A POSTERIORI $L^\infty(L^2)$-ERROR BOUNDS IN FINITE ELEMENT APPROXIMATION OF THE WAVE EQUATION

EMMANUIL H. GEORGOULIS, OMAR LAKKIS, AND CHARALAMBOS MAKRIDAKIS

Abstract. We address the error control of Galerkin discretization (in space) of linear second order hyperbolic problems. More specifically, we derive a posteriori error bounds in the $L^\infty(L^2)$-norm for finite element methods for the linear wave equation, under minimal regularity assumptions. The theory is developed for both the space-discrete case, as well as for an implicit fully discrete scheme. The derivation of these bounds relies crucially on carefully constructed space- and time-reconstructions of the discrete numerical solutions, in conjunction with a technique introduced by Baker (1976, SIAM J. Numer. Anal., 13) in the context of a priori error analysis of Galerkin discretization of the wave problem in weaker-than-energy spatial norms.

1. Introduction

In computing approximate solutions of evolution initial-boundary value problems mesh-adaptivity plays an important role, in that it drives variable resolution requirements, thereby contributing reduction in computational cost. Adaptive strategies are often based on a posteriori error estimates, i.e., computable quantities which estimate the error of the finite element method measured in a suitable norm (or other functionals of interest).

A posteriori error bounds are well developed for stationary boundary value problems (e.g., [36, 2, 4, 16, 20, 32, 17] and the references therein). Adaptivity and error estimation for parabolic problems has also been an active area of research for the last two decades (e.g., [22, 35, 31, 24, 29, 13, 15, 27, 23] and the references therein).

Surprisingly, there has been considerably less work on the error control of finite element methods for second order hyperbolic problems, despite the substantial amount of research in the design of finite element methods for the wave problem (e.g., [5, 6, 8, 7, 21, 25, 28, 9, 19, 12, 26] and the references therein). A posteriori bounds for standard implicit time-stepping finite element approximations to the linear wave equation have been proposed and analyzed (but only in very specific situations) by Adjerid [1]. Bernardi and Süli [14] derive rigorous a posteriori error bounds, for the same fully-discrete method we analyze in this work. Also in [30] a posteriori bounds

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for the wave equation descritized in time were derived. In [14], [30] the analysis was based on the first order in time formulation of the wave equation. We note that goal-oriented error estimation for wave problems (via duality techniques) is also available [10, 11], while some earlier work on a posteriori estimates for first order hyperbolic systems have been studied in the time semidiscrete setting [30], as well as in the fully discrete one [25, 33, 34].

In this work, we derive a posteriori bounds in the $L^\infty(L^2)$-norm of the error using reconstructions in space and time. The theory is developed for both the space-discrete case, as well as for the practically relevant case of an implicit fully discrete scheme. The derivation of these bounds relies crucially on *reconstruction* techniques, used earlier for parabolic problems [29, 27, 3]. Such estimates make use on any given estimator for the corresponding elliptic problem, and are capable to treat various time discretization methods. A particular difficulty in the wave equation is the need of special time reconstructions for *two-step* methods and the corresponding analysis in the one field equation, as opposed to treatments using the first order in time formulation, see [14] and [30]. A key tool in our analysis is the special testing procedure due to Baker [5], who used it in the a priori error analysis of Galerkin discretization of the wave problem, in their standard one field formulation, in weaker-than-energy spatial norms.

While for the proof of a posteriori bounds for the semidiscrete case, the *elliptic reconstruction* previously considered in [29, 27] suffices, the fully discrete analysis necessitates the careful introduction of a novel space-time reconstruction, satisfying a crucial *local vanishing moment property* in time. Our approach is based on the one-field formulation of the wave equation and, thus, non-trivial three-point time reconstructions are required. A further challenge presented by the wave equation is the special treatment of deriving bounds for the "elliptic error" of the reconstruction framework, to obtain practically implementable residual estimators. The derived a posteriori estimators are formally of optimal order, i.e., of the same order as the error on uniform space- and time-meshes.

The rest of this work is organized as follows. In §2 we present the model problem and the necessary basic definitions along with the finite element methods for the wave equations considered in this work. In §3 we consider the case of a posteriori bounds for the space-discrete problem. In §4, we derive abstract a posteriori error bounds for the fully-discrete implicit finite element method, while in §5 the case of a posteriori bounds of residual type are presented. In §6, we draw some final concluding remarks.

2. Preliminaries

2.1. **Model problem and notation.**

We denote by $L^p(\omega)$, $1 \le p \le +\infty$, $\omega \subset \mathbb{R}^d$, the Lebesgue spaces, with corresponding norms $\|\cdot\|_{L^p(\omega)}$. The norm of $L^2(\omega)$, denoted by $\|\cdot\|_\omega$, corresponds to the $L^2(\omega)$-inner product $\langle\cdot,\cdot\rangle_\omega$. We denote by $H^s(\omega)$, the Hilbertian Sobolev space of order $s \ge 0$ of real-valued functions defined on $\omega \subset \mathbb{R}^d$; in particular $H^1_0(\omega)$ signifies the space of functions in $H^1(\omega)$ that vanish on the boundary $\partial\omega$ (boundary values are taken in the sense of traces). Negative order Sobolev spaces $H^{-s}(\omega)$, for $s > 0$, are defined through duality. In the case $s = 1$, the definition of $\langle\cdot,\cdot\rangle_\omega$ is extended to the standard duality pairing between $H^{-1}(\omega)$ and

