \int_0^{t_\star} \mathbb{E}\left[|F(Y(s))|^{4\kappa} \right] \, ds + \int_0^{t_\star} \mathbb{E}\left[|U(f(s,s))|^{4\kappa} \right] \, ds \right\},$$

$$(3.16) \qquad \begin{aligned} \mathbb{E}\left[|U(f(t,t))|^{2m} \right] &\leq (C_U)^{2m} \, 3^{2m-1} \left(1 + |f_0(t)|^{2m p_U} + \mathbb{E}\left[|g(t,t)|^{2m p_U} \right] \right) \\ &\leq (C_U)^{2m} \, 3^{2m-1} \left(1 + |f_0(t)|^{2m p_U} + C_{2m p_U, 0}^M \right), \quad \forall\, t \in [0, t_\star], \quad \forall\, m \in \mathbb{N}, \end{aligned}$$

$$(3.17) \qquad\qquad \mathbb{E}\left[|G(\Lambda(\Psi(\Upsilon)))|^{2\kappa} \right] \leq (C_G)^{2\kappa} \left(1 + \mathbb{E}\left[|\Lambda(\Psi(\Upsilon))|^{2\kappa p_G} \right] \right),$$

$$(3.18) \qquad \mathbb{E}\left[|\Lambda(\Psi(\Upsilon))|^{2m} \right] \leq (\tau_\star - \tau_a)^{2m-1} \int_{\tau_a}^{\tau_{\max}} \mathbb{E}\left[|\Psi(\Upsilon(\tau))|^{2m} \right] \, d\tau, \quad \forall\, m \in \mathbb{N},$$

and

$$(3.19) \qquad \begin{aligned} \mathbb{E}\left[|\Psi(\Upsilon(\tau))|^{2m} \right] &\leq (C_\Psi)^{2m} \, 3^{2m-1} \left(1 + \mathbb{E}\left[|g(t_\star, \tau)|^{2m p_\Psi} \right] + |f_0(\tau)|^{2m p_\Psi} \right) \\ &\leq (C_\Psi)^{2m} \, 3^{2m-1} \left(1 + |f_0(\tau)|^{2m p_\Psi} + C_{2m p_\Psi, 0}^M \right), \quad \forall\, \tau \in [0, \tau_\star], \quad \forall\, m \in \mathbb{N}. \end{aligned}$$

Thus, we obtain (3.12) combining the inequalities (3.13)-(3.19) above. $\qquad\qquad\qquad\square$

In Lemma 3.3 below, we show boundness for the moments of the numerical approximations produced by the (EFD) and the (EFM) method.

Lemma 3.3. *Let $\mathcal{I} := \{0, \ldots, N\} \times \{0, \ldots, L-1\}$ and $(\bar{\bar{g}}_{n,\ell})_{(n,\ell) \in \mathcal{I}}$ be the numerical approximations produced by the (EFD) or the (EFM) method. Then, for $\kappa \in \mathbb{N}$, there exists a nonnegative constant $C_{D,\kappa}^M$, depending on κ, $(\lambda_j)_{j=1}^J$, $\widetilde{\lambda}$, f_0, $C_{\xi,1}$, $\tau_{\max}$ and $t_{\max}$, such that*

$$(3.20) \qquad\qquad \max_{(n,\ell) \in \mathcal{I}} \mathbb{E}\left[|\bar{\bar{g}}_{n,\ell}|^{2\kappa} \right] \leq C_{D,\kappa}^M,$$

where $C_{\xi,1}$ is the constant in (3.1).

Proof. Let $D_\star \equiv [0, t_{\max}] \times [0, \tau_{\max}]$, $\kappa \in \mathbb{N}$, $(n, \ell) \in \mathcal{I}$ with $n \geq 1$. Then, from (2.6) and (2.8), we conclude that

$$(3.21) \qquad \bar{\bar{g}}_{n,\ell} = \sum_{m=0}^{n-1} \Delta t_m \, \xi^2 (\bar{\bar{g}}_{m,\ell_m} + f_0^m) \, \nu^{m,\ell} + \sum_{m=0}^{n-1} \sum_{j=1}^{J} \xi(\bar{\bar{g}}_{m,\ell_m} + f_0^m) \, \mu_j^{m,\ell} \, \Delta W_m^j,$$

where $f_0^m := f_0(t_m)$, $\nu^{m,\ell} = \widetilde{\lambda}(t_m, \tau_\ell)$ and $\mu_j^{m,\ell} = \lambda_j(t_m, \tau_\ell)$ for the (EFD) method, and $\nu^{m,\ell} = \Pi\widetilde{\lambda}(t_m; \tau_\ell)$ and $\mu_j^{m,\ell} = \Pi\lambda_j(t_m; \tau_\ell)$ for the (EFE) method. Thus, we obtain

$$(3.22) \qquad \mathbb{E}\left[\, |\bar{\bar{g}}_{n,\ell}|^{2\kappa} \,\right] \leq (J+1)^{2\kappa-1} \left(\, T_{1,\kappa}^{n,\ell} + T_{2,\kappa}^{n,\ell} \,\right)$$

where

$$T_{1,\kappa}^{n,\ell} := \mathbb{E}\left[\, \left(\sum_{m=0}^{n-1} \Delta t_m \, \nu^{m,\ell} \, \xi^2 (\bar{\bar{g}}_{m,\ell_m} + f_0^m) \right)^{2\kappa} \,\right],$$

$$T_{2,\kappa}^{n,\ell} := \sum_{j=1}^{J} \mathbb{E}\left[\, \left(\sum_{m=0}^{n-1} \mu_j^{m,\ell} \, \xi(\bar{\bar{g}}_{m,\ell_m} + f_0^m) \, \Delta W_m^j \right)^{2\kappa} \,\right].$$

Using (3.1) we bound $T_{1,\kappa}^{n,\ell}$ as follows

$$T_{1,\kappa}^{n,\ell} \leq (C_{\xi,1})^{2\kappa} \, \mathbb{E}\left[\, \left(\sum_{m=0}^{n-1} \Delta t_m \, |\nu^{m,\ell}| \, (1 + |f_0^m| + |\bar{\bar{g}}_{m,\ell_m}|) \right)^{2\kappa} \,\right]$$

$$\leq 2^{2\kappa-1} \, (C_{\xi,1})^{2\kappa} \, \mathbb{E}\left[\, \left(\sum_{m=0}^{n-1} \Delta t_m \, |\nu^{m,\ell}| \, (1 + |f_0^m|) \right)^{2\kappa} + \left(\sum_{m=0}^{n-1} \Delta t_m |\nu^{m,\ell}| \, |\bar{\bar{g}}_{m,\ell_m}| \right)^{2\kappa} \,\right]$$

which, after applying the Hölder inequality, yields

$$(3.23) \qquad T_{1,\kappa}^{n,\ell} \leq C_{D,1,\kappa} \left(t_{\max} + \sum_{m=0}^{n-1} \Delta t_m \, \mathbb{E}\left[\, |\bar{\bar{g}}_{m,\ell_m}|^{2\kappa} \,\right] \right).$$

where $C_{D,1,\kappa} = (2\,C_{\xi,1})^{2\kappa} \, (t_{\max})^{2\kappa-1} \, \max_{D_\star} [\, |\widetilde{\lambda}| \, (1 + |f_0|) \,]^{2\kappa}$. Also, using the properties of independent Gaussian random variables and (3.1), we obtain

$$T_2^{n,\ell} \leq (2\kappa - 1)!! \sum_{j=1}^{J} \left(\sum_{m=0}^{n-1} \Delta t_m \, (\mu_j^{m,\ell})^2 \, \mathbb{E}\left[\, \xi^2(\bar{\bar{g}}_{m,\ell_m} + f_0^m) \,\right] \right)^{\kappa}$$

$$\leq (2\kappa - 1)!! \, (C_{\xi,1})^{\kappa} \, \max_{D_\star} |\lambda|^{2\kappa} \left(\sum_{m=0}^{n-1} \Delta t_m \, \left(2 + |f_0^m| + \mathbb{E}\left[\, |\bar{\bar{g}}_{m,\ell_m}|^2 \,\right] \right) \right)^{\kappa}$$

$$\leq (2\kappa - 1)!! \, (C_{\xi,1})^{\kappa} \, \max_{D_\star} |\lambda|^{2\kappa} \left[t_{\max} \max_{[0,t_{\max}]} (2 + |f_0|) + \sum_{m=0}^{n-1} \Delta t_m \, \mathbb{E}\left[\, |\bar{\bar{g}}_{m,\ell_m}|^2 \,\right] \right]^{\kappa}$$

which yields that

$$(3.24) \qquad T_2^{n,\ell} \leq C_{D,2,\kappa} \left[(t_{\max})^{\kappa} + \left(\sum_{m=0}^{n-1} \Delta t_m \, \mathbb{E}\left[\, |\bar{\bar{g}}_{m,\ell_m}|^2 \,\right] \right)^{\kappa} \right]$$

where $C_{D,2,\kappa} = 2^{\kappa-1} \, (2\kappa - 1)!! \, (C_{\xi,1})^{\kappa} \, \max_{D_\star} |\lambda|^{2\kappa} \, \max_{[0,t_{\max}]} (2 + |f_0|)^{\kappa}$. Now, combining (3.22), (3.23) and (3.24) we obtain

$$(3.25) \qquad \mathbb{E}\left[\, |\bar{\bar{g}}_{n,\ell}|^{2\kappa} \,\right] \leq C_\kappa^{I,D} + C_\kappa^{II,D} \sum_{m=0}^{n-1} \Delta t_m \, \mathbb{E}\left[\, |\bar{\bar{g}}_{m,\ell_m}|^{2\kappa} \,\right] + C_\kappa^{III,D} \left(\sum_{m=0}^{n-1} \Delta t_m \, \mathbb{E}\left[\, |\bar{\bar{g}}_{m,\ell_m}|^2 \,\right] \right)^{\kappa},$$

where $C_\kappa^{I,D}$, $C_\kappa^{II,D}$ and $C_\kappa^{III,D}$ are constants that depend on J, κ, $t_{\max}$, $C_{D,1,\kappa}$ and $C_{D,2,\kappa}$.

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First, let us consider the case $\kappa = 1$. Then, setting $\ell = \ell_n$ in (3.25), we obtain

$$(3.26) \qquad \mathbb{E}\left[\,|\bar{\bar{g}}_{n,\ell_n}|^2\,\right] \leq C_1^{I,D} + C_1^{IV,D} \sum_{m=0}^{n-1} \Delta t_m\, \mathbb{E}\left[\,|\bar{\bar{g}}_{m,\ell_m}|^2\,\right], \quad n = 1, \ldots, N,$$

where $C_1^{IV,D} = C_1^{II,D} + C_1^{III,D}$. Setting $\beta_n := \frac{1}{C_1^{I,D}}\, \mathbb{E}\left[\,|\bar{\bar{g}}_{n,\ell_n}|^2\,\right]$ for $n = 0, \ldots, N$, (3.26) is written equivalently as follows

$$(3.27) \qquad \beta_n \leq 1 + C_1^{IV,D} \sum_{m=0}^{n-1} \Delta t_m\, \beta_m, \quad n = 1, \ldots, N.$$

Now, setting $\rho_1 := 1$ and $\rho_n := 1 + C_1^{IV,D} \sum_{m=1}^{n-1} \Delta t_m\, \rho_m$ for $n = 2, \ldots, N$ and observing that $\beta_0 = 0$, we use (3.27) and apply a simple induction argument to get

$$(3.28) \qquad \beta_n \leq \rho_n, \quad n = 1, \ldots, N.$$

Since $\rho_n = (1 + C_1^{IV,D}\, \Delta t_{n-1})\, \rho_{n-1}$ for $n = 2, \ldots, N$, we use the inequality $e^x \geq 1 + x$ for $x \geq 0$, and a simple induction argument to conclude that

$$(3.29) \qquad \rho_n \leq \exp(C_1^{IV,D}\, t_n), \quad n = 1, \ldots, N.$$

Thus, (3.28) and (3.29) yield

$$(3.30) \qquad \max_{0 \leq m \leq N} \mathbb{E}\left[\,|\bar{\bar{g}}_{m,\ell_m}|^2\,\right] \leq C_1^{I,D}\, \exp(C_1^{IV,D}\, t_{\max}),$$

which, along with (3.25), establishes (3.20) for $\kappa = 1$.

Now, we assume that $\kappa \geq 2$. Then, we combine (3.25) and (3.30) to obtain

$$(3.31) \qquad \mathbb{E}\left[\,|\bar{\bar{g}}_{n,\ell_n}|^{2\kappa}\,\right] \leq C_\kappa^{V,D} + C_\kappa^{II,D} \sum_{m=0}^{n-1} \Delta t_m\, \mathbb{E}\left[\,|\bar{\bar{g}}_{m,\ell_m}|^{2\kappa}\,\right], \quad n = 1, \ldots, N,$$

where $C_\kappa^{V,D} = C_\kappa^{I,D} + C_\kappa^{III,D}\, (t_{\max})^\kappa\, (C_1^{I,D}\, \exp(C_1^{IV,D}\, t_{\max}))^\kappa$. Then, proceeding as in obtaining (3.30) from (3.26), we arrive at

$$(3.32) \qquad \max_{0 \leq m \leq N} \mathbb{E}\left[\,|\bar{\bar{g}}_{m,\ell_m}|^{2\kappa}\,\right] \leq C_1^{V,D}\, \exp(C_1^{II,D}\, t_{\max}),$$

which, along with (3.25) and (3.30), yields (3.20) for $\kappa \geq 2$. $\qquad\square$

3.2. **Estimates for the consistency error.** In Lemmas 3.4 and 3.5 below, we show that some Lipschitz-type properties for the solution g to the problem (1.9)–(1.10) hold.

Lemma 3.4. *Let $\kappa \in \mathbb{N}$ and g be the solution of (1.9)–(1.10). Then, it holds that*

$$(3.33) \qquad \mathbb{E}\left[\,|g(t,\tau_1) - g(t,\tau_2)|^{2\kappa}\,\right] \leq C_{\kappa,1}^M\, |\tau_1 - \tau_2|^{2\kappa}, \quad \forall\, \tau_1, \tau_2 \in [0, \tau_{\max}], \quad \forall\, t \in [0, t_{\max}],$$

where $C_{\kappa,1}^M$ is the constant in (3.3) for $\ell = 1$.