$H^1_0(\omega)$. For $1 \le p \le +\infty$, we also define the spaces $L^p(0,T,X)$, with X being a real Banach space with norm $\|\cdot\|_X$, consisting of all measurable functions $v:(0,T)\to X$, for which

$$\begin{aligned} \|v\|_{L^p(0,T;X)} &:= \Big(\int_0^T \|v(t)\|_X^p \mathrm{d}t\Big)^{1/p} < +\infty, \quad \text{for} \quad 1 \le p < +\infty, \\ \|v\|_{L^\infty(0,T;X)} &:= ess \sup_{0\le t\le T} \|v(t)\|_X < +\infty, \quad \text{for} \quad p = +\infty. \end{aligned} \tag{2.1}$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded open polygonal domain with Lipschitz boundary $\partial\Omega$. For brevity, the standard inner product on $L^2(\Omega)$ will be denoted by $\langle\cdot,\cdot\rangle$ and the corresponding norm by $\|\cdot\|$.

For time $t \in (0,T]$, we consider the linear second order hyperbolic initial-boundary value problem of finding $u \in L^2(0,T;H^1_0(\Omega))$, with $u_t \in L^2(0,T;L^2(\Omega))$ and $u_{tt} \in L^2(0,T;H^{-1}(\Omega))$ such that

$$u_{tt} - \nabla\cdot(a\nabla u) = f \quad \text{in } (0,T)\times\Omega, \tag{2.2}$$

where $f \in L^2(0,T;L^2(\Omega))$ and a is a scalar-value function in $\in C(\bar{\Omega})$, with $0 < \alpha_{\min} \le a \le \alpha_{\max}$, such that

$$\begin{aligned} u(x,0) &= u_0(x) \text{ on } \Omega\times\{0\}, \\ u_t(x,0) &= u_1(x) \text{ on } \Omega\times\{0\} \\ u(0,t) &= 0 \text{ on } \partial\Omega\times(0,T], \end{aligned} \tag{2.3}$$

where $u_0 \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega)$.

We identify a function $v \in \Omega\times[0,T]\to\mathbb{R}$ with the function $v:[0,T]\to H^1_0(\Omega)$ and we use the shorthand $v(t)$ to indicate $v(\cdot,t)$.

2.2. **Finite element method.** Let $\mathcal{T}$ be a shape-regular subdivision of Ω into disjoint open simplicial or quadrilateral elements. Each element $\kappa\in\mathcal{T}$ is constructed via mappings $F_\kappa:\hat{\kappa}\to\kappa$, where $\hat{\kappa}$ is the reference simplex or reference square, so that $\bar{\Omega} = \cup_{\kappa\in\mathcal{T}}\bar{\kappa}$ [18].

For a nonnegative integer p, we denote by $\mathcal{P}_p(\hat{\kappa})$ either the set of all polynomials on $\hat{\kappa}$ of degree p or less, when $\hat{\kappa}$ is the simplex, or the set of polynomials of at most degree p in each variable, when $\hat{\kappa}$ is the reference square (or cube). We consider p fixed and use the finite element space

$$V_h := \{v \in H^1_0(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{P}_p(\hat{\kappa}),\ \kappa\in\mathcal{T}\}. \tag{2.4}$$

Further, we denote by $\Gamma := \cup_{\kappa\in\mathcal{T}}(\partial\kappa\backslash\partial\Omega)$, i.e., the union of all $(d-1)$-dimensional element edges (or faces) e in Ω associated with the subdivision $\mathcal{T}$ excluding the boundary. We introduce the mesh-size function $h:\Omega\to\mathbb{R}$, defined by $h(x) = diam\kappa$, if $x\in\kappa$ and $h(x) = diam(e)$, if $x \in e$ when e is an edge.

The semidiscrete finite element method for the initial-boundary value problem (2.2)–(2.3) consists in finding $U \in L^2(0,T;V_h)$ such that

$$\langle U_{tt}, V\rangle + a(U,V) = \langle f, V\rangle \quad \forall V \in L^2(0,T;V_h), \tag{2.5}$$

where the bilinear form a is defined for each $z, v \in H_0^1(\Omega)$ by

$$a(z,v) = \int_\Omega a\nabla z \cdot \nabla v \,\mathrm{d}x, \tag{2.6}$$

and the corresponding energy norm is defined for $v \in H_0^1(\Omega)$ by

$$\|v\|_a = \|\sqrt{a}\nabla v\|. \tag{2.7}$$

To introduce the fully-discrete implicit scheme approximating (2.2)–(2.3), we consider a subdivision of the time interval $(0,T]$ into subintervals $(t^{n-1}, t^n]$, $n = 1, \dots, N$, with $t^0 = 0$ and $t^N = T$, and we define $k_n := t^n - t^{n-1}$, the local time-step. Associated with the time-subdivision, let $\mathcal{T}^n$, $n = 0, \dots, N$, be a sequence of meshes which are assumed to be *compatible*, in the sense that for any two consecutive meshes $\mathcal{T}^{n-1}$ and $\mathcal{T}^n$, $\mathcal{T}^n$ can be obtained from $\mathcal{T}^{n-1}$ by locally coarsening some of its elements and then locally refining some (possibly other) elements. The finite element space corresponding to $\mathcal{T}^n$ will be denoted by V_h^n.