Proof. Let $t \in [0, t_{\max}]$ and $\tau_1, \tau_2 \in [0, \tau_{\max}]$ with $\tau_2 \geq \tau_1$. Then, applying the Hölder inequality, we have

$$\mathbb{E}\left[\,|g(t,\tau_1) - g(t,\tau_2)|^{2\kappa}\,\right] = \mathbb{E}\left[\left|\int_{\tau_1}^{\tau_2} \partial_\tau g(t,\tau)\, d\tau\right|^{2\kappa}\right]$$

$$\leq |\tau_2 - \tau_1|^{2\kappa - 1} \int_{\tau_1}^{\tau_2} \mathbb{E}\left[\,|\partial_\tau g(t,\tau)|^{2\kappa}\,\right]\, d\tau$$

$$\leq |\tau_2 - \tau_1|^{2\kappa} \max_{\tau \in [\tau_1, \tau_2]} \mathbb{E}\left[\,|\partial_\tau g(t,\tau)|^{2\kappa}\,\right].$$

Thus, we obtain (3.33) combining the inequality above and (3.3) for $\ell = 1$. $\qquad\square$

Lemma 3.5. *Let $\kappa \in \mathbb{N}$ and g be the solution of (1.9)–(1.10). Then, there exists a nonnegative constant C_{Lip}, depending on κ, J, λ, $\widetilde{\lambda}$, f_0, $C_{\xi,1}$, $\tau_{\max}$ and $t_{\max}$, such that*

$$(3.34) \qquad \mathbb{E}\left[\,|g(t_1,\tau) - g(t_2,\tau)|^{2\kappa}\,\right] \leq C_{\mathrm{Lip}}\, |t_1 - t_2|^\kappa, \quad \forall\, t_1, t_2 \in [0, t_{\max}], \quad \forall\, \tau \in [0, \tau_{\max}].$$

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Proof. Let $D_\star \equiv [0, t_{\max}] \times [0, \tau_{\max}]$, $\tau \in [0, \tau_{\max}]$ and $t_1, t_2 \in [0, t_{\max}]$ with $t_2 \geq t_1$. Proceeding as in the proof of Lemma 3.1 we obtain

$$(3.35) \qquad \mathbb{E}\left[\, |g(\tau, t_1) - g(\tau, t_2)|^{2\kappa} \,\right] \leq (J+1)^{2\kappa - 1} \left[\, B_{\kappa, I}(\tau; t_1, t_2) + B_{\kappa, II}(\tau; t_1, t_2) \,\right]$$

where

$$B_{\kappa, I}(\tau; t_1, t_2) = \mathbb{E}\left[\left(\int_{t_1}^{t_2} \widetilde{\lambda}(s, \tau)\, \xi^2(g(s,s) + f_0(s))\, ds\right)^{2\kappa}\right],$$

$$B_{\kappa, II}(\tau; t_1, t_2) = (2\kappa - 1)!! \sum_{j=1}^{J} \left(\int_{t_1}^{t_2} (\lambda_j(s, \tau))^2\, \mathbb{E}\left[\xi^2(g(s,s) + f_0(s))\right]\, ds\right)^{\kappa}.$$

Using the Hölder inequality and (3.1) we obtain

$$(3.36) \qquad \begin{aligned} B_{\kappa, I}(\tau; t_1, t_2) &\leq (2\, C_{\xi,1})^{2\kappa} \max_{D_\star} |\widetilde{\lambda}|^{2\kappa}\, \mathbb{E}\left[\left(\int_{t_1}^{t_2} (1 + |f_0(s)|)\, ds\right)^{2\kappa} + \left(\int_{t_1}^{t_2} |g(s,s)|\, ds\right)^{2\kappa}\right] \\ &\leq (2\, C_{\xi,1})^{2\kappa} \max_{D_\star} |\widetilde{\lambda}|^2\, |t_1 - t_2|^{2\kappa} \left(\max_{s \in [t_1, t_2]} (1 + |f_0(s)|)^{2\kappa} + \max_{s \in [t_1, t_2]} \mathbb{E}\left[\, |g(s,s)|^{2\kappa} \,\right]\right) \end{aligned}$$

and

$$(3.37) \qquad \begin{aligned} B_{\kappa, II}(\tau; t_1, t_2) &\leq (2\kappa - 1)!! \, (C_{\xi,1})^{\kappa} \left(\sum_{j=1}^{J} \max_{D_\star} |\lambda_j|^{2\kappa}\right) \left(\int_{t_1}^{t_2} \left(2 + |f_0(s)| + \mathbb{E}\left[\, |g(s,s)|^2 \,\right]\right) ds\right)^{\kappa} \\ &\leq (2\kappa - 1)!! \, (C_{\xi,1})^{\kappa} \left(\sum_{j=1}^{J} \max_{D_\star} |\lambda_j|^{2\kappa}\right) |t_2 - t_1|^{\kappa} \max_{s \in [t_1, t_2]} \left(2 + |f_0(s)| + \mathbb{E}\left[\, |g(s,s)|^2 \,\right]\right)^{\kappa}. \end{aligned}$$

Thus, (3.34) follows easily from (3.35), (3.36), (3.37) and (3.3) for $\ell = 0$. $\qquad\square$

In Proposition 3.1 that follows, we prove a consistency result for the (EFD) and (EFE) methods defined in Section 2.

Proposition 3.1. *Let $\kappa \in \mathbb{N}$, g be the solution of (1.9)–(1.10), $f_0^m := f_0(t_m)$ for $m = 0, \ldots, N$, and $\mathcal{K}_{n,\ell}$ be defined by*

$$(3.38) \qquad \widehat{g}_{n+1, \ell} = \widehat{g}_{n, \ell} + \Delta t_n\, \nu^{n, \ell}\, \xi^2(\widehat{g}_{n, \ell_n} + f_0^n) + \sum_{j=1}^{J} \mu_j^{n, \ell}\, \xi(\widehat{g}_{n, \ell_n} + f_0^n)\, \Delta W_n^j + \mathcal{K}_{n, \ell},$$

for $n = 0, \ldots, N-1$ and $\ell = 0, \ldots, L-1$, where $\nu^{n, \ell} = \widetilde{\lambda}(t_n, \tau_\ell)$, $\mu_j^{n, \ell} = \lambda_j(t_m, \tau_\ell)$ and $\widehat{g}_{m, \ell} = g(t_m, \tau_\ell)$ for the (EFD) method, and $\nu^{m, \ell} = \Pi\widetilde{\lambda}(t_m; \tau_\ell)$ and $\mu_j^{m, \ell} = \Pi\lambda_j(t_m; \tau_\ell)$ and $\widehat{g}_{m, \ell} = \Pi g(t_m; \tau_\ell)$ for the (EFE) method. Also, we assume that $f_0 \in C^1([0, \tau_{\max}]; \mathbb{R})$ and $\partial_t \widetilde{\lambda}$, $\partial_{t\tau} \widetilde{\lambda}$, $(\partial_t \lambda_j)_{j=1}^{J}$, $(\partial_{t\tau} \lambda_j)_{j=1}^{J}$ are well-defined and continuous on $[0, t_{\max}] \times [0, \tau_{\max}]$. Then, there exists a nonnegative constant $C_{cn,1}$, independent of the partitions of the intervals $[0, t_{\max}]$ and $[0, \tau_{\max}]$, such that

$$(3.39) \qquad \mathbb{E}\left[\left|\sum_{m=0}^{n} \mathcal{K}_{m, \ell}\right|^{2\kappa}\right] \leq C_{cn,1} \left[(\Delta t)^{\kappa} + (\Delta \tau)^{2\kappa}\right]$$

for $n = 0, \ldots, N-1$ and $\ell = 0, \ldots, L-1$. In addition, for the (EFD) method there exists a nonnegative constant $C_{cn,2}$, independent of the partitions of the intervals $[0, t_{\max}]$ and $[0, \tau_{\max}]$, such that

$$(3.40) \qquad \mathbb{E}\left[\left|\sum_{m=0}^{n} \mathcal{K}_{m, \ell+1} - \mathcal{K}_{m, \ell}\right|^{2\kappa}\right] \leq C_{cn,2}\, (\Delta \tau_\ell)^{2\kappa} \left[(\Delta t)^{\kappa} + (\Delta \tau)^{2\kappa}\right]$$

for $n = 0, \ldots, N-1$ and $\ell = 0, \ldots, L-2$.

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Proof. Here, we set $D_\star := [0, t_{\max}] \times [0, \tau_{\max}]$ and use the symbol C for a generic constant independent of the partitions of the intervals $[0, t_{\max}]$ and $[0, \tau_{\max}]$. First, we observe that (1.9) yields that

$$
\begin{aligned}
\widehat{g}_{n+1,\ell} = \widehat{g}_{n,\ell} &+ \int_{t_n}^{t_{n+1}} \nu^\ell(s)\,\xi^2(g(s,s) + f_0(s))\,ds \\
&+ \sum_{j=1}^{J} \int_{t_n}^{t_{n+1}} \mu_j^\ell(s)\,\xi(g(s,s) + f_0(s))\,dW^j(s),
\end{aligned}
$$

(3.41)

for $n = 0, \ldots, N-1$ and $\ell = 0, \ldots, L-1$, where $\nu^\ell(s) = \widetilde{\lambda}(s, \tau_\ell)$ and $\mu_j^\ell(s) = \lambda_j(s, \tau_\ell)$ for the (EFD) method and $\nu^\ell(s) = \Pi\widetilde{\lambda}(s; \tau_\ell)$ and $\mu_j^\ell(s) = \Pi\lambda_j(s; \tau_\ell)$ for the (EFE) method. Then, subtracting (3.41) from (3.38) we obtain $\sum_{m=0}^{n} \mathcal{K}_{m,\ell} = \sum_{i=1}^{4} E_{i,C}^{n,\ell}$ for $\ell = 0, \ldots, L-1$ and $n = 0, \ldots, N-1$, where

$$
E_{1,C}^{n,\ell} \equiv \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} (\nu^\ell(s) - \nu^{m,\ell})\,\xi^2(g(s,s) + f_0(s))\,ds,
$$

$$
E_{2,C}^{n,\ell} \equiv \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} \nu^{m,\ell} \left[\xi^2(g(s,s) + f_0(s)) - \xi^2(\widehat{g}_{m,\ell_m} + f_0^m) \right] ds,
$$

$$
E_{3,C}^{n,\ell} \equiv \sum_{j=1}^{J} \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} (\mu_j^\ell(s) - \mu_j^{n,\ell})\,\xi(g(s,s) + f_0(s))\,dW^j(s),
$$

$$
E_{4,C}^{n,\ell} \equiv \sum_{j=1}^{J} \sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} \mu_j^{n,\ell} \left[\xi(g(s,s) + f_0(s)) - \xi(\widehat{g}_{m,\ell_m} + f_0^m) \right] dW^j(s).
$$

Next, using (3.1), the Hölder inequality and (3.3), we obtain

$$
\begin{aligned}
\mathbb{E}\left[(E_{1,C}^{n,\ell})^{2\kappa} \right] &\leq (C_{\xi,1})^{2\kappa}\,(\Delta t)^{2\kappa}\,\max_{D_\star} |\partial_t \widetilde{\lambda}|^{2\kappa}\,\mathbb{E}\left[\left(\int_0^{t_{\max}} (1 + |f_0(s)| + |g(s,s)|)\,ds \right)^{2\kappa} \right] \\
&\leq C\,(\Delta t)^{2\kappa}\,\mathbb{E}\left[\left(\int_0^{t_{\max}} (1 + |f_0(s)|\,ds \right)^{2\kappa} + \left(\int_0^{t_{\max}} |g(s,s)|\,ds \right)^{2\kappa} \right] \\
&\leq C\,(\Delta t)^{2\kappa}\,\left[1 + (t_{\max})^{2\kappa}\,\max_{s \in [0,t_{\max}]} \mathbb{E}\left[(g(s,s))^{2\kappa} \right] \right] \\
&\leq C\,(\Delta t)^{2\kappa}\,\left[1 + (t_{\max})^{2\kappa}\,C_{\kappa,0}^M \right]
\end{aligned}
$$

(3.42)

and

$$
\begin{aligned}
\mathbb{E}\left[(E_{3,C}^{n,\ell})^{2\kappa} \right] &\leq C\,\sum_{j=1}^{J} \mathbb{E}\left[\sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} (\mu_j^\ell(s) - \mu_j^{m,\ell})\,\xi(g(s,s) + f_0(s))\,dW^j(s) \right]^{2\kappa} \\
&\leq C\,\sum_{j=1}^{J} \left[\sum_{m=0}^{n} \int_{t_m}^{t_{m+1}} (\mu_j^\ell(s) - \mu_j^{m,\ell})^2\,\mathbb{E}\left[\xi^2(g(s,s) + f_0(s)) \right] ds \right]^{\kappa} \\
&\leq C\,(\Delta t)^{2\kappa} \left(\int_0^{t_{\max}} (1 + |f_0(s)| + \mathbb{E}\left[|g(s,s)| \right])\,ds \right)^{\kappa} \\
&\leq C\,(\Delta t)^{2\kappa} \left(\int_0^{t_{\max}} (2 + |f_0(s)| + \mathbb{E}\left[(g(s,s))^2 \right])\,ds \right)^{\kappa} \\
&\leq C\,(\Delta t)^{2\kappa} \left(\int_0^{t_{\max}} (2 + |f_0(s)| + C_{1,0}^M)\,ds \right)^{\kappa}
\end{aligned}
$$

(3.43)

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Now, we apply (3.2) and the Hölder inequality, to get

$$\mathbb{E}\left[(E_{2,C}^{n,\ell})^{2\kappa}\right] \leq C\,\mathbb{E}\left[\left(\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\left(|g(s,s)-\widehat{g}_{m,\ell_m}|+|f_0(s)-f_0^m|\right)\,ds\right)^{2\kappa}\right]$$

$$\leq C\,\mathbb{E}\left[\left(\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}|f_0(s)-f_0^m|\,ds\right)^{2\kappa}+\left(\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}|g(s,s)-\widehat{g}_{m,\ell_m}|\,ds\right)^{2\kappa}\right]$$

(3.44)

$$\leq C\left[\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}|f_0(s)-f_0^m|^{2\kappa}\,ds+\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\mathbb{E}\left[|g(s,s)-\widehat{g}_{m,\ell_m}|^{2\kappa}\right]\,ds\right]$$

$$\leq C\left[(\Delta t)^{2\kappa}\max_{[0,\tau_{\max}]}|f_0'|^{2\kappa}+\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\mathbb{E}\left[|g(s,s)-\widehat{g}_{m,\ell_m}|^{2\kappa}\right]\,ds\right]$$

and

$$\mathbb{E}\left[(E_{4,C}^{n,\ell})^{2\kappa}\right] \leq C\sum_{j=1}^{J}\mathbb{E}\left[\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\mu_j^{n,\ell}\left[\xi(g(s,s)+f_0(s))-\xi(\widehat{g}_{m,\ell_m}+f_0^m)\right]\,dW^j(s)\right]^{2\kappa}$$

$$\leq C\left[\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}(\mu_j^{n,\ell})^2\,\mathbb{E}\left[\left(\xi(g(s,s)+f_0(s))-\xi(\widehat{g}_{m,\ell_m}+f_0^m)\right)^2\right]\,ds\right]^{\kappa}$$