We consider the fully discrete scheme for the wave problem (2.2), (2.3)

$$\begin{aligned}&\text{for each } n = 1, \dots, N, \text{ find } U^n \in V_h^n \text{ such that}\\ &\langle \partial^2 U^n, V\rangle + a(U^n, V) = \langle f^n, V\rangle \quad \forall V \in V_h^n,\end{aligned} \tag{2.8}$$

where $f^n := f(t^n, \cdot)$, the backward second and first finite differences

$$\partial^2 U^n := \frac{\partial U^n - \partial U^{n-1}}{k_n}, \tag{2.9}$$

with

$$\partial U^n := \begin{cases} \dfrac{U^n - U^{n-1}}{k_n}, & \text{for } n = 1, 2, \dots, N,\\ V^0 := \pi^0 u_1 & \text{for } n = 0,\end{cases} \tag{2.10}$$

where $U^0 := \pi^0 u_0$, and $\pi^0 : L^2(\Omega) \to V_h^0$ a suitable projection onto the finite element space (e.g., the orthogonal L^2-projection operator).

3. A posteriori error bounds for the semi-discrete problem

We derive here a posteriori error bound for the error $\|u - U\|_{L^\infty(0,T;L^2(\Omega))}$ between the exact solution of (2.2), (2.3) and that of the semidiscrete scheme 2.5.

Definition 3.1 (elliptic reconstruction and error splitting)**.** *Let U be the (semidiscrete) finite element solution to the problem (2.5). Let also $\Pi : L^2(\Omega) \to V_h$ be the orthogonal L^2-projection operator onto the finite element space V_h. We define the* elliptic reconstruction *$w = w(t) \in H_0^1(\Omega)$, $t \in [0,T]$, of U to be the solution of the elliptic problem*

$$a(w,v) = \langle g, v\rangle \quad \forall v \in H_0^1(\Omega) \tag{3.1}$$

where

$$g := AU - \Pi f + f, \tag{3.2}$$

and $A : V_h \to V_h$ is the discrete elliptic operator *defined by*

$$\text{for } q \in V_h, \quad \langle Aq, \chi\rangle = a(q,\chi) \quad \forall \chi \in V_h. \tag{3.3}$$

We decompose the error as follows:

$$e := U - u = \rho - \epsilon, \text{ where } \epsilon := w - U, \text{ and } \rho := w - u. \tag{3.4}$$

Lemma 3.2 (error relation). *With reference to the notation in (3.4) we have*

$$\langle e_{tt}, v\rangle + a(\rho, v) = 0 \quad \forall v \in H_0^1(\Omega). \tag{3.5}$$

Proof. We have, respectively,

$$\begin{aligned}\langle e_{tt}, v\rangle + a(\rho, v) &= \langle U_{tt}, v\rangle + a(w, v) - \langle f, v\rangle \\ &= \langle U_{tt}, \Pi v\rangle + a(w, v) - \langle f, v\rangle \\ &= -a(U, \Pi v) + a(w, v) + \langle \Pi f - f, v\rangle = 0,\end{aligned} \tag{3.6}$$

observing the identity $a(U, \Pi v) - \langle \Pi f - f, v\rangle = a(w, v)$, due to the construction of w. □

Theorem 3.3 (abstract semidiscrete error bound). *With the notation introduced in (3.4), the following error bound holds:*

$$\begin{aligned}\|e\|_{L^\infty(0,T;L^2(\Omega))} \le &\|\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big) \\ &+ 2\int_0^T \|\epsilon_t\| + C_{a,T}\|u_1 - U_t(0)\|,\end{aligned} \tag{3.7}$$

where $C_{a,T} := \min\{2T, \sqrt{2C_\Omega/\alpha_{\min}}\}$*, where* C_Ω *is the constant of the Poincaré–Friedrichs inequality* $\|v\|^2 \le C_\Omega\|\nabla v\|^2$*, for* $v \in H_0^1(\Omega)$*.*

Proof. We use a testing procedure due to Baker [5]. Let $\tilde v : [0, T] \times \Omega \to \mathbb{R}$ with

$$\tilde v(t, \cdot) = \int_t^\tau \rho(s, \cdot)\mathrm{d}s, \quad t \in [0, T], \tag{3.8}$$

from some fixed $\tau \in [0, T]$. Clearly $\tilde v \in H_0^1(\Omega)$ as $\rho \in H_0^1(\Omega)$. Also, we observe that:

$$\tilde v(\tau, \cdot) = 0, \quad \nabla \tilde v(\tau, \cdot) = 0, \quad \text{and} \quad \tilde v_t(t, \cdot) = -\rho(t, \cdot), \quad \text{a.e. in } [0, T]. \tag{3.9}$$

Set $v = \tilde v$ in (3.5), integrate between 0 and τ with respect to the variable t and integrate by parts the first term on the left-hand side, to obtain

$$-\int_0^\tau \langle e_t, \tilde v_t\rangle + \langle e_t(\tau), \tilde v(\tau)\rangle - \langle e_t(0), \tilde v(0)\rangle + \int_0^\tau a(\rho, \tilde v) = 0. \tag{3.10}$$

Using (3.9), we have

$$\int_0^\tau \frac{1}{2}\frac{d}{dt}\|\rho\|^2 - \int_0^\tau \frac{1}{2}\frac{d}{dt}a(\tilde v, \tilde v) = \int_0^\tau \langle \epsilon_t, \rho\rangle + \langle e_t(0), \tilde v(0)\rangle, \tag{3.11}$$

which implies

$$\frac{1}{2}\|\rho(\tau)\|^2 - \frac{1}{2}\|\rho(0)\|^2 + \frac{1}{2}a(\tilde v(0), \tilde v(0)) = \int_0^\tau \langle \epsilon_t, \rho\rangle + \langle e_t(0), \tilde v(0)\rangle. \tag{3.12}$$

Hence, we deduce

$$\frac{1}{2}\|\rho(\tau)\|^2 - \frac{1}{2}\|\rho(0)\|^2 + \frac{1}{2}a(\tilde v(0), \tilde v(0)) \le \max_{0\le t\le T}\|\rho(t)\|\int_0^\tau \|\epsilon_t\| + \|e_t(0)\|\|\tilde v(0)\|. \tag{3.13}$$

Now, we select $\tau = \hat{\tau}$ such that $\|\rho(\hat{\tau})\| = \max_{0\le t\le T}\|\rho(t)\|$, and we present two alternative, but complementary, ways to complete the proof.

In the first way, we start by observing that $\|\tilde{v}(0)\| \le \tau\|\rho(\hat{\tau})\|$, gives

$$\frac{1}{4}\|\rho(\tau)\|^2 - \frac{1}{2}\|\rho(0)\|^2 \le \Big(\int_0^\tau \|\partial_t\epsilon\| + \tau\|e_t(0)\|\Big)^2. \tag{3.14}$$

Using the bound $\|\rho(0)\| \le \|e(0)\| + \|\epsilon(0)\|$, $e(0) = U(0) - u_0$ and $e_t(0) = U_t(0) - u_1$, we conclude that

$$\begin{aligned}\|e\|_{L^\infty(0,T;L^2(\Omega))} \le & \|\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big)\\ & + 2\Big(\int_0^T \|\epsilon_t\| + T\|u_1 - \partial U(0)\|\Big).\end{aligned} \tag{3.15}$$

The second alternative, described next, consists in a different treatment of the last term on the right-hand side of (3.13). The Poincaré–Friedrichs inequality and the positivity of the diffusion coefficient a imply $\|\tilde{v}(0)\|^2 \le C_\Omega \alpha_{\min}^{-1}\|\tilde{v}(0)\|_a^2$, for some constant C_Ω depending on the domain Ω only. Combining this bound with (3.13), we arrive to

$$\frac{1}{2}\|\rho(\tau)\|^2 - \frac{1}{2}\|\rho(0)\|^2 \le \max_{0\le t\le T}\|\rho(t)\| \int_0^\tau \|\epsilon_t\| + \frac{1}{2}C_\Omega\alpha_{\min}^{-1}\|e_t(0)\|^2, \tag{3.16}$$

which implies

$$\begin{aligned}\|e\|_{L^\infty(0,T;L^2(\Omega))} \le & \|\epsilon\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big)\\ & + 2\int_0^T \|\epsilon_t\| + \sqrt{2C_\Omega/\alpha_{\min}}\|u_1 - U_t(0)\|.\end{aligned} \tag{3.17}$$

Taking the minimum of the bounds (3.15) and (3.17) yields the result. □

Remark 3.4 (short and long integration times)**.** *The use of two alternative arguments in the last step of the proof of Lemma 3.2 improves the "reliability constant" $C_{a,T}$ that works for both the short-time and the long-time integration regimes.*

Remark 3.5 (Completing the a posteriori estimation)**.** *To obtain a practical a posteriori bound, we need to estimate the norms involving the elliptic error ϵ. By construction, the elliptic reconstruction w is the exact solution to the elliptic boundary-value problem (3.1) whose finite element solution is U. Indeed, inserting $v = V \in V_h$ in (3.1), we have*

$$a(w, V) = \langle AU - \Pi f + f, V\rangle = a(U, V), \tag{3.18}$$

which implies the Galerkin orthogonality property $a(w - U, V) = 0$. Therefore, by construction, ϵ is the error of the finite element method on V_h for the elliptic problem

$$-\nabla\cdot(a\nabla w) = g, \tag{3.19}$$

with homogeneous Dirichlet boundary conditions, with g defined by (3.2).

Definition 3.6. *For every element face $e \subset \Gamma$, we define the* jump *across e of a field $\mathbf{w}$, defined in an open neighborhood of e, by*

$$\llbracket \mathbf{w} \rrbracket(x) = \lim_{\delta\to 0} \Big(\mathbf{w}(x+\delta\mathbf{n}_e) - \mathbf{w}(x-\delta\mathbf{n}_e)\Big)\cdot \mathbf{n}_e, \tag{3.20}$$

for $x \in e$, where $\mathbf{n}_e$ denotes one of the two normal vectors to e (the definition of jump is independent of the choice).