(3.45)

$$\leq C\left[\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\left(\mathbb{E}\left[|g(s,s)-\widehat{g}_{m,\ell_m}|^2\right]+|f_0(s)-f_0^m|^2\right)\,ds\right]^{\kappa}$$

$$\leq C\left[(\Delta t)^2\max_{[0,\tau_{\max}]}|f_0'|^2+\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\mathbb{E}\left[|g(s,s)-\widehat{g}_{m,\ell_m}|^2\right]\right]^{\kappa}$$

Using (3.33), (3.34) and (2.4), we have

$$\mathbb{E}\left[|g(s,s)-\widehat{g}_{m,\ell_m}|^{2\kappa}\right] \leq C\left(\mathbb{E}\left[|g(s,s)-g(s,\tau_{\ell_m})|^{2\kappa}\right]+\mathbb{E}\left[|g(s,\tau_{\ell_m})-g(t_m,\tau_{\ell_m})|^{2\kappa}\right]\right.$$

$$\left.+\,\mathbb{E}\left[|g(t_m,\tau_{\ell_m})-\widehat{g}_{m,\ell_m}|^{2\kappa}\right]\right)$$

(3.46)

$$\leq C\left(|s-\tau_{\ell_m}|^{2\kappa}+|s-t_m|^{\kappa}+\mathbb{E}\left[|g(t_m,\tau_{\ell_m})-\widehat{g}_{m,\ell_m}|^{2\kappa}\right]\right)$$

$$\leq C\left(|s-t_m|^{2\kappa}+|t_m-\tau_{\ell_m}|^{2\kappa}+(\Delta t)^{\kappa}+\mathbb{E}\left[|g(t_m,\tau_{\ell_m})-\widehat{g}_{m,\ell_m}|^{2\kappa}\right]\right)$$

$$\leq C\left(|\tau_{\ell_m+1}-\tau_{\ell_m}|^{2\kappa}+(\Delta t)^{\kappa}+\mathbb{E}\left[|g(t_m,\tau_{\ell_m})-\widehat{g}_{m,\ell_m}|^{2\kappa}\right]\right)$$

$$\leq C\left((\Delta\tau)^{2\kappa}+(\Delta t)^{\kappa}+\mathbb{E}\left[|g(t_m,\tau_{\ell_m})-\widehat{g}_{m,\ell_m}|^{2\kappa}\right]\right)$$

for $s\in[t_m,t_{m+1}]$ and $m=0,\ldots,N-1$. For the (EFE) method, after using (3.33), we have

$$\mathbb{E}\left[|g(t_m,\tau_{\ell_m})-\widehat{g}_{m,\ell_m}|^{2\kappa}\right] \leq (\Delta\tau_{\ell_m})^{-1}\int_{\tau_{\ell_m}}^{\tau_{\ell_m+1}}\mathbb{E}\left[(g(t_m,\tau_{\ell_m})-g(t_m,\tau))^{2\kappa}\right]\,d\tau$$

(3.47)

$$\leq C\,(\Delta\tau_{\ell_m})^{-1}\int_{\tau_{\ell_m}}^{\tau_{\ell_m+1}}|\tau_{\ell_m}-\tau|^{2\kappa}\,d\tau$$

$$\leq C\,(\Delta\tau)^{2\kappa},\quad m=0,\ldots,N-1,$$

while for the (EFD) method the term we estimate above vanishes. Finally, (3.46) and (3.47) yield

(3.48)
$$\mathbb{E}\left[|g(s,s)-\widehat{g}_{m,\ell_m}|^{2\kappa}\right] \leq C\left[(\Delta t)^{\kappa}+(\Delta\tau)^{2\kappa}\right]$$

for $s\in[t_m,t_{m+1}]$ and $m=0,\ldots,N-1$. Observing that $\mathbb{E}\left[\left(\sum_{\ell=0}^{n}\mathcal{K}_{m,\ell}\right)^{2\kappa}\right]\leq 4^{2\kappa-1}\sum_{i=1}^{4}\mathbb{E}\left[(E_{i,C}^{n,\ell})^{2\kappa}\right]$ for $\ell=0,\ldots,L-1$ and $n=0,\ldots,N-1$, and that estimate (3.48) holds for $\kappa=1$, the estimate (3.39) for the consistency error follows easily in view of (3.42), (3.43), (3.44), (3.45) and (3.48).

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Since, $\mathbb{E}\left[\left(\sum_{\ell=0}^n \mathcal{K}_{m,\ell+1} - \mathcal{K}_{m,\ell+1}\right)^{2\kappa}\right] \leq 4^{2\kappa-1}\sum_{i=1}^4 \mathbb{E}\left[(E_{i,C}^{n,\ell+1} - (E_{i,C}^{n,\ell})^{2\kappa}\right]$ for $\ell = 0, \ldots, L-2$ and $n = 0, \ldots, N-1$, we obtain (3.40) for the (EFD) nethod, observing that

$$|y^{m,\ell+1} - y^{m,\ell}| \leq C\,\Delta\tau_\ell$$

(3.49)
$$\left|(y^{\ell+1}(s) - y^{m,\ell+1}) - (y^\ell(s) - y^{m,\ell})\right| = \left|\int_{t_m}^s \int_{\tau_\ell}^{\tau_{\ell+1}} \partial_{t\tau} y(t', \tau') dt' d\tau'\right|$$
$$\leq C\,\Delta t\,\Delta\tau_\ell, \quad \forall\, s \in [t_m, t_{m+1}],$$

where $y^{m,\ell} = \nu^{m,\ell}$ or $\mu_j^{n,\ell}$ and $y = \widetilde{\lambda}$ or λ_j, respectively, and proceeding as above. $\qquad\square$

3.3. **Error estimation.** In this section we derive an error estimate for the strong approximation error $\mathcal{G}(g) - \overline{\mathcal{G}}(\overline{g})$ by splitting it as sum of the strong discretization error $\mathcal{G}(g) - \mathcal{G}(\overline{g})$ which we estimate in Theorem 3.2 and of the strong mumerical quadrature error $\mathcal{G}(\overline{g}) - \overline{\mathcal{G}}(\overline{g})$ which we estimate in Theorem 3.3.

Theorem 3.2. *Let g, Y and Z be the solution of (1.9)–(1.10), $\mathcal{M} := \{0, \ldots, N\} \times \{0, \ldots, L-2\}$ $\mathcal{J} := \{0, \ldots, N\} \times \{0, \ldots, L-1\}$, $\mathcal{I} := \{0, \ldots, N\} \times \{0, \ldots, L+1\}$ and $(\overline{g}_{n,\ell})_{(n,\ell)\in\mathcal{I}}$ be the numerical approximations produced by the (EFD) or the (EFM) method. Also, we assume that the functions Ψ', F, F', G, G', U, $U' : \mathbb{R} \to \mathbb{R}$ have polynomial growth, and we define $\Lambda_\Psi(w) := \Lambda(\Psi(w + f_0))$ for $w \in S_{\Delta\tau}$ or $w \in C([0, \tau_{\max}]; \mathbb{R})$. Then, there exist nonnegative constants $(C_i^{CV})_{i=1}^6$, independent of the partitions of the intervals $[0, t_{\max}]$ and $[0, \tau_{\max}]$, such that*

(3.50)
$$\max_{(n,\ell)\in\mathcal{J}} \left(\mathbb{E}\left[\left|g(t_n, \tau_\ell) - \overline{\overline{g}}_{n,\ell}\right|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_1^{CV}\left[(\Delta t)^{\frac{1}{2}} + \Delta\tau\right],$$

(3.51)
$$\max_{0 \leq n \leq N} \left(\mathbb{E}\left[|Y(t_n) - \overline{\overline{g}}_{n,L}|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_2^{CV}\left[(\Delta t)^{\frac{1}{2}} + \Delta\tau\right],$$

(3.52)
$$\left(\mathbb{E}\left[|Z(t_{\max}) - \overline{\overline{g}}_{N,L+1}|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_3^{CV}\left[(\Delta t)^{\frac{1}{2}} + \Delta\tau\right],$$

(3.53)
$$\left(\mathbb{E}\left[\left|\Lambda_\Psi(g) - \Lambda_\Psi(\overline{g})\right|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_4^{CV}\left[(\Delta t)^{\frac{1}{2}} + \Delta\tau\right],$$

(3.54)
$$\left(\mathbb{E}\left[|\mathcal{G}(g) - \mathcal{G}(\overline{g})|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_5^{CV}\left[(\Delta t)^{\frac{1}{2}} + \Delta\tau\right]$$

and, for the (EFD) method,

(3.55)
$$\max_{(n,\ell)\in\mathcal{M}} \left(\mathbb{E}\left[\left|\frac{g(t_n, \tau_{\ell+1}) - g(t_n, \tau_\ell)}{\Delta\tau_\ell} - \frac{\overline{\overline{g}}_{n,\ell+1} - \overline{\overline{g}}_{n,\ell}}{\Delta\tau_\ell}\right|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_6^{CV}\left[(\Delta t)^{\frac{1}{2}} + \Delta\tau\right].$$

Proof. Here, we set $D_\star := [0, t_{\max}] \times [0, \tau_{\max}]$ and will use the symbol C for a generic constant independent of the partitions of the intervals $[0, t_{\max}]$ and $[0, \tau_{\max}]$. Let $E_{m,\ell} = \widehat{g}_{m,\ell} - \overline{\overline{g}}_{m,\ell}$ for $m = 0, \ldots, N$ and $\ell = 0, \ldots, L-1$, where $\widehat{g}_{m,\ell} = g(t_m, \tau_\ell)$ for the (EFD) method and $\widehat{g}_{m,\ell} = \Pi g(t_m; \tau_\ell)$ for the (EFE) method. First, subtract (2.6) or (2.8) from (3.38), and then sum with respect to n, to obtain

$$E_{n,\ell} = A_{n,\ell} + B_{n,\ell} + \sum_{m=0}^{n-1} \mathcal{K}_{m,\ell}, \quad \ell = 0, \ldots, L-1, \quad n = 1, \ldots, N,$$

where

$$A_{n,\ell} := \sum_{m=0}^{n-1} \Delta t_m\, \nu^{m,\ell}\left[\xi^2(\widehat{g}_{m,\ell_m} + f_0^m) - \xi^2(\overline{\overline{g}}_{m,\ell_m} + f_0^m)\right],$$

$$B_{n,\ell} := \sum_{m=0}^{n-1}\sum_{j=1}^J \mu_j^{m,\ell}\left[\xi(\widehat{g}_{m,\ell_m} + f_0^m) - \xi(\overline{\overline{g}}_{m,\ell_m} + f_0^m)\right]\Delta W_m^j,$$

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$f_0^m := f_0(t_m)$, $\nu^{m,\ell} = \widetilde{\lambda}(t_m, \tau_\ell)$ and $\mu_j^{m,\ell} = \lambda_j(t_m, \tau_\ell)$ for the (EFD) method and $\nu^{m,\ell} = \Pi\widetilde{\lambda}(t_m; \tau_\ell)$ and $\mu_j^{m,\ell} = \Pi\lambda_j(t_m; \tau_\ell)$ for the (EFE) method. Thus, we have

$$(3.56) \qquad \mathbb{E}\left[\,|E_{n,\ell}|^{2\kappa}\,\right] = 3^{2\kappa-1}\left(\mathbb{E}\left[\,|A_{n,\ell}|^{2\kappa}\,\right] + \mathbb{E}\left[\,|B_{n,\ell}|^{2\kappa}\,\right] + \mathbb{E}\left[\left|\sum_{m=1}^{n-1}\mathcal{K}_{m,\ell}\right|^{2\kappa}\right]\right)$$

for $\ell = 0, \ldots, L-1$ and $n = 1, \ldots, N$. First, using (3.2) and the Hölder inequality, we obtain

$$(3.57) \qquad \begin{aligned} \mathbb{E}\left[\,|A_{n,\ell}|^{2\kappa}\,\right] &\leq (C_{\xi,2})^2 \max_{D_\star}|\widetilde{\lambda}|^{2\kappa}\,\mathbb{E}\left[\left(\sum_{m=0}^{n-1}\Delta t_m\,|E_{m,\ell_m}|\right)^{2\kappa}\right] \\ &\leq C\left(\sum_{m=0}^{n-1}\Delta t_m\,\mathbb{E}\left[\,(E_{m,\ell_m})^{2\kappa}\,\right]\right) \end{aligned}$$

and

$$(3.58) \qquad \begin{aligned} \mathbb{E}\left[\,|B_{n,\ell}|^{2\kappa}\,\right] &\leq C\sum_{j=1}^{J}\left(\sum_{m=0}^{n-1}\Delta t_m\,(\mu_j^{m,\ell})^2\,\mathbb{E}\left[\,\left|\xi(\widehat{g}_{m,\ell_m} + f_0^m) - \xi(\overline{\overline{g}}_{m,\ell_m} + f_0^m)\right|^2\,\right]\right)^\kappa \\ &\leq C\left(\sum_{m=0}^{n-1}\Delta t_m\,\mathbb{E}\left[\,\left|\xi(\widehat{g}_{m,\ell_m} + f_0^m) - \xi(\overline{\overline{g}}_{m,\ell_m} + f_0^m)\right|^2\,\right]\right)^\kappa \\ &\leq C\left(\sum_{m=0}^{n-1}\Delta t_m\,\mathbb{E}\left[\,(E_{m,\ell_m})^2\,\right]\right)^\kappa \end{aligned}$$

for $\ell = 0, \ldots, L-1$ and $n = 1, \ldots, N$. Combining, (3.56), (3.57) and (3.58) and (3.39), we have

$$(3.59) \quad \mathbb{E}\left[\,|E_{n,\ell}|^{2\kappa}\,\right] \leq C\left[\left((\Delta t)^\kappa + (\Delta\tau)^{2\kappa}\right) + \sum_{m=0}^{n-1}\Delta t_m\,\mathbb{E}\left[\,|E_{m,\ell_m}|^{2\kappa}\,\right] + \left(\sum_{m=0}^{n-1}\Delta t_m\,\mathbb{E}\left[\,|E_{m,\ell_m}|^2\,\right]\right)^\kappa\right]$$

for $\ell = 0, \ldots, L-1$ and $n = 1, \ldots, N$. Considering the case $\kappa = 1$ and proceeding as in the proof of Lemma 3.3, from (3.59) we arrive at the estimate

$$(3.60) \qquad \max_{0 \leq n \leq N}\mathbb{E}\left[\,|E_{n,\ell_n}|^2\,\right] \leq C\left(\Delta t + (\Delta\tau)^2\right).$$