Theorem 3.7 (elliptic a posteriori residual bounds [36, 2])**.** *Let $z \in H_0^1(\Omega)$ be the solution to the elliptic problem:*

$$-\nabla\cdot(a\nabla z) = r \tag{3.21}$$

$r \in L^2(\Omega)$ and Ω convex, and let $Z \in V_h$ be the finite element approximation of z satisfying

$$a(Z,V) = \langle r, V\rangle \quad \forall V \in V_h. \tag{3.22}$$

Then, there exists a positive constant C_{el}, independent of $\mathcal{T}$, h, z and Z, so that

$$\|z - Z\|^2 \le C_{\mathrm{el}}\,\mathcal{E}(Z, r, \mathcal{T}), \tag{3.23}$$

where

$$\mathcal{E}(Z,r,\mathcal{T}) := \Big(\sum_{\kappa\in\mathcal{T}}\Big(\|h^2(r+\nabla\cdot(a\nabla Z)\|_\kappa^2 + \sum_{e\subset\Gamma}\|h^{3/2}\llbracket a\nabla Z\rrbracket\|_e^2\Big)\Big)^{1/2}. \tag{3.24}$$

Corollary 3.8 (semidiscrete residual-type a posteriori error bound)**.** *Assume that the hypotheses of Theorems 3.3 and 3.7 hold. Assume further that f is differentiable with respect to time. Then the following error bound holds:*

$$\begin{aligned}\|e\|_{L^\infty(0,T;L^2(\Omega))} \le & C_{\mathrm{el}}\|\mathcal{E}(U,g,\mathcal{T})\|_{L^\infty(0,T)} + 2C_{\mathrm{el}}\int_0^T \mathcal{E}(U_t, g_t, \mathcal{T}) \\ & + \sqrt{2}C_{\mathrm{el}}\mathcal{E}(U(0), g(0), \mathcal{T}) \\ & + \sqrt{2}\|u_0 - U(0)\| + C_{a,T}\|u_1 - U_t(0)\|.\end{aligned} \tag{3.25}$$

Proof. Using (3.18), $\|\epsilon\|$ and $\|\epsilon_t\|$ can be bounded from above using (3.23). □

Remark 3.9. *A bound of the form (3.23) is only required to to hold for Corollary 3.8 to be valid. Therefore, other available a posteriori bounds for elliptic problems* [36, 2] *can be also used.*

4. A posteriori error bounds for the fully discrete problem

The analysis of §3 is now extended to the case of a fully-discrete implicit scheme with the aid of a novel three point space-time reconstruction, satisfying a crucial *vanishing moment property* in the time variable.

Definition 4.1 (space-time reconstruction)**.** *Let U^n, $n = 0,\dots,N$, be the fully discrete solution computed by the method (2.8), $\Pi^n : L^2(\Omega) \to V_h^n$ be the orthogonal L^2-projection, and $A^n : V_h^n \to V_h^n$ to be the discrete operator defined by*

$$\text{for } q \in V_h^n, \quad \langle A^n q, \chi\rangle = a(q,\chi) \quad \forall \chi \in V_h^n. \tag{4.1}$$

We define the elliptic reconstruction $w^n \in H_0^1(\Omega)$, *of* U^n *to be the solution of the elliptic problem*

$$a(w^n, v) = \langle g^n, v\rangle \quad \forall v \in H_0^1(\Omega), \tag{4.2}$$

with

$$g^n := A^n U^n - \Pi^n f^n + \bar{f}^n, \tag{4.3}$$

where $\bar{f}^0(\cdot) := f(0,\cdot)$ *and* $\bar{f}^n(\cdot) := k_n^{-1}\int_{t^{n-1}}^{t^n} f(t,\cdot)\mathrm{d}t$ *for* $n = 1,\dots,N$. *Finally, we need to define the* elliptic reconstruction $\partial w^0 \in H_0^1(\Omega)$, *of* V^0 *to be the solution of the elliptic problem*

$$a(\partial w^0, v) = \langle \partial g^0, v\rangle \quad \forall v \in H_0^1(\Omega), \tag{4.4}$$

with

$$\partial g^0 := A^0 V^0 - \Pi^0 f^0 + f^0. \tag{4.5}$$

The *time-reconstruction* $U : [0,T]\times\Omega \to \mathbb{R}$ of $\{U^n\}_{n=0}^N$, is defined by

$$U(t) := \frac{t - t^{n-1}}{k_n}U^n + \frac{t^n - t}{k_n}U^{n-1} - \frac{(t-t^{n-1})(t^n-t)^2}{k_n}\partial^2 U^n, \tag{4.6}$$

for $t \in (t^{n-1}, t^n]$, $n = 1,\dots,N$, noting that ∂U^0 is well defined. We note that $\hat{U}$ is a C^1-function in the time variable, with $\hat{U}(t^n) = U^n$ and $\hat{U}_t(t^n) = \partial U^n$ for , $n = 0,1,\dots,N$.

We shall also use the time-continuous elliptic reconstruction w, defined by

$$w(t) := \frac{t - t^{n-1}}{k_n}w^n + \frac{t^n - t}{k_n}w^{n-1} - \frac{(t-t^{n-1})(t^n-t)^2}{k_n}\partial^2 w^n, \tag{4.7}$$

noting that ∂w^0 is well defined. By construction, this is also a C^1-function in the time variable.

We decompose the error as follows:

$$e := U - u = \rho - \epsilon, \text{ where } \epsilon := w - U, \text{ and } \rho := w - u. \tag{4.8}$$

Remark 4.2 (notation overload)**.** *In this section we use symbols, e.g.,* U, w, e, ϵ, ρ, *that where used in §3, but with a slightly different meaning. Indeed, these are now fully-discrete constructs, corresponding in aim and meaning, but different, to their semidiscrete counterpart. It is hoped that this overload of notation should not create any confusion.*

Proposition 4.3 (fully-discrete error relation)**.** *For* $t \in (t^{n-1}, t^n]$, $n = 1,\dots,N$, *we have*

$$\langle e_{tt}, v\rangle + a(\rho, v) = \langle (I - \Pi^n)U_{tt}, v\rangle + \mu^n\langle \partial^2 U^n, \Pi^n v\rangle + a(w - w^n, v) + \langle \bar{f}^n - f, v\rangle, \tag{4.9}$$

for all $v \in H_0^1(\Omega)$, *with* $\Pi^n : L^2(\Omega) \to V_h^n$ *denoting the orthogonal* L^2*-projection operator onto* V_h^n, I *is the identity mapping in* $L^2(\Omega)$, *and*

$$\mu^n(t) := -6k_n^{-1}(t - t^{n-\frac{1}{2}}), \tag{4.10}$$

where $t^{n-\frac{1}{2}} := \frac{1}{2}(t^n + t^{n-1})$.

Proof. Noting that $U_{tt}(t) = (1+\mu^n(t))\partial^2 U^n$, for $t \in (t^{n-1}, t^n]$, $n = 1, \dots, N$, and the identity $a(U^n, \Pi^n v) - \langle \Pi^n f^n - \bar{f}^n, v\rangle = a(w^n, v)$, we deduce

$$\begin{aligned}
&\langle e_{tt}, v\rangle + a(\rho, v) = \langle U_{tt}, v\rangle + a(w, v) - \langle f, v\rangle,\\
&\quad = \langle (I - \Pi^n)U_{tt}, v\rangle + \langle U_{tt}, \Pi^n v\rangle + a(w, v) - \langle f, v\rangle,\\
&\quad = \langle (I - \Pi^n)U_{tt}, v\rangle + \mu^n(t)\langle \partial^2 U^n, \Pi^n v\rangle\\
&\qquad - a(U^n, \Pi^n v) + a(w, v) + \langle \Pi^n f^n - f, v\rangle\\
&\quad = \langle (I - \Pi^n)U_{tt}, v\rangle + \mu^n(t)\langle \partial^2 U^n, \Pi^n v\rangle + a(w - w^n, v) + \langle \bar{f}^n - f, v\rangle.
\end{aligned} \tag{4.11}$$

□

Remark 4.4 (vanishing moment property)**.** *The particular form of the remainder $\mu^n(t)$ satisfies the vanishing moment property*

$$\int_{t^{n-1}}^{t^n} \mu^n(t)\,\mathrm{d}t = 0, \tag{4.12}$$

which appears to be of crucial importance for the optimality of the a posteriori bounds presented below.

Definition 4.5 (a posteriori error indicators)**.** *We define in a list form the error indicators which will form error estimator the fully discrete bounds.*

mesh change indicator: $\eta_1(\tau) := \eta_{1,1}(\tau) + \eta_{1,2}(\tau)$, *with*

$$\eta_{1,1}(\tau) := \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \|(I - \Pi^j)U_t\| + \int_{t^{m-1}}^{\tau} \|(I - \Pi^m)U_t\|, \tag{4.13}$$

and

$$\eta_{1,2}(\tau) := \sum_{j=1}^{m-1} (\tau - t^j)\|(\Pi^{j+1} - \Pi^j)\partial U^j\| + \tau\|(I - \Pi^0)V^0(0)\|, \tag{4.14}$$

evolution error indicator:

$$\eta_2(\tau) := \int_0^{\tau} \|\mathcal{G}\|, \tag{4.15}$$

where $\mathcal{G} : (0, T] \to \mathbb{R}$ with $\mathcal{G}|_{(t^{j-1}, t^j]} := \mathcal{G}^j$, $j = 1, \dots, N$ and

$$\mathcal{G}^j(t) := \frac{(t^j - t)^2}{2}\partial g^j - \Big(\frac{(t^j - t)^4}{4k_j} - \frac{(t^j - t)^3}{3}\Big)\partial^2 g^j - \gamma_j, \tag{4.16}$$

with g^j as in Definition 4.1 and $\gamma_j := \gamma_{j-1} + \frac{k_j^2}{2}\partial g^j + \frac{k_j^3}{12}\partial^2 g^j$, $j = 1, \dots, N$, with $\gamma_0 = 0$;

data error indicator:

$$\eta_3(\tau) := \frac{1}{2\pi} \sum_{j=1}^{m-1} \Big(\int_{t^{j-1}}^{t^j} k_j^3 \|\bar{f}^j - f\|^2\Big)^{1/2} + \Big(\int_{t^{m-1}}^{\tau} k_m^3 \|\bar{f}^m - f\|^2\Big)^{1/2}; \tag{4.17}$$

time reconstruction error indicator:

$$\eta_4(\tau) := \frac{1}{2\pi}\sum_{j=1}^{m-1}\Big(\int_{t^{j-1}}^{t^j} k_j^3\|\mu^j\partial^2U^j\|^2\Big)^{1/2} + \Big(\int_{t^{m-1}}^{\tau} k_m^3\|\mu^m\partial^2U^m\|^2\Big)^{1/2}. \tag{4.18}$$

Theorem 4.6 (abstract fully-discrete error bound)**.** *Recalling the notation of Definition 4.1 and the indicators of Definition 4.5 we have the bound*

$$\begin{aligned}\|e\|_{L^\infty(0,t^N;L^2(\Omega))} \leq & \|\epsilon\|_{L^\infty(0,t^N;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U(0)\| + \|\epsilon(0)\|\Big)\\ & + 2\Big(\int_0^{t^N}\|\epsilon_t\| + \sum_{i=1}^4 \eta_i(t^N)\Big) + C_{a,N}\|u_1 - V^0\|,\end{aligned} \tag{4.19}$$

where $C_{a,N} := \min\{2t^N, \sqrt{2C_\Omega/\alpha_{\min}}\}$*,* C_Ω *is Poincaré–Friedrichs inequality constant.*