Letting $\kappa \geq 2$, under the view of (3.60), the inequality (3.59) yields

$$(3.61) \qquad \mathbb{E}\left[\,|E_{n,\ell}|^{2\kappa}\,\right] \leq C\left[\left((\Delta t)^\kappa + (\Delta\tau)^{2\kappa}\right) + \sum_{m=0}^{n-1}\Delta t_m\,\mathbb{E}\left[\,|E_{m,\ell_m}|^{2\kappa}\,\right]\right]$$

for $\ell = 0, \ldots, L-1$ and $n = 1, \ldots, N$. Now, proceeding again as in the proof of Lemma 3.3, from (3.61) we conclude that

$$(3.62) \qquad \max_{0 \leq n \leq N}\mathbb{E}\left[\,|E_{n,\ell_n}|^{2\kappa}\,\right] \leq C\left((\Delta t)^\kappa + (\Delta\tau)^{2\kappa}\right).$$

Thus, combining (3.61) and (3.62) we arrive at

$$(3.63) \qquad \max_{(n,\ell)\in\mathcal{I}_{N,L}}\mathbb{E}\left[\,|E_{n,\ell}|^{2\kappa}\,\right] \leq C\left((\Delta t)^\kappa + (\Delta\tau)^{2\kappa}\right).$$

The estimate (3.50) for the (EFD) method follows directly from (3.63). For the (EFE) method, (3.50) follows combining (3.63) with the following estimate (cf. (3.47))

$$\begin{aligned} \max_{(n,\ell)\in\mathcal{I}_{N,L}}\mathbb{E}\left[\,|g(t_n,\tau_\ell) - \Pi g(t_n;\tau_\ell)|^{2\kappa}\,\right] &\leq \max_{(n,\ell)\in\mathcal{I}}\max_{\tau\in[\tau_\ell,\tau_{\ell+1}]}\mathbb{E}\left[\,|g(t_m,\tau) - g(t_m,\tau_\ell)|^{2\kappa}\,\right] \\ &\leq C\,(\Delta\tau)^{2\kappa}. \end{aligned}$$

Since $\frac{E_{n,\ell+1} - E_{n,\ell}}{\Delta\tau_\ell} = \frac{A_{n,\ell+1} - A_{n,\ell}}{\Delta\tau_\ell} + \frac{B_{n,\ell+1} - B_{n,\ell}}{\Delta\tau_\ell} + \sum_{m=0}^{n-1}\frac{\mathcal{K}_{m,\ell+1} - \mathcal{K}_{m,\ell}}{\Delta\tau_\ell}$ for $\ell = 0, \ldots, L-2$ and $n = 1, \ldots, N$, to obtain the estimate (3.55) for the (EFD) method we proceed as above using (3.40) and (3.49).

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In order to get the second error estimate, we use (1.6) and (2.6) or (2.8), to conclude that

$$
(3.64) \qquad \mathbb{E}\left[\left|Y(t_n) - \bar{\bar{g}}_{n,L}\right|^{2\kappa}\right] \le C \sum_{i=1}^{3} \mathbb{E}\left[|\zeta_i^n|^{2\kappa}\right], \quad n = 1, \dots, N,
$$

where

$$
\zeta_1^n := \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} (f_0(s) - f_0(t_m))\, ds, \quad \zeta_2^n := \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} (g(s,s) - g(t_m, \tau_{\ell_m}))\, ds,
$$

$$
\zeta_3^n := \sum_{m=0}^{n-1} \Delta t_m \left(g(t_m, \tau_{\ell_m}) - \bar{\bar{g}}_{m,\ell_m}\right).
$$

First, we observe that

$$
(3.65) \qquad |\zeta_1^n|^{2\kappa} \le C\, (\Delta t)^{2\kappa} \max_{[0, t_{\max}]} |f_0'|^{2\kappa}, \quad n = 1, \dots, N.
$$

Next, we use the Hölder inequality, (3.34), (3.33) and (3.50) to obtain

$$
\begin{aligned}
\mathbb{E}\left[|\zeta_2^n|^{2\kappa}\right] &\le C \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \mathbb{E}\left[|g(s,s) - g(t_m, \tau_{\ell_m})|^{2\kappa}\right]\, ds \\
&\le C \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \mathbb{E}\left[|g(s,s) - g(t_m, s)|^{2\kappa} + |g(t_m, s) - g(t_m, \tau_{\ell_m})|^{2\kappa}\right]\, ds \\
(3.66) \qquad &\le C \left[(\Delta t)^{\kappa} + \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \left(|s - t_m|^{2\kappa} + |t_m - \tau_{\ell_m}|^{2\kappa}\right)\, ds\right] \\
&\le C \left[(\Delta t)^{\kappa} + \sum_{m=0}^{n-1} \Delta t_m\, |\tau_{\ell_m+1} - \tau_{\ell_m}|^{2\kappa}\right] \\
&\le C \left[(\Delta t)^{\kappa} + (\Delta \tau)^{2\kappa}\right], \quad n = 1, \dots, N,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.67) \qquad \mathbb{E}\left[|\zeta_3^n|^{2\kappa}\right] &\le C \sum_{m=0}^{n-1} \Delta t_m\, \mathbb{E}\left[|g(t_m, \tau_{\ell_m}) - \bar{\bar{g}}_{m,\ell_m}|^{2\kappa}\right] \\
&\le C \left[(\Delta t)^{\kappa} + (\Delta \tau)^{2\kappa}\right], \quad n = 1, \dots, N.
\end{aligned}
$$

Thus, (3.51) follows easily from (3.68), (3.65), (3.66) and (3.67).

In order to prove our third error estimate, we use (1.6), (2.6) or (2.8), and the mean value theorem for scalar fields, to conclude that

$$
(3.68) \qquad \mathbb{E}\left[\left|Z(t_{\max}) - \bar{\bar{g}}_{N,L+1}\right|^{2\kappa}\right] \le C \sum_{i=1}^{3} \mathbb{E}\left[|\Gamma_i|^{2\kappa}\right],
$$

where

$$
\Gamma_1 := \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} F'(A_m(s))\, U(B_m(s))\, (Y(s) - \bar{\bar{g}}_{m,L})\, ds,
$$

$$
\Gamma_2 := \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} F(A_m(s))\, U'(B_m(s))\, (g(s,s) - \bar{\bar{g}}_{m,\ell_m})\, ds,
$$

$$
\Gamma_3 := \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} F(A_m(s))\, U'(B_m(s))\, (f_0(s) - f_0^m)\, ds,
$$

and

$$
A_m(s) := \delta_m(s)\, \left(Y(s) - \bar{\bar{g}}_{m,L}\right) + \bar{\bar{g}}_{m,L},
$$

$$
B_m(s) := \tilde{\delta}_m(s)\, (g(s,s) + f_0(s)) + (1 - \tilde{\delta}_m(s))\, (\bar{\bar{g}}_{m,\ell_m} + f_0^m),
$$

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with $\delta_m(s)$, $\tilde{\delta}_m(s) \in [0,1]$. Let $\tilde{m} \in \mathbb{N}$. Since F, F', U and U' have polynomial growth, we use (3.51), (3.12), (3.20) and (3.3) to conclude that there exist a nonnegative constant $C_\star^m$ such that

$$(3.69) \qquad \max_{0 \leq m \leq N-1} \sup_{s \in (t_m, t_{m+1})} \left[\mathbb{E}\left[|F'(A_m(s))\, U(B_m(s))|^{2\tilde{m}} \right] + \mathbb{E}\left[|F(A_m(s))\, U'(B_m(s))|^{2\tilde{m}} \right] \right] \leq C_\star^{\tilde{m}}.$$

Also, we use the Hölder inequality and (3.3) to arrive at

$$(3.70) \qquad \begin{aligned} \mathbb{E}\left[|Y(t_b) - Y(t_a)|^{2\tilde{m}} \right] &\leq (t_b - t_a)^{2\tilde{m}-1} \int_{t_a}^{t_b} \mathbb{E}\left[|g(s,s) + f_0(s)|^{2\tilde{m}} \right]\, ds \\ &\leq C\, (t_b - t_a)^{2\tilde{m}} \end{aligned}$$

for all t_a, $t_b \in [0, t_{\max}]$ with $t_a \leq t_b$. Now, we are ready to estimare the quantities at the right hand side of (3.68). First, we use the Hölder inequality and (3.69) to arrive at

$$(3.71) \qquad \begin{aligned} \mathbb{E}\left[|\Gamma_1|^{2\kappa} \right] &\leq C \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} \mathbb{E}\left[|F'(A_m(s))\, U(B_m(s))|^{2\kappa}\, |Y(s) - \bar{\bar{g}}_{m,L}|^{2\kappa} \right]\, ds, \\ &\leq C \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} \left(\mathbb{E}\left[|F'(A_m(s))\, U(B_m(s))|^{4\kappa} \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[|Y(s) - \bar{\bar{g}}_{m,L}|^{4\kappa} \right] \right)^{\frac{1}{2}}\, ds, \\ &\leq C \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} \left(\mathbb{E}\left[|Y(s) - Y(t_m)|^{4\kappa} + |Y(t_m) - \bar{\bar{g}}_{m,L}|^{4\kappa} \right] \right)^{\frac{1}{2}}\, ds, \end{aligned}$$

$$(3.72) \qquad \begin{aligned} \mathbb{E}\left[|\Gamma_2|^{2\kappa} \right] &\leq C \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} \left(\mathbb{E}\left[|F(A_m(s))\, U'(B_m(s))|^{4\kappa} \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[|g(s,s) - \bar{\bar{g}}_{m,\ell_m}|^{4\kappa} \right] \right)^{\frac{1}{2}}\, ds \\ &\leq C \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} \left(\mathbb{E}\left[|g(s,s) - g(t_m, \tau_{\ell_m})|^{4\kappa} + |g(t_m, \tau_{\ell_m}) - \bar{\bar{g}}_{m,\ell_m}|^{4\kappa} \right] \right)^{\frac{1}{2}}\, ds, \end{aligned}$$

and

$$(3.73) \qquad \mathbb{E}\left[|\Gamma_3|^{2\kappa} \right] \leq C \sum_{m=0}^{N-1} \int_{t_m}^{t_{m+1}} |f_0(s) - f_0^m|^{2\kappa}\, ds.$$

Next, we combining (3.72), (3.73), (3.48) and (3.50) we obtain

$$(3.74) \qquad \mathbb{E}\left[|\Gamma_2|^{2\kappa} \right] + \mathbb{E}\left[|\Gamma_3|^{2\kappa} \right] \leq C\left[(\Delta t)^\kappa + (\Delta \tau)^{2\kappa} \right].$$

Finally, we combine (3.71), (3.70) and (3.51) to obtain

$$(3.75) \qquad \mathbb{E}\left[|\Gamma_1|^{2\kappa} \right] \leq C\left[(\Delta t)^\kappa + (\Delta \tau)^{2\kappa} \right].$$

Thus, the error estimate (3.52) is a simple consequence of (3.68), (3.74) and (3.75).

To derive our fourth error estimate, first we set $E := \Lambda_\Psi(g) - \Lambda_\Psi(\bar{g})$, and then we use the Hölder inequality to obtain

$$(3.76) \qquad \mathbb{E}\left[|E|^{2\kappa} \right] \leq C\, \sqrt{E_A}\, \sqrt{E_B},$$

where

$$E_A := \sum_{\ell=\ell_a}^{L-1} \int_{\tau_\ell}^{\tau_{\ell+1}} \mathbb{E}\left[\sup_{\epsilon \in [0,1]} \left| \Psi'\left(f_0(\tau) + \epsilon\, g(t_{\max}, \tau) + (1-\epsilon)\bar{\bar{g}}_{N,\ell} \right) \right|^{4\kappa} \right]\, d\tau,$$

$$E_B := \sum_{\ell=\ell_a}^{L-1} \int_{\tau_\ell}^{\tau_{\ell+1}} \mathbb{E}\left[|g(t_{\max}, \tau) - \bar{\bar{g}}_{N,\ell}|^{4\kappa} \right]\, d\tau.$$

Since Ψ' has polynomial growth, the use of (3.20) and (3.3) yields that

$$(3.77) \qquad E_A \leq C.$$

Also, using (3.33) and (3.50) we obtain

$$(3.78) \qquad E_B \leq C\left[(\Delta t)^{2\kappa} + (\Delta \tau)^{4\kappa} \right].$$

Thus, the estimate (3.53) follows after combining (3.76), (3.77) and (3.78).

To obtain our fifth error estimate, first we set $E_{\mathcal{G}} := \mathcal{G}(g) - \mathcal{G}(\overline{g})$ and use the error bound (3.52) to obtain

(3.79)
$$\mathbb{E}\left[\,|E_{\mathcal{G}}|^{2\kappa}\,\right] \leq C\left[\,\sqrt{\mathcal{G}_{A_1}}\,\sqrt{\mathcal{G}_{A_2}} + \sqrt{\mathcal{G}_{B_1}}\,\sqrt{\mathcal{G}_{B_2}} + (\Delta t)^{\kappa} + (\Delta \tau)^{2\kappa}\,\right],$$

where

$$\mathcal{G}_{A_1} := \mathbb{E}\left[\,\left|G(\Lambda_{\Psi}(g))\right|^{4\kappa}\,\right], \quad \mathcal{G}_{A_2} := \mathbb{E}\left[\,\left|F(Y(t_{\max})) - F(\overline{\overline{g}}_{N,L})\right|^{4\kappa}\,\right]$$

$$\mathcal{G}_{B_1} := \mathbb{E}\left[\,\left|F(\overline{\overline{g}}_{N,L})\right|^{4\kappa}\,\right], \quad \mathcal{G}_{B_2} := \mathbb{E}\left[\,\left|G(\Lambda_{\Psi}(g)) - G(\Lambda_{\Psi}(\overline{g}))\right|^{4\kappa}\,\right].$$

Since F and G have polynomial growth, we combine (3.12) and (3.51) to get

(3.80)
$$\mathcal{G}_{A_1} + \mathcal{G}_{B_1} \leq C.$$

Since F' has polynomial growth, we use the mean value theorem, the Cauchy-Schwarz inequality, (3.12) and the error bound (3.51) to have

(3.81)
$$\mathcal{G}_{A_2} \leq \left(\mathbb{E}\left[\max_{\epsilon \in [0,1]}\left|F'(\epsilon\, Y(t_{\max}) + (1-\epsilon)\,\overline{\overline{g}}_{N,L})\right|^{8\kappa}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\,|Y(t_{\max}) - \overline{\overline{g}}_{N,L}|^{8\kappa}\right]\right)^{\frac{1}{2}}$$
$$\leq C\left[(\Delta t)^{2\kappa} + (\Delta \tau)^{4\kappa}\right].$$