Proof. The proof of Theorem 4.6, is spread in this and the following paragraphs up to

Next we set $v = \tilde{v}$ in (4.9) with $\tilde{v}$ defined by (3.8) where ρ is defined as in (4.8) (i.e., the fully discrete ρ), assuming that $t^{m-1} < \tau \leq t^m$ for some integer m with $1 \leq m \leq N$. We integrate the resulting equation with respect to t between 0 and τ, to arrive to

$$\int_0^\tau \langle e_{tt}, \tilde{v}\rangle + \int_0^\tau a(\rho, \tilde{v}) = \mathcal{I}_1(\tau) + \mathcal{I}_2(\tau) + \mathcal{I}_3(\tau) + \mathcal{I}_4(\tau), \tag{4.20}$$

where

$$\begin{aligned}\mathcal{I}_1(\tau) &:= \sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\langle (I - \Pi^j)U_{tt}, \tilde{v}\rangle + \int_{t^{m-1}}^{\tau}\langle (I - \Pi^m)U_{tt}, \tilde{v}\rangle,\\ \mathcal{I}_2(\tau) &:= \sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j} a(w - w^j, \tilde{v}) + \int_{t^{m-1}}^{\tau} a(w - w^m, \tilde{v})\\ \mathcal{I}_3(\tau) &:= \sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\langle \bar{f}^j - f, \tilde{v}\rangle + \int_{t^{m-1}}^{\tau}\langle \bar{f}^m - f, \tilde{v}\rangle,\\ \mathcal{I}_4(\tau) &:= \sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\mu^j\langle \partial^2U^j, \Pi^j\tilde{v}\rangle + \int_{t^{m-1}}^{\tau}\mu^m\langle \partial^2U^m, \Pi^m\tilde{v}\rangle.\end{aligned} \tag{4.21}$$

In Lemmas 4.7, 4.8, 4.9, and 4.11 we will derive bounds of the form

$$\mathcal{I}_i(\tau) \leq \eta_i(\tau)\max_{0\leq t\leq T}\|\rho(t)\|, \tag{4.22}$$

for $i = 1, 2, 3, 4$. With the help of these, we will conclude the proof in §4.12. □

Lemma 4.7 (mesh change error estimate)**.** *Under the assumptions of Theorem 4.6 and with the notation (4.21) we have*

$$\mathcal{I}_1(\tau) \leq \eta_1(\tau)\max_{0\leq t\leq T}\|\rho(t)\|. \tag{4.23}$$

Proof. Observing that the projections Π^j, $j=1,\dots,N$, commute with time-differentiation, we integrate by parts with respect to t, arriving to

$$\begin{aligned}\mathcal{I}_1(\tau) = &\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\langle (I-\Pi^j)U_t,\rho\rangle + \int_{t^{m-1}}^{\tau}\langle (I-\Pi^m)U_t,\rho\rangle \\ &+\sum_{j=1}^{m-1}\langle(\Pi^{j+1}-\Pi^j)U_t(t^j),\tilde v(t^j)\rangle - \langle (I-\Pi^0)U_t(0),v(0)\rangle.\end{aligned}\tag{4.24}$$

The first two terms on the right-hand side of (4.24) are bounded by

$$\max_{0\le t\le T}\|\rho(t)\|\Big(\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\|(I-\Pi^j)U_t\| + \int_{t^{m-1}}^{\tau}\|(I-\Pi^m)U_t\|\Big).\tag{4.25}$$

Recalling the definition of $\tilde v$ and that $U(t^j)=\partial U^j$, $j=0,1,\dots,N$, we can bound the last two terms on the right-hand side of (4.24) by

$$\max_{0\le t\le T}\|\rho(t)\|\Big(\sum_{j=1}^{m-1}(\tau-t^j)\|(\Pi^{j+1}-\Pi^j)\partial U^j\| + \tau\|(I-\Pi^0)V^0(0)\|\Big).\tag{4.26}$$

□

Lemma 4.8 (evolution error bound)**.** *Under the assumptions of Theorem 4.6 and with the notation (4.21) we have*

$$\mathcal{I}_2(\tau)\le \eta_2(\tau)\max_{0\le t\le T}\|\rho(t)\|.\tag{4.27}$$

Proof. First, we observe the identity

$$w-w^j = -(t^j-t)\partial w^j + \Big(k_j^{-1}(t^j-t)^3-(t^j-t)^2\Big)\partial^2 w^j,\tag{4.28}$$

on each $(t^{j-1},t^j]$, $j=2,\dots,m$. Hence, from Definition 4.1, we deduce

$$a(w-w^j,\tilde v) = \langle -(t^j-t)\partial g^j + \Big(k_j^{-1}(t^j-t)^3-(t^j-t)^2\Big)\partial^2 g^j,\tilde v\rangle\tag{4.29}$$

The integral of the first component in the inner product on the right-hand side of (4.29) with respect to t between $(t^{j-1},t^j]$ is then given by $\mathcal{G}$. The choice of constants in $\mathcal{G}$ implies that $\mathcal{G}$ is continuous on t^j, $j=1,2,\dots,N$ and $\mathcal{G}(0)=0$.