Similarly, since G' has polynomial growth, we use the mean value theorem, the Cauchy-Schwarz inequality, (3.12), and the error bound (3.53) to have

(3.82)
$$\mathcal{G}_{B_2} \leq \left(\mathbb{E}\left[\max_{\epsilon \in [0,1]}\left|G'\left(\epsilon\, \Lambda_{\Psi}(g) + (1-\epsilon)\,\Lambda_{\Psi}(\overline{g})\right)\right|^{8\kappa}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\,|\Lambda_{\Psi}(g) - \Lambda_{\Psi}(\overline{g})|^{8\kappa}\right]\right)^{\frac{1}{2}}$$
$$\leq C\left[(\Delta t)^{2\kappa} + (\Delta \tau)^{4\kappa}\right].$$

Thus, the error bound (3.54) is a simple consequence of (3.79), (3.80), (3.81) and (3.82). $\qquad\square$

Theorem 3.3. *Let* $\mathcal{I} := \{0,\dots,N\} \times \{0,\dots,L+1\}$, $(\overline{\overline{g}}_{n,\ell})_{(n,\ell)\in\mathcal{I}}$ *be the numerical approximations produced by the* (EFD) *or the* (EFM) *method,* $\Lambda_{\Psi}(\overline{g})$ *be defined as in the Theorem 3.2 and* $\overline{\Lambda}_{\Psi,Q}(\overline{g})$ *be the quantity defined by* (2.9). *We assume that the quadrature rule* Q *used in* (2.9) *is of order* p_Q, $\Psi \in C^{p_Q}(\mathbb{R};\mathbb{R})$ *and* $f_0 \in C^{p_Q}([0,\tau_{\max}];\mathbb{R})$. *Also, we assume that* Ψ *and all its derivatives up to order* p_Q, *along with the functions* F *and* G', *have polynomial growth. Then, for* $\kappa \in \mathbb{N}$, *there exist constants* $C_{\kappa}^{Q,A}$ *and* $C_{\kappa}^{Q,B}$, *independent of the partitions of the intervals* $[0,t_{\max}]$ *and* $[0,\tau_{\max}]$, *such that*

(3.83)
$$\left(\mathbb{E}\left[\,\left|\Lambda_{\Psi}(\overline{g}) - \overline{\Lambda}_{\Psi,Q}(\overline{g})\right|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_{\kappa}^{Q,A}\,(\Delta\tau)^{p_Q}$$

and

(3.84)
$$\left(\mathbb{E}\left[\,\left|\mathcal{G}(\overline{g}) - \overline{\mathcal{G}}(\overline{g})\right|^{2\kappa}\right]\right)^{\frac{1}{2\kappa}} \leq C_{\kappa}^{Q,B}\,(\Delta\tau)^{p_Q}.$$

Proof. For $\ell = \ell_a,\dots,L-1$, we set $v_{\ell}(s) := \Psi(\overline{\overline{g}}_{N,\ell} + f_0(\tau_{\ell} + s\,\Delta\tau_{\ell}))$ for $s \in [0,1]$. Since the quadrature rule Q has order p_Q, applying a standard argument from the error analysis for quadrature rules based on the Taylor formula (see, e.g., [2]), we obtain

(3.85)
$$\mathbb{E}\left[\,\left|\Lambda_{\Psi}(\overline{g}) - \overline{\Lambda}_{\Psi,Q}(\overline{g})\right|^{2\kappa}\right] \leq C\,(\Delta\tau)^{2\kappa\, p_Q}\,\mathbb{E}\left[\max_{\ell_a \leq \ell \leq L-1}\max_{[0,1]}\left|\partial_s^{p_Q} v_{\ell}\right|^{2\kappa}\right].$$

Observing that $\partial_s^{p_Q} v_{\ell}(s) = \sum_{j=0}^{p_Q}\binom{p_Q}{j}\Psi^{(j)}(\overline{\overline{g}}_{N,\ell} + f_0(\tau_{\ell} + s\,\Delta\tau_{\ell}))\,f_0^{(p_Q - j)}(\tau_{\ell} + s\,\Delta\tau_{\ell})$, assuming that $\Psi^{(j)}$ has polynomial growth p_j for $j = 1,\dots,p_Q$, and using (3.20), we obtain

(3.86)
$$\mathbb{E}\left[\max_{\ell_a \leq \ell \leq L-1}\sup_{[0,1]}\left|\partial_s^{p_Q} v_{\ell}\right|^{2\kappa}\right] \leq C\max_{\ell_a \leq \ell \leq L-1}\sum_{j=0}^{p_Q}\left(1 + |\overline{\overline{g}}_{N,\ell}|^{2\kappa p_j}\right)$$
$$\leq C.$$

Now, combine (3.85) and (3.86) to arrive at (3.83).

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Since F and G' have polynomial growth, using the Cauchy-Schwarz inequality, the mean value theorem, (3.51), (3.12), (3.83) and (3.53), we obtain

$$\mathbb{E}\left[\left|\mathcal{G}(\overline{g})-\overline{\mathcal{G}}(\overline{g})\right|^{2\kappa}\right]\leq C\,\left(\mathbb{E}\left[\left|F(\overline{g}_{N,L})\right|^{4\kappa}\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|G(\Lambda_{\Psi}(\overline{g}))-G(\overline{\Lambda}_{\Psi,Q}(\overline{g}))\right|^{4\kappa}\right]\right)^{\frac{1}{2}}$$

$$\leq C\,\left(\mathbb{E}\left[\sup_{\epsilon\in[0,1]}\left|G'\left(\epsilon\,\Lambda_{\Psi}(\overline{g})+(1-\epsilon)\,\overline{\Lambda}_{\Psi,Q}(\overline{g})\right)\right|^{8\kappa}\right]\right)^{\frac{1}{4}}\left(\mathbb{E}\left[\left|\Lambda_{\Psi}(\overline{g})-\overline{\Lambda}_{\Psi,Q}(\overline{g})\right|^{8\kappa}\right]\right)^{\frac{1}{4}}$$

$$\leq C\,(\Delta\tau)^{2\kappa p_Q}$$

which yields the estimate (3.84). $\qquad\square$

4. Computable Weak Error Approximation

In this section we present a computable approximation for the weak $t-$ and $\tau-$ discretization error E_D defined in (2.14) for the (EFD) method. In Theorem 4.1 below we give an estimate of E_D which, as the step size of both the time and maturity time partitions go to zero and the number of realizations goes to infinity, is asymptotically correct. On the other hand, the statistical error E_S can be analyzed by the Central Limit Theorem or Berry-Esseen Theorem, a standard procedure in Monte Carlo methods (cf. Section 5). While, in Theorem 3.3 we have estimated the quadrature error E_Q, concluding that when the order p_Q of the quadrature rule Q we use in (2.9) is sufficiently large, the quadrature error, E_Q, is a higher order term in the expansion of the computational error.

To have an easier access to the results and the techniques of [21], we reformulate problem (1.9)-(1.10), letting the process $g=g(t,\tau)$ be the solution of the problem

$$(4.1)\qquad\begin{aligned}dg(t,\tau)&=a(t,\tau,g(t,t))\,dt+b(t,\tau,g(t,t))\cdot dW(t),\quad\forall t\in[0,t_{\max}],\\ g(0,\tau)&=0,\end{aligned}$$

for $\tau\in[0,\tau_{\max}]$, where $a:[0,t_{\max}]\times[0,\tau_{\max}]\times\mathbb{R}\to\mathbb{R}$, $b:[0,t_{\max}]\times[0,\tau_{\max}]\times\mathbb{R}\to\mathbb{R}^J$ given by

$$a(t,\tau,x)\equiv\xi^2(x+f_0(t))\,\widetilde{\lambda}(t,\tau),$$
$$b(t,\tau,x)\equiv\xi(x+f_0(t))\,\lambda(t,\tau).$$

We approximate the unknown process $g(t,\tau)$ by a time and maturity discretization $\overline{\overline{g}}(t,\tau)$, with $t\in(t_n)_{n=0}^N$ and $\tau\in(\tau_\ell)_{\ell=0}^{L-1}$, based on the (EFD) method, which, for $n=0,\ldots,N-1$, reads

$$(4.2)\qquad\begin{aligned}\overline{\overline{g}}(t_{n+1},\tau_\ell)&=\overline{\overline{g}}(t_n,\tau_\ell)+a(t_n,\tau_\ell,\overline{\overline{g}}(t_n,\tau_{\ell_n}))\,\Delta t_n+b(t_n,\tau_\ell,\overline{\overline{g}}(t_n,\tau_{\ell_n}))\cdot\Delta W_n,\quad\ell=0,\ldots,L-1,\\ \overline{\overline{g}}(0,\tau_\ell)&=0,\quad\ell=0,\ldots,L.\end{aligned}$$

For the analysis of the (EFD) method, it is useful to extend its definition for all times t and all maturities τ as follows: for $n=0,\ldots,N-1$ and $\ell=0,\ldots,L-1$, set

$$(4.3)\qquad\begin{aligned}\overline{\overline{g}}(t,\tau)&=\overline{\overline{g}}(t_n,\tau_\ell)+a(t_n,\tau_\ell,\overline{\overline{g}}(t_n,\tau_{\ell_n}))(t-t_n)+b(t_n,\tau_\ell,\overline{\overline{g}}(t_n,\tau_{\ell_n}))\cdot(W(t)-W(t_n))\\ &=\overline{\overline{g}}(t_n,\tau_\ell)+\int_{t_n}^t\overline{\overline{a}}(s,\tau,\overline{\overline{g}}(t_n,\tau_{\ell_n}))ds+\int_{t_n}^t\overline{\overline{b}}(s,\tau,\overline{\overline{g}}(t_n,\tau_{\ell_n}))\cdot dW(s),\quad\forall t\in[t_n,t_{n+1}),\\ \overline{\overline{g}}(0,\tau)&=0,\end{aligned}$$

for $\tau\in[\tau_\ell,\tau_{\ell+1})$, where $\overline{\overline{a}}$ and $\overline{\overline{b}}$ are the piecewise constant approximations

$$(4.4)\qquad\begin{aligned}\overline{\overline{a}}(t,\tau,x)\big|_{(t,\tau)\in[t_n,t_{n+1})\times[\tau_\ell,\tau_{\ell+1})}&\equiv a(t_n,\tau_\ell,x)=\xi^2(x+f_0(t_n))\,\widetilde{\lambda}(t_n,\tau_\ell),\\ \overline{\overline{b}}(t,\tau,x)\big|_{(t,\tau)\in[t_n,t_{n+1})\times[\tau_\ell,\tau_{\ell+1})}&\equiv b(t_n,\tau_\ell,x)=\xi(x+f_0(t_n))\,\lambda(t_n,\tau_\ell).\end{aligned}$$

Thus, the extension above results in $\overline{\overline{g}}(t,.)\in S_{\Delta\tau}$ for any time $t\in[0,t_{\max}]$.

Theorem 4.1. *Let $\mathcal{I}:=\{0,\ldots,N\}\times\{0,\ldots,L-1\}$, $(\overline{\overline{g}}_{n,\ell})_{(n,\ell)\in\mathcal{I}}$ be the numerical approximations produced by the (EFD) method. Also, we assume that the functions F, U, Ψ, G along with their derivatives have polynomial growth. Also, we set*

$$(4.5)\qquad d(t,\tau,\widetilde{\tau},x):=\tfrac{1}{2}\,\xi^2(x+f_0(t))\,\lambda(t,\widetilde{\tau})\cdot\lambda(t,\tau),$$

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for $x \in \mathbb{R}$, $t \in [0, t_{\max}]$ and τ, $\widetilde{\tau} \in [0, \tau_{\max}]$. Then the computational error of the (EFD) method has the expansion

$$(4.6) \qquad E_D := \mathbb{E}\left[\mathcal{G}(g)\right] - \mathbb{E}\left[\mathcal{G}(\overline{g})\right] = E_{D,\mathrm{tau}} + E_{D,\mathrm{tim}} + \mathcal{O}((\Delta t)^2 + (\Delta \tau)^2),$$

where

$$(4.7) \qquad \begin{aligned} E_{D,\mathrm{tau}} &= \sum_{n=0}^{N-1} \Delta t_n \left\{ \sum_{\ell=0}^{L-1} \Delta \tau_\ell \, \mathbb{E}\left[\frac{a(t_n, \tau_{\ell+1}, \overline{\overline{g}}(t_n, \tau_{\ell_n})) - a(t_n, \tau_\ell, \overline{\overline{g}}(t_n, \tau_{\ell_n}))}{2} \, \overline{\overline{\varphi}}_{n,\ell} \right] \right\} \\ &+ \sum_{n=0}^{N-1} \Delta t_n \left\{ \sum_{\ell=0}^{L-1} \sum_{\ell'=0}^{L-1} \Delta \tau_\ell \, \Delta \tau_{\ell'} \, \mathbb{E}\left[\frac{d(t_n, \tau_{\ell+1}, \tau_{\ell'+1}, \overline{\overline{g}}(t_n, \tau_{\ell_n})) - d(t_n, \tau_\ell, \tau_{\ell'}, \overline{\overline{g}}(t_n, \tau_{\ell_n}))}{2} \, \overline{\overline{\varphi}}'_{n,\ell,\ell'} \right] \right\}, \end{aligned}$$

and

$$(4.8) \qquad \begin{aligned} E_{D,\mathrm{tim}} &= \sum_{n=0}^{N-1} \frac{\Delta t_n}{2} \left\{ \mathbb{E}\left[\left(F(\overline{g}_{n+1,L}) U(\overline{r}_{n+1}) - F(\overline{g}_{n,L}) U(\overline{r}_n) \right) \overline{\overline{\varphi}}_{n+1,L+1} \right] \right. \\ &\qquad + \mathbb{E}\left[\left(\overline{\overline{r}}_{n+1} - \overline{\overline{r}}_n \right) \overline{\overline{\varphi}}_{n+1,L} \right] \\ &\qquad \left. + \sum_{\ell=0}^{L-1} \mathbb{E}\left[\left(a(t_{n+1}, \tau_\ell, \overline{\overline{g}}(t_{n+1}, t_{n+1})) - a(t_n, \tau_\ell, \overline{\overline{g}}(t_n, t_n)) \right) \overline{\overline{\varphi}}_{n+1,\ell} \right] \right\} \\ &+ \sum_{n=0}^{N-1} \frac{\Delta t_n}{2} \left\{ \sum_{\ell,\ell'=0}^{L-1} \mathbb{E}\left[\left(d(t_{n+1}, \tau_\ell, \tau_{\ell'}, \overline{\overline{g}}(t_{n+1}, t_{n+1})) - d(t_n, \tau_\ell, \tau_{\ell'}, \overline{\overline{g}}(t_n, t_n)) \right) \overline{\overline{\varphi}}'_{n+1,\ell,\ell'} \right] \right\} \end{aligned}$$

with

$$\overline{\overline{r}}_n := \overline{\overline{g}}(t_n, t_n) + f_0(t_n) = \overline{\overline{g}}_{n,\ell_n} + f_0(t_n).$$

The two leading order terms $E_{D,\mathrm{tau}}$ and $E_{D,\mathrm{tim}}$ in the right hand side of (4.6) are in a posteriori form and based on the discrete duals $\overline{\overline{\varphi}}_n \in \mathbb{R}^{L+2}$ and $\overline{\overline{\varphi}}'_n \in \mathbb{R}^{(L+2) \times (L+2)}$ which are determined as follows. First, set

$$\overline{\Lambda}'_{\Psi,Q,\ell}(\overline{g}) := \Delta \tau_\ell \sum_{i=1}^{N_Q} w_{Q,i} \, \Psi'\left(\overline{g}_{N,\ell} + f_0(\tau_\ell + s_{Q,i} \, \Delta \tau_\ell) \right),$$

$$\overline{\Lambda}''_{\Psi,Q,\ell}(\overline{g}) := \Delta \tau_\ell \sum_{i=1}^{N_Q} w_{Q,i} \, \Psi''\left(\overline{g}_{N,\ell} + f_0(\tau_\ell + s_{Q,i} \, \Delta \tau_\ell) \right)$$

for $\ell = \ell_a, \dots, L-1$, and

$$c_{n,j}(x) := a(t_n, \tau_j, x) \, \Delta t_n + b(t_n, \tau_j, x) \cdot \Delta W_n$$

for $x \in \mathbb{R}$ and $j = 0, \dots, L-1$. Then, the first dual $\overline{\overline{\varphi}}$ is defined by the dual backward problem with final datum

$$(4.9) \qquad \overline{\overline{\varphi}}_{N,\ell} = \begin{cases} 0, & \ell = 0, \dots, \ell_a - 1, \\ F(\overline{g}_{N,L}) \, G'(\overline{\Lambda}_{\Psi,Q}(\overline{g})) \, \overline{\Lambda}'_{\Psi,Q,\ell}(\overline{g}), & \ell = \ell_a, \dots, L-1, \\ F'(\overline{g}_{N,L}) \, G(\overline{\Lambda}_{\Psi,Q}(\overline{g})), & \ell = L, \\ 1, & \ell = L+1, \end{cases}$$

and

$$(4.10) \qquad \overline{\overline{\varphi}}_{n,\ell} = \begin{cases} \overline{\overline{\varphi}}_{n+1,\ell}, & \ell \in \{0, \dots, L-1\} \setminus \{\ell_n\}, \\ \Delta t_n \, \overline{\overline{\varphi}}_{n+1,L} + \Delta t_n \, F(\overline{g}_{n,L}) \, U'\left(\overline{\overline{r}}_n \right) \overline{\overline{\varphi}}_{n+1,L+1} \\ \quad + \sum_{j=0}^{L-1} c'_{n,j}(\overline{g}_{n,\ell_n}) \, \overline{\overline{\varphi}}_{n+1,j} + \overline{\overline{\varphi}}_{n+1,\ell_n}, & \ell = \ell_n \\ \overline{\overline{\varphi}}_{n+1,L} + \Delta t_n \, F'(\overline{g}_{n,L}) \, U\left(\overline{\overline{r}}_n \right) \overline{\overline{\varphi}}_{n+1,L+1}, & \ell = L, \\ \overline{\overline{\varphi}}_{n+1,L+1}, & \ell = L+1, \end{cases}$$

for $n = N-1, \ldots, 0$. The second dual, $\overline{\overline{\varphi}}'$, has final datum

$$
(4.11) \quad \overline{\overline{\varphi}}'_{N,\ell,\ell'} = \begin{cases}
0, & \ell = 0, \ldots, \ell_a - 1,\ \ell' = 0, \ldots, L+1, \\[4pt]
F(\overline{\overline{g}}_{N,L})\, G''(\overline{\Lambda}_{\Psi,Q}(\overline{g}))\, \overline{\Lambda}_{\Psi,Q,\ell}(\overline{g})\, \overline{\Lambda}_{\Psi,Q,\ell'}(\overline{g}), & \ell, \ell' \in \{\ell_a, \ldots, L-1\},\ \ell \neq \ell', \\[4pt]
F(\overline{\overline{g}}_{N,L}) \left[G''(\overline{\Lambda}_{\Psi,Q}(\overline{g}))(\overline{\Lambda}'_{\Psi,Q,\ell}(\overline{g}))^2 \right. & \\
\qquad\qquad \left. + G'(\overline{\Lambda}_{\Psi,Q}(\overline{g}))\, \overline{\Lambda}''_{\Psi,Q,\ell}(\overline{g}) \right], & \ell \in \{\ell_a, \ldots, L-1\},\ \ell' = \ell, \\[4pt]
F'(\overline{\overline{g}}_{N,L})\, G'(\overline{\Lambda}_{\Psi,Q}(\overline{g}))\, \overline{\Lambda}'_{\Psi,Q,\ell}(\overline{g}), & \ell = \ell_a, \ldots, L-1,\ \ell' = L, \\[4pt]
0, & \ell = \ell_a, \ldots, L,\ \ell' = L+1, \\[4pt]
0, & \ell = \ell_a, \ldots, L+1,\ \ell' = 0, \ldots, \ell_a - 1, \\[4pt]
F'(\overline{\overline{g}}_{N,L})\, G'(\overline{\Lambda}_{\Psi,Q}(\overline{g}))\, \overline{\Lambda}'_{\Psi,Q,\ell'}(\overline{g}), & \ell = L,\ \ell' = \ell_a, \ldots, L-1, \\[4pt]
F''(\overline{\overline{g}}_{N,L})\, G(\overline{\Lambda}_{\Psi,Q}(\overline{g})), & \ell = L,\ \ell' = L, \\[4pt]
0, & \ell = L+1,\ \ell' = \ell_a, \ldots, L+1,
\end{cases}
$$

and solves the recursion

$$
(4.12) \quad \overline{\overline{\varphi}}'_{n,\ell,\ell'} = \begin{cases}
\overline{\overline{\varphi}}'_{n+1,\ell,\ell'}, & \ell, \ell' \in \{0, \ldots, L-1\} \setminus \{\ell_n\}, \\[6pt]
\displaystyle\sum_{j,p=0}^{L-1} (\delta_{j,\ell_n} + c'_{n,j}(\overline{\overline{g}}_{n,\ell_n}))\, (\delta_{p,\ell_n} + c'_{n,p}(\overline{\overline{g}}_{n,\ell_n}))\, \overline{\overline{\varphi}}'_{n+1,j,p} & \\[6pt]
\quad + 2\,\Delta t_n \displaystyle\sum_{j=0}^{L-1} (\delta_{\ell_n,j} + c'_{n,j}(\overline{\overline{g}}_{n,\ell_n}))\, \overline{\overline{\varphi}}'_{n+1,j,L} & \\[6pt]
\quad + 2\,\Delta t_n \displaystyle\sum_{j=0}^{L-1} (\delta_{\ell_n,j} + c'_{n,j}(\overline{\overline{g}}_{n,\ell_n}))\, F(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,j,L+1} & \\[6pt]
\quad + \overline{\overline{\varphi}}'_{n+1,L,L} (\Delta t_n)^2 + \overline{\overline{\varphi}}'_{n+1,L+1,L+1} (\Delta t_n)^2 (F(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n))^2 & \\[6pt]
\quad + 2\,\Delta t_n\, F(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L,L+1} & \\[6pt]
\quad + \displaystyle\sum_{j=0}^{L-1} c''_{n,j}(\overline{\overline{g}}_{n,\ell_n})\, \overline{\varphi}_{n+1,j} + \Delta t_n\, F(\overline{\overline{g}}_{n,L})\, U''(\overline{\overline{r}}_n)\, \overline{\varphi}_{n+1,L+1}, & \ell = \ell' = \ell_n, \\[6pt]
\displaystyle\sum_{j=0}^{L-1} (\delta_{j,\ell_n} + c'_{n,j}(\overline{\overline{g}}_{n,\ell_n}))\, \overline{\overline{\varphi}}'_{n+1,j,\ell'} + \Delta t_n\, \overline{\overline{\varphi}}'_{n+1,L,\ell'} & \\[6pt]
\quad + \Delta t_n\, F(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L+1,\ell'}, & \ell = \ell_n,\ \ell' \in \{0, \ldots, L-1\} \setminus \{\ell_n\} \\[6pt]
\overline{\overline{\varphi}}'_{n,\ell',\ell}, & \ell \in \{0, \ldots, L-1\} \setminus \{\ell_n\},\ \ell' = \ell_n,
\end{cases}
$$

$$
(4.13) \quad \overline{\overline{\varphi}}'_{n,\ell,\ell'} = \begin{cases}
\overline{\overline{\varphi}}'_{n+1,L,\ell'} + \Delta t_n\, F'(\overline{\overline{g}}_{n,L})\, U(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L+1,\ell'}, & \ell = L,\ \ell' \in \{0, \ldots, L-1\} \setminus \{\ell_n\}, \\[4pt]
\overline{\overline{\varphi}}'_{n,\ell',\ell}, & \ell \in \{0, \ldots, L-1\} \setminus \{\ell_n\},\ \ell' = L,
\end{cases}
$$

$$
\begin{aligned}
(4.14) \quad \overline{\overline{\varphi}}'_{n,\ell,\ell'} =\ & \Delta t_n\, F'(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}_{n+1,L+1} + \sum_{j=0}^{L-1} (\delta_{j,\ell_n} + c'_{n,j}(\overline{\overline{g}}_{n,\ell_n}))\, \overline{\overline{\varphi}}'_{n+1,L,j} \\
& + \Delta t_n\, F'(\overline{\overline{g}}_{n,L})\, U(\overline{\overline{r}}_n))\, \sum_{j=0}^{L-1} (\delta_{j,\ell_n} + c'_{n,j}(\overline{\overline{g}}_{n,\ell_n}))\, \overline{\overline{\varphi}}'_{n+1,L+1,j} \\
& + (\Delta t_n)^2\, F'(\overline{\overline{g}}_{n,L})\, U(\overline{\overline{r}}_n))\, \left[\overline{\overline{\varphi}}'_{n+1,L+1,L} + F(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L+1,L+1} \right] \\
& + \Delta t_n \left[\overline{\overline{\varphi}}'_{n+1,L,L} + F(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L,L+1} \right], \quad (\ell, \ell') \in \{(L, \ell_n), (\ell_n, L)\},
\end{aligned}
$$

$$
\begin{aligned}
\overline{\overline{\varphi}}'_{n,\ell,\ell'} =& \Delta t_n\, F''(\overline{\overline{g}}_{n,L})\, U(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}_{n+1,L+1} + \overline{\overline{\varphi}}'_{n+1,L,L} + 2\,\Delta t_n\, F'(\overline{\overline{g}}_{n,L})\, U(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L+1,L} \\
& + (\Delta t_n)^2\, (F'(\overline{\overline{g}}_{n,L})\, U(\overline{\overline{r}}_n))^2\, \overline{\overline{\varphi}}'_{n+1,L+1,L+1}, \quad \ell = \ell' = L,
\end{aligned}
\tag{4.15}
$$

$$
\overline{\overline{\varphi}}'_{n,\ell,\ell'} = \overline{\overline{\varphi}}'_{n+1,L,L+1} + \Delta t_n\, F'(\overline{\overline{g}}_{n,L})\, U(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L+1,L+1}, \quad (\ell,\ell') \in \{(L,L+1),(L+1,L)\},
\tag{4.16}
$$

and

$$
\overline{\overline{\varphi}}'_{n,\ell,\ell'} =
\begin{cases}
\overline{\overline{\varphi}}'_{n+1,L+1,\ell'}, & \ell = L+1,\ \ell' \in \{0,\dots,L-1,L+1\}\setminus\{\ell_n\}, \\[2mm]
\displaystyle\sum_{j=0}^{L-1}(\delta_{j,\ell_n} + c'_{n,j}(\overline{\overline{g}}_{n,\ell_n}))\,\overline{\overline{\varphi}}'_{n+1,L+1,j} + \Delta t_n\, \overline{\overline{\varphi}}'_{n+1,L+1,L} & \\[2mm]
\quad + \Delta t_n\, F(\overline{\overline{g}}_{n,L})\, U'(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}'_{n+1,L+1,L+1}, & \ell = L+1,\ \ell' = \ell_n, \\[2mm]
\overline{\overline{\varphi}}'_{n,\ell',\ell}, & \ell \in \{0,\dots,L-1,L+1\},\ \ell' = L+1.
\end{cases}
\tag{4.17}
$$

Proof. The proof is an application of Theorem 2.2 in [21]. To be able to split the time and maturity time discretization errors, introduce the semidiscretized fluxes $\overline{a}$ and $\overline{b}$ that, for $\tau_\ell \le \tau < \tau_{\ell+1}$, are defined as $\overline{a}(t,\tau,x) \equiv \xi^2(x + f_0(t))\, \widetilde{\lambda}(t,\tau_\ell)$, $\overline{b}(t,\tau,x) \equiv \xi(x + f_0(t))\, \lambda(t,\tau_\ell)$ and denote by $\overline{g}$ the corresponding semidiscrete in τ solution. As a first step, replace the exact solution of (4.1), g, by a finite dimensional approximation: a piecewise constant $g_*(t,\cdot)$, which is an Euler approximation with a much finer discretization, both in time t and maturity time τ, than $\overline{\overline{g}}$. Thus, g_* uses a time grid $(\hat{t}_n)_{n=0}^P$ much finer than $(t_n)_{n=0}^N$, and a maturity time grid, $(\hat{\tau}_\ell)_{\ell=0}^M$ much finer than $(\tau_\ell)_{\ell=0}^L$. Consequently, the number of time steps satisfy $P \gg N$, $M \gg L$, respectively, and $\Delta\hat{t} := \max_{0 \le m \le P-1}\hat{t}_{m+1} - \hat{t}_m \ll \Delta t$, $\Delta\hat{\tau} := \max_{0 \le m \le M-1}\hat{\tau}_{m+1} - \hat{\tau}_m \ll \Delta\tau$. In the application of Theorem 2.2 in [21], include the τ-discretization error terms $a - \overline{a}$, $b - \overline{b}$ as well as the t-discretization terms $\overline{a} - \overline{\overline{a}}$, $\overline{b} - \overline{\overline{b}}$ in the error expansion, following Lemmata 2.1-2.5 in [21], to obtain (4.6-4.8) for g replaced by the piecewise constant process g_*. For this purpose, observe that $\overline{g}$ can be also thought of as a piecewise constant function on the finer τ-partition that defines g_*. The second step is to let $M, P \to \infty$ and $\Delta\hat{\tau}, \Delta\hat{t} \to 0$, using

$$
\left(\mathbb{E}\left[\max_{[0,\tau_{\max}]}|g(t,\cdot) - g_*(t,\cdot)|^2 + \left|\tfrac{[g(t,\hat{\tau}_{m+1})-g(t,\hat{\tau}_m)]-[g_*(t,\hat{\tau}_{m+1})-g_*(t,\hat{\tau}_m)]}{\hat{\tau}_{m+1}-\hat{\tau}_m}\right|\right]\right)^{\frac{1}{2}} = \mathcal{O}\left(\Delta\hat{\tau} + (\Delta\hat{t})^{\frac{1}{2}}\right),
$$

for $t \in [0, t_{\max}]$ and $m = 0, \dots, M-2$, along with similar estimates for the corresponding dual functions $\overline{\overline{\varphi}}, \overline{\overline{\varphi}}', \dots$, to control the higher order terms in the error expansion. The latter strong convergence estimates follow moving along the lines of the analysis of Section 3. $\qquad\square$

Remark 4.1. *In the* (EFD) *method the τ-discretization error of* (4.6) *and* (4.7) *can, by* (4.4), (4.5), *be expressed by*

$$
\begin{aligned}
E_{D,\text{tau}} =& \sum_{n=0}^{N-1}\Delta t_n\Bigg\{\sum_{\ell=0}^{L-1}\Delta\tau_\ell\, \mathbb{E}\left[\xi^2(\overline{\overline{r}}_n)\, \overline{\overline{\varphi}}_{n,\ell}\right]\, \tfrac{\widetilde{\lambda}(t_n,\tau_{\ell+1}) - \widetilde{\lambda}(t_n,\tau_\ell)}{2} \\
& + \tfrac{1}{4}\sum_{\ell,\ell'=0}^{L-1}\mathbb{E}\left[\xi^2(r_n)\overline{\overline{\varphi}}'_{n,\ell,\ell'}\right]\, [\lambda(t_n,\tau_{\ell+1})\cdot\lambda(t_n,\tau_{\ell'+1}) - \lambda(t_n,\tau_\ell)\cdot\lambda(t_n,\tau_{\ell'})]\,\Delta\tau_\ell\Delta\tau_{\ell'}\Bigg\}
\end{aligned}
\tag{4.18}
$$

and the time discretization is

$$
\begin{aligned}
E_{D,\text{tim}} =& \sum_{n=0}^{N-1}\tfrac{\Delta t_n}{2}\Bigg\{\mathbb{E}\left[F(\overline{\overline{g}}_{n+1,L})U(\overline{\overline{r}}_{n+1}) - F(\overline{\overline{g}}_{n,L})U(\overline{\overline{r}}_n)\right] + \mathbb{E}\left[(\overline{\overline{r}}_{n+1} - \overline{\overline{r}}_n)\overline{\overline{\varphi}}_{n+1,L}\right] \\
& + \sum_{\ell=0}^{L-1}\mathbb{E}\left[(\xi^2(\overline{\overline{r}}_{n+1})\widetilde{\lambda}(t_{n+1},\tau_\ell) - \xi^2(\overline{\overline{r}}_n)\widetilde{\lambda}(t_n,\tau_\ell))\overline{\overline{\varphi}}_{n+1,\ell}\right] \\
& + \tfrac{1}{2}\sum_{\ell,\ell'=0}^{L-1}\mathbb{E}\Big[\left(\xi^2(\overline{\overline{r}}_{n+1})\lambda(t_{n+1},\tau_\ell)\cdot\lambda(t_{n+1},\tau_{\ell'})\right. \\
& \qquad\qquad\left. - \xi^2(\overline{\overline{r}}_n)\lambda(t_n,\tau_\ell)\cdot\lambda(t_n,\tau_{\ell'})\right)\overline{\overline{\varphi}}'_{n+1,\ell,\ell'}\Big]\Bigg\}.
\end{aligned}
\tag{4.19}
$$

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In Monte Carlo computations all the expected values in (4.18) and (4.19) are naturally approximated by sample averages.

Remark 4.2. *The analysis of the* (EFE) *method follows a similar line as the estimates of the* (EFD) *method. The difference lies in the τ-discretization error, which by virtue of the orthogonality of both $\widetilde{\lambda} - \Pi\widetilde{\lambda}$ and $\lambda - \Pi\lambda$ to the subspace of piecewise constant functions $S_{\Delta\tau}$, becomes second order accurate. Therefore, more careful expansions, including interpolation estimates, need to be carried out in order to capture the second order contributions from the τ-discretization.*

5. Numerical experiments

In this section we provide numerical evidence for the weak computational error (2.12) of the numerical methods defined in Section 2 approximating the quantity of interest $\mathbb{E}[\mathcal{F}(f)] = \mathbb{E}[\mathcal{G}(g)]$ described in (1.4-1.6). In particular, we show results from numerical experiments with examples that have known exact solution which permit a straightforward derivation of an exact solution to compare with. The implementation uses double precision `FORTRAN 77` and simulates the increments of the J independent Wiener processes by a double precision modification of the functions `ran1` and `gasdev` proposed in [18]. The numerical quadrature approximation $\overline{\Lambda}_{\Psi,Q}(\overline{g})$ of $\Lambda_{\Psi}(\overline{g})$ in (2.9) is done via the use of Simpson's quadrature rule. For the particular case of the (EFD) method, the estimates for the computational error developed in Theorem 4.1 are compared with the exact computational error. The numerical results obtained are in agreement with the theory and the work to compute these estimates is small.

5.1. Control of the statistical error.
For M independent samples $\{Y(\omega_j)\}_{j=1}^{M}$ of a random variable Y, with $\mathbb{E}[|Y|^6] < \infty$, define the sample average $\mathcal{A}(Y;M)$ and the sample standard deviation $\mathcal{S}(Y;M)$ of Y by

$$\mathcal{A}(Y;M) \equiv \frac{1}{M}\sum_{j=1}^{M} Y(\omega_j) \quad \text{and} \quad \mathcal{S}(Y;M) \equiv \left[\mathcal{A}(Y^2;M) - (\mathcal{A}(Y;M))^2\right]^{\frac{1}{2}}.$$

Let $\sigma \equiv \sqrt{\mathbb{E}[|Y - \mathbb{E}[Y]|^2]}$ and consider the random variable

$$Z_M \equiv \frac{\sqrt{M}}{\sigma}\left(\mathcal{A}(Y;M) - \mathbb{E}[Y]\right)$$

with cumulative distribution function $F_{Z_M}(x) \equiv P(Z_M \leq x)$, for $x \in \mathbb{R}$. Let

$$\lambda \equiv \frac{1}{\sigma}\left(\mathbb{E}\left[|Y - \mathbb{E}[Y]|^3\right]\right)^{\frac{1}{3}} < \infty,$$

then the Berry-Esseen theorem (cf. [10] p. 126), gives the following estimate in the central limit theorem

$$\sup_{x\in\mathbb{R}} |F_{Z_M}(x) - \Phi(x)| \leq \frac{3}{\sqrt{M}}\lambda^3$$

for the rate of convergence of F_{Z_M} to the distribution function, Φ, of a normal random variable with mean zero and variance one, i.e.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}\,ds.$$

Since in the examples below M is sufficiently large, i.e. $M \gg 36\,\lambda^6$, the statistical error

$$\mathcal{E}_s(Y;M) \equiv \mathbb{E}[Y] - \mathcal{A}(Y;M)$$

satisfies, by the Berry-Esseen theorem, the following probability approximation

$$P\left(\left[|\mathcal{E}_s(Y;M)| \leq c_0\,\frac{\sigma}{\sqrt{M}}\right]\right) \simeq 2\Phi(c_0) - 1.$$

In practice choose some constant $c_0 \geq 1.65$, so the normal distribution satisfies

$$1 > 2\Phi(c_0) - 1 \geq 0.901$$

and the event

(5.1) $$|\mathcal{E}_s(Y;M)| \leq \mathrm{E}_s(Y;M) \equiv c_0\,\frac{\mathcal{S}(Y;M)}{\sqrt{M}}$$

has probability close to one, which involves the additional step to approximate σ by $\mathcal{S}(Y;M)$, cf. [11]. Thus, in the computations $\mathrm{E}_s(Y;M)$ is a good approximation of the statistical error $\mathcal{E}_s(Y;M)$.

For a given TOL > 0, the goal is to find M such that $\mathrm{E}_S(Y; M) \leq$ TOL. The algorithm described in [21] adaptively finds the number of realizations M to compute the sample average $\mathcal{A}(Y; M)$ as an approximation to $\mathbb{E}[Y]$. With large probability, depending on c_0, the statistical error in the approximation is then bounded by TOL. For more details on the implementation of an adaptive algorithm to control the statistical error, see [21].

Remark 5.1 (Computational cost of the error estimates)**.** *The work to approximate $\mathbb{E}[\mathcal{G}(g)] = \mathbb{E}[X]$ within an accuracy* TOL *is $\mathcal{O}\left(\frac{\mathrm{Var}[X]}{\mathrm{TOL}^4}\right)$, provided we use the Monte Carlo version of the EFD method as in* (2.11)*. It is therefore important to try to use both variance reduction techniques and adaptive methods to save computational effort. On the other hand, the work needed to compute sufficiently sharp error estimates as described in* Theorem 4.1 *is only $\mathcal{O}(\mathrm{TOL}^{-3})$. The number of realizations needed to have a statistical error in the error bound much smaller than* TOL *is only $\mathcal{O}(\mathrm{TOL}^{-1})$ instead of the $\mathcal{O}\left(\frac{\mathrm{Var}[X]}{\mathrm{TOL}^2}\right)$ realizations we need to compute an approximation of $\mathcal{F}(g)$ using* (2.11)*, while the work to compute the error estimate for each realization is still $\mathcal{O}(\mathrm{TOL}^{-2})$, including the computation of the duals $\overline{\overline{\varphi}}$ and $\overline{\overline{\varphi}}'$. This surprising reduction of work for $\overline{\overline{\varphi}}$ and $\overline{\overline{\varphi}}'$ is special for the* HJM *model studied here. For general SDEs the corresponding work would be $\mathcal{O}(\mathrm{TOL}^{-4})$ instead of $\mathcal{O}(\mathrm{TOL}^{-2})$. Thus, cheap and sharp error bounds are obtained by the use of the a posteriori error estimates in* Theorem 4.1*. Observe that if variance reduction techniques are applied to the approximation of $\mathbb{E}[\mathcal{G}(g)]$, it is natural to try to use them also to reduce the variance in the error estimators.*

Remark 5.2 (Variance reduction techniques)**.** *The use of variance reduction techniques can decrease substantially the statistical errors. In particular the so called antithetic variates technique introduced in* [12] *reduces the variance in a sample estimator $\mathcal{A}(M; Y)$ by using another estimator $\mathcal{A}(M; \widetilde{Y})$ with the same expectation as the first one, but which is negatively correlated with the first. Then, the improved estimator is $\mathcal{A}(M; \frac{Y+\widetilde{Y}}{2})$. Here, the choice of Y and $\widetilde{Y}$ relates to the Wiener process W and its reflection along the time axis, $-W$, which is also a Wiener process. If a realization of the Wiener process, $W(\cdot, \omega_j)$, yields, using one of the numerical discretizations* (2.5-2.8)*, a realization $\overline{\overline{g}}(\cdot, \cdot, \omega_j)$ and $-W(\cdot, \omega_j)$ yields $\overline{\overline{\overline{g}}}(\cdot, \cdot, \omega_j)$ respectively, then we choose*

$$\frac{1}{M}\sum_{j=1}^{M} \frac{F\left(\overline{\overline{g}}_{N,L}(\omega_j)\right)G\left(\overline{\Lambda}_{\Psi,Q}(\overline{\overline{g}}(\cdot,\cdot,\omega_j))\right)+\overline{\overline{g}}_{N,L+1}(\omega_j)+F\left(\overline{\overline{\overline{g}}}_{N,L}(\omega_j)\right)G\left(\overline{\Lambda}_{\Psi,Q}(\overline{\overline{\overline{g}}}(\cdot,\cdot,\omega_j))\right)+\overline{\overline{\overline{g}}}_{N,L+1}(\omega_j)}{2}$$

as a better estimate. All the numerical results presented below use antithetic variates. In general, the use of control variates, see [5]*, can be also combined with other variance reduction methods. For example, the control variates technique is based on the knowledge of an estimator $Y_\star$, positively correlated with Y, whose expected value $\mathbb{E}[Y_\star]$ is known and relatively close to the desired $\mathbb{E}[Y]$, yielding $Y - Y_\star + \mathbb{E}[Y_\star]$ as an improved estimator. The estimates presented in this work do not preclude the use of control variates, and even though it is not applied here, it can be a valuable tool in practical computations.*

5.2. Numerical results. Now let us introduce some notation to be used later in the description of our numerical results. $\mathrm{E}_{\mathrm{tau}}$ denotes the sample average approximating the τ-discretization error (4.7) and $\mathrm{E}_{\mathrm{tim}}$ denotes the sample average approximation to the t-discretization error (4.8). Beside this, denote by E_S the approximation (5.1) to the statistical error E_S introduced in (2.14) and by $\mathrm{E}_{\mathrm{tau},S}$ the approximation (5.1) to the statistical error in the estimation of the τ-discretization error (4.7) by sample averages. Similarly, $\mathrm{E}_{\mathrm{tim},S}$ denotes the corresponding approximation to the statistical error in the estimation of the expected values in t-discretization error (4.8).

5.2.1. *Ho-Lee model.* The Ho-Lee model has $\xi(x) = \sigma$ and $\lambda_0(x) = 1$ so $\widetilde{\lambda}_0(x) = x$ and (1.2)-(1.3) takes the form

$$(5.2) \qquad \begin{aligned} df(t, \tau) &= \sigma^2 (\tau - t)\, dt + \sigma\, dW(t), \quad 0 \leq t \leq \tau, \\ f(0, \tau) &= f_0(\tau) \end{aligned}$$

for $\tau \in [0, \tau_{\max}]$. In this example the initial condition is $f_0(\tau) = r_0 - \frac{\sigma^2}{2}\tau^2 + \int_0^\tau \vartheta(s)ds$, where r_0 and σ are real positive constants and $\vartheta : \mathbb{R}^+ \to \mathbb{R}$ is a given function. Then, the exact solution of (5.2) is

$$f(t, \tau) = r_0 - \tfrac{\sigma^2}{2}(\tau - t)^2 + \int_0^\tau \vartheta(s)\, ds + \sigma\, W(t), \quad 0 \le t \le \tau,$$

which follows the normal distribution and therefore, yields bond prices which are log-normal distributed, allowing the use of Black and Scholes formulas for the pricing of call and put options on bonds.

Setting $\tau_a = t_{\max}$, $F(x) = 1 - x$, $G(x) = x$, $\Psi(x) = x$ and $U(x) = 0$ in (1.5)-(1.6), the functional to be computed has the form

$$(5.3) \qquad \mathbb{E}\left[\mathcal{F}(f)\right] = \mathbb{E}\left[\left(1 - \int_0^{t_{\max}} f(s, s)ds\right)\left(\int_{t_{\max}}^{\tau_{\max}} f(t_{\max}, \tau)\, d\tau\right)\right].$$

In the numerical experiments we choose $r_0 = 0.05$, $\sigma = 0.01$, $\vartheta(s) = \frac{1}{10}e^{-s}$. Then $\mathbb{E}\left[\mathcal{F}(f)\right]$ is a known function of $t_{\max}$ and $\tau_{\max}$. The first experiment sets $t_{\max} = 1.0$ and $\tau_{\max} = 2.0$, comparing the efficiency of the (EFD) and (EFE) methods. Table 5.2.1 shows the computational error for both methods and compares the a posteriori approximation of the error with the true computational error for the (EFD) method. Here, a confidence interval for the ratio between the error approximation and the exact computational error, $\mathcal{E}_c$, introduced in (2.12), is $[A - B, A + B]$, with $A \equiv \frac{\mathbf{E}_{\mathrm{tim}} + \mathbf{E}_{\mathrm{tau}}}{|\mathcal{E}_c|}$ and $B \equiv \frac{\mathbf{E}_S + \mathbf{E}_{\mathrm{tim},S} + \mathbf{E}_{\mathrm{tau},S}}{|\mathcal{E}_c|}$. Whenever we use the (EFD) method we call $\mathcal{E}_{C,EFD} \equiv \mathcal{E}_c$ and if we use the (EFE) method we call $\mathcal{E}_{C,EFE} \equiv \mathcal{E}_c$. Observe that the ratio $A \pm B$ of the *a posteriori* approximation of the error over the computational error becomes closer and closer to one as we refine the time and maturity partitions, provided that the statistical error is small compared to the t-discretization error and the τ-discretization error. In this example, the t-discretization gives the largest contribution to the computational error, and there is no practical advantage in the use of the (EFE) method.

iseed $= -1$	(EFE)	(EFD)	
$N = L$	$\mathcal{E}_{C,EFE}$	$\mathcal{E}_{C,EFD}$	$[A - B, A + B]$
5	-8.40×10^{-4}	-8.25×10^{-4}	$[0.97, 0.97]$
10	-4.16×10^{-4}	-4.08×10^{-4}	$[0.98, 0.99]$
20	-2.07×10^{-4}	-2.03×10^{-4}	$[0.98, 1.00]$

Table 5.2.1. Comparing the (EFD) and (EFE) methods in the Ho-Lee model approximating functional (5.3) with $M = 5000$ and $c_0 = 1.65$.

5.2.2. *Vasicek model.* The Vasicek model has $\xi(x) = \sigma$ and $\lambda_0(x) = e^{-\alpha x}$, so

$$\widetilde{\lambda}_0(x) = \tfrac{1}{\alpha}e^{-\alpha x}\left(1 - e^{-\alpha x}\right)$$

and the forward rate equation (1.2-1.3) becomes

$$(5.4) \qquad \begin{aligned} &df(t, \tau) = \tfrac{\sigma^2}{\alpha}\left(1 - e^{-\alpha(\tau - t)}\right)e^{-\alpha(\tau - t)}\, dt + \sigma\, e^{-\alpha(\tau - t)}\, dW(t), \quad 0 \le t \le \tau, \\ &f(0, \tau) = f_0(\tau) \end{aligned}$$

for $\tau \in [0, \tau_{\max}]$. In this example the initial condition is

$$f_0(\tau) = \left(r_0 - \tfrac{\vartheta}{\alpha}\right)e^{-\alpha\tau} + \tfrac{\vartheta}{\alpha} - \tfrac{\sigma^2}{2\alpha^2}\left(1 - e^{-\alpha\tau}\right)^2, \quad \tau \in [t, \tau_{\max}],$$

where r_0, σ, α and ϑ are given positive constants. The solution of (5.4) is then

$$f(t, \tau) = e^{-\alpha(\tau - t)}\left[e^{-\alpha t}\left(r_0 - \tfrac{\vartheta}{\alpha}\right) + \sigma\int_0^t e^{-\alpha(t - s)}\, dW(s)\right]$$

$$+ \tfrac{\vartheta}{\alpha} - \tfrac{\sigma^2}{2\alpha^2}\left(1 - e^{-\alpha(\tau - t)}\right)^2, \quad 0 \le t \le \tau,$$

which is normally distributed and yields bond prices that are lognormal, as in the Ho-Lee model.

Here we set $\tau_a = t_{\max} = 0.3$, $\tau_{\max} = 6.0$, and approximate again the functional defined in (5.3). In addition, we take $r_0 = 0.03$, $\alpha = 1.0$, $\sigma = 0.01$ and $\vartheta = 0.05$. Table 5.2.2 displays the computational

errors for the (EFD) and (EFE) methods and compares the *a posteriori* approximation of the error with the true error for the (EFD) method. Observe that the ratio $A \pm B$ of the a posteriori approximation of the error over the computational error becomes closer and closer to 1 as we refine the time and maturity partitions, provided that the statistical error is small compared to the t- and τ-discretization error.

iseed $= -1$	(EFE)	(EFD)	
$N = L$	$\mathcal{E}_{C,EFE}$	$\mathcal{E}_{C,EFD}$	$[A - B, A + B]$
5	-2.30×10^{-5}	-2.07×10^{-5}	$[1.92, 1.95]$
10	-2.05×10^{-5}	-1.95×10^{-5}	$[1.03, 1.05]$
20	-1.06×10^{-5}	-1.00×10^{-5}	$[0.99, 1.02]$

Table 5.2.2. Comparing the (EFD) and (EFE) methods in the Vasicek model approximating functional (5.3) with $M = 5000$ and $c_0 = 1.65$.

5.2.3. *The Cox-Ingersoll-Ross (CIR) model.* Consider the following (CIR) short rate model

$$(5.5) \qquad r(t) = r_0 + \int_0^t (\vartheta - \alpha\, r(s))\, ds + \int_0^t \sigma \sqrt{r(s)}\, dW(s), \quad t \geq 0,$$

where ϑ, α and σ are real constants. To connect the solution $r(t)$ of (5.5) to the diagonal value $f(t,t)$ of the solution of an HJM problem, consider, first, the solution $B = B(t;\tau)$ of the following Riccati differential equation (see [1]):

$$\partial_t B(t;\tau) = \tfrac{1}{2}\,\sigma^2\, B^2(t;\tau) + \alpha\, B(t;\tau) - 1, \quad t \in [0,\tau], \quad \tau \geq 0,$$
$$B(\tau;\tau) = 0,$$

which has the form $B(t;\tau) = \psi(\tau - t)$ where

$$\psi(x) = -\tfrac{\alpha}{\sigma^2} + \tfrac{2}{\sigma^2}\ \widetilde{\gamma}_0\ \frac{\sinh(\widetilde{\gamma}_0 x) + \frac{\alpha}{2\widetilde{\gamma}_0}\cosh(\widetilde{\gamma}_0 x)}{\cosh(\widetilde{\gamma}_0 x) + \frac{\alpha}{2\widetilde{\gamma}_0}\sinh(\widetilde{\gamma}_0 x)} \quad \text{and} \quad \widetilde{\gamma}_0 := \tfrac{1}{2}\ \sqrt{2\sigma^2 + \alpha^2}.$$

Provided $\xi(x) = \sigma \sqrt{\max\{x,0\}}$ and $\lambda_0(x) = \psi'(x)$, then $\widetilde{\lambda}_0(x) = \psi'(x)\psi(x)$ and the stochastic function

$$f(t,\tau) = r(t)\,\psi'(\tau - t) + \vartheta\,\psi(\tau - t)$$

solves (1.2)-(1.3) with the initial condition $f_0(\tau) = r_0\,\psi'(\tau) + \vartheta\,\psi(\tau)$. Taking into account that $\psi'(0) = 1$ and $\psi(0) = 0$, it follows that $f(t,t) = r(t)$.

Setting $\tau_a = t_{\max}$, $F(x) = e^{-x}$, $G(x) = \max\{e^{-x} - K_0, 0\}$, $\Psi(x) = x$ and $U(x) = 0$ in (1.5-1.6), the functional to compute in this example takes the form

$$(5.6) \qquad \mathbb{E}\,[\mathcal{F}(f)] = \mathbb{E}\left[\exp\left(-\int_0^{t_{\max}} f(s,s)\, ds\right) \max\left\{\exp\left(-\int_{t_{\max}}^{\tau_{\max}} f(t_{\max},\tau)\, d\tau\right) - K_0, 0\right\}\right].$$

In the numerical experiments we choose $r_0 = 0.15$, $\alpha = 1.0$, $\sigma = 0.1$, $\vartheta = 0.05$, $t_{\max} = 5.0$, $\tau_{\max} = 8.0$ and $K_0 = 0.5$. Table 5.2.3 shows the computational errors for the (EFD) and (EFE) methods and the ratio between the approximation of the computational error and the exact computational error for (EFD) method. There is no practical difference in this case between the (EFD) and the (EFE) method since the computational error is mainly t-discretization error and the τ-discretization error is relatively unimportant.

In order to have smooth coefficients in the HJM model (1.2-1.3) we approximate the function $\sqrt{\max\{x,0\}}$ in the diffusion term by a Lipschitz function globally defined in $\mathbb{R}$ (cf. [9] p. 252),

$$\sqrt{\max\{x,0\}} \approx \sqrt{\tfrac{1}{2}(x + \sqrt{x^2 + \delta})}$$

where δ is a small positive constant. Observe that after this regularization the value of the functional $\mathbb{E}\,[\mathcal{F}(f)]$ depends on δ. In the computations δ has been taken small enough to make this dependence negligible with respect to the size of the computational error. In this example we compute an accurate

iseed $= -1$	(EFE)	(EFD)	
$N = L$	$\mathcal{E}_{C,EFE}$	$\mathcal{E}_{C,EFD}$	$[A-B, A+B]$
5	1.23×10^{-2}	1.21×10^{-2}	$[0.31, 0.44]$
10	5.83×10^{-3}	5.39×10^{-3}	$[0.91, 0.95]$
20	2.76×10^{-3}	2.79×10^{-3}	$[0.89, 0.94]$

Table 5.2.3. Comparing the (EFD) and (EFE) methods in the (CIR) model approximating functional (5.6) with $M = 2000$ and $c_0 = 1.65$.

numerical approximation of the exact $\mathbb{E}\left[\mathcal{F}(f)\right]$ from (5.6), via the Feynman-Kac representation formula, using a numerical solution of the following backward PDE (cf. [20] p. 313),

$$v_t + (\vartheta - \alpha\, r)\, v_r + \tfrac{1}{2}\sigma^2\, r\, v_{rr} - r\, v = 0, \quad t \in [0, t_{\max}], \quad r \in [0, r_{\max}],$$

with final datum $v(t_{\max}, r) = \left(B(r, t_{\max}, \tau_{\max}) - K_0\right)^+$, where $B(r, t_{\max}, \tau_{\max})$ denotes the (CIR) value for a bond with contracting time $t_{\max}$, maturity time $\tau_{\max}$ and short rate at $t_{\max}$ equal to r. We also use the boundary conditions

$$v_t(t, 0) + \alpha\, v_r(t, 0) = 0, \quad v(t, r_{\max}) = 0,$$

for $t \in [0, t_{\max}]$. The value of $r_{\max} \gg \frac{\vartheta}{\alpha}$ is taken sufficiently large so that the homogeneous Dirichlet boundary at $r = r_{\max}$ has a negligible effect on the numerical approximation for $v(0, 0.15) = \mathbb{E}\left[\mathcal{F}(f)\right]$. The spatial discretization is a centered finite differences scheme and the time stepping is done by a diagonally implicit Runge Kutta method, namely the DIRK2 method, see [8]. Another way to estimate the exact solution with high accuracy is to use a formula based on the χ^2 distribution (see [19], pp. 187-193 for details).

5.2.4. *A two-factor Gaussian model.* A two-factor model has randomness introduced by two scalar independent Wiener processes W_1, W_2. In particular, for a two-factor Gaussian model we have $\xi(x) = 1$, $\lambda_{0,1}(x) = \sigma_1$ and $\lambda_{0,2}(x) = \sigma_2 e^{-\frac{a_2 x}{2}}$, where σ_1, σ_2 and a_2 are real positive constants. Thus (1.2)-(1.3) takes the form

(5.7)
$$df(t, \tau) = \left[(\sigma_1)^2\ (\tau - t) + \frac{2(\sigma_2)^2\, e^{-\frac{a_2(\tau-t)}{2}}}{a_2}\left(1 - e^{-\frac{a_2(\tau-t)}{2}}\right)\right] dt$$
$$+ \sigma_1\, dW_1(t) + \sigma_2\, e^{-\frac{a_2(\tau-t)}{2}}\, dW_2(t), \quad 0 \le t \le \tau,$$
$$f(0, \tau) = f_0(\tau)$$

for $\tau \in [0, \tau_{\max}]$. Here the initial condition is $f_0(\tau) = b_0 + b_1\, e^{-k\,\tau}$ where b_0, b_1 and k are real constants. Then, the exact solution of (5.7) is normal distributed as in the Ho-Lee and Vasicek models, so explicit formulas are available for the pricing of put and call options with bonds as underlyings.

In the numerical experiment we take $\sigma_1 = 0.02$, $\sigma_2 = 0.01$, $a_2 = 0.5$, and compute with the functional defined in (5.6) with strike $K_0 = 0.5$, $t_{\max} = 1$ and $\tau_{\max} = 3$. For the initial condition we set $b_0 = 0.0759$, $b_1 = -0.0439$ and $k = 0.4454$. Table 5.2.4 shows the computational errors for the (EFD) and (EFE) methods and the ratio between the approximation of the computational error and the exact computational error for method (EFD).

iseed $= -1$	(EFE)	(EFD)	
$N = L$	$\mathcal{E}_{C,EFE}$	$\mathcal{E}_{C,EFD}$	$[A-B, A+B]$
5	-5.15×10^{-4}	-6.90×10^{-4}	$[0.98, 1.02]$
10	-2.78×10^{-4}	-3.50×10^{-4}	$[0.96, 1.05]$

Table 5.2.4. Comparing the (EFD) and (EFE) methods in the two-factor Gaussian model approximating functional (5.6) with $M = 40000$ and $c_0 = 1.65$.

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