Hence, integrating by parts on each interval $(t^{j-1},t^j]$, $j=1,\dots,m$, we obtain

$$\mathcal{I}_2(\tau) = \int_0^\tau\langle\mathcal{G},\rho\rangle,\tag{4.30}$$

which already implies the result. □

Lemma 4.9 (data approximation error bound)**.** *Under the assumptions of Theorem 4.6 and with the notation (4.21) we have*

$$\mathcal{I}_3(\tau)\le \eta_3(\tau)\max_{0\le t\le T}\|\rho(t)\|.\tag{4.31}$$

Proof. We begin by observing that

$$\int_{t^{j-1}}^{t^j} (\bar{f}^j - f) = 0, \tag{4.32}$$

for all $j = 1, \dots, m-1$. Hence, we have

$$\sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \langle \bar{f}^j - f, \tilde{v} \rangle = \sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \langle \bar{f}^j - f, \tilde{v} - \bar{\tilde{v}}^j \rangle, \tag{4.33}$$

where $\bar{\tilde{v}}^j(\cdot) := k_j^{-1} \int_{t^{j-1}}^{t^j} \tilde{v}(t, \cdot) \mathrm{d}t$. Using the inequality

$$\int_{t^{j-1}}^{t^j} \|\tilde{v} - \bar{\tilde{v}}^j\|^2 \le \frac{k_j^2}{4\pi^2} \int_{t^{j-1}}^{t^j} \|\tilde{v}_t\|^2, \tag{4.34}$$

and recalling that $\tilde{v}_t = \rho$, we have, respectively,

$$\begin{aligned}
\sum_{j=1}^{m-1} \int_{t^{j-1}}^{t^j} \langle \bar{f}^j - f, \tilde{v} \rangle &\le \sum_{j=1}^{m-1} \Big(\int_{t^{j-1}}^{t^j} \|\bar{f}^j - f\|^2 \Big)^{1/2} \Big(\int_{t^{j-1}}^{t^j} \|\tilde{v} - \bar{\tilde{v}}^j\|^2 \Big)^{1/2} \\
&\le \frac{1}{2\pi} \sum_{j=1}^{m-1} \Big(\int_{t^{j-1}}^{t^j} \|\bar{f}^j - f\|^2 \Big)^{1/2} \Big(\int_{t^{j-1}}^{t^j} k_j^2 \|\rho\|^2 \Big)^{1/2} \\
&\le \frac{1}{2\pi} \sum_{j=1}^{m-1} \Big(\int_{t^{j-1}}^{t^j} k_j^3 \|\bar{f}^j - f\|^2 \Big)^{1/2} \max_{0 \le t \le T} \|\rho(t)\|.
\end{aligned} \tag{4.35}$$

For the remaining term in $\mathcal{I}_3$, we first observe that

$$\int_{t^{m-1}}^{\tau} \|\tilde{v}\|^2 \mathrm{d}t \le \int_{t^{m-1}}^{\tau} k_m \int_t^{\tau} \|\rho\|^2 \mathrm{d}s \mathrm{d}t \le k_m^3 \max_{0 \le s \le T} \|\rho(t)\|^2, \tag{4.36}$$

which implies

$$\int_{t^{m-1}}^{\tau} \langle \bar{f}^m - f, \tilde{v} \rangle \le \Big(\int_{t^{m-1}}^{\tau} k_m^3 \|\bar{f}^m - f\|^2 \Big)^{1/2} \max_{0 \le t \le T} \|\rho(t)\|. \tag{4.37}$$

Recalling η_3 from Definition 4.5 we conclude the proof. □

Remark 4.10 (the order of the data approximation indicator)**.** *The choice of the particular combination of functions involving the right-hand side data f in the definition of g^n in the elliptic reconstruction, results to the property (4.32). When f is differentiable, we have $\eta_3(\tau) = O(k^2)$ as $k := \max_{1 \le j \le m} k_j \to 0$, and the convergence is of second order with respect to the maximum time-step. In this case, η_3 is, therefore, a higher order term.*

Lemma 4.11 (time-reconstruction error bound)**.** *Under the assumptions of Theorem 4.6 and with the notation (4.21) we have*

$$\mathcal{I}_4(\tau) \le \eta_4(\tau) \max_{0 \le t \le T} \|\rho(t)\|. \tag{4.38}$$

Proof. The method of bounding $\mathcal{I}_4(\tau)$ is similar to that of Lemma 4.9, so we shall only highlight the differences.

Recalling the vanishing moment property (4.12) and noting that $\partial^2 U^j$ is piecewise constant in time, we have

$$\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\mu^j\langle\partial^2U^j,\Pi^j\tilde{v}\rangle=\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\mu^j\langle\partial^2U^j,\Pi^j(\tilde{v}-\bar{\tilde{v}}^j)\rangle,\tag{4.39}$$

where $\bar{\tilde{v}}^j(\cdot)=k_j^{-1}\int_{t^{j-1}}^{t^j}\tilde{v}(t,\cdot)\mathrm{d}t$. Hence, since Π^j commutes with time integration, we obtain

$$\begin{aligned}\sum_{j=1}^{m-1}\int_{t^{j-1}}^{t^j}\mu^j\langle\partial^2U^j,\Pi^j(\tilde{v}-\bar{\tilde{v}}^j)\rangle&\le\frac{1}{2\pi}\sum_{j=1}^{m-1}\Big(\int_{t^{j-1}}^{t^j}\|\mu^j\partial^2U^j\|^2\Big)^{1/2}\Big(\int_{t^{j-1}}^{t^j}k_j^2\|\Pi^j\rho\|^2\Big)^{1/2}\\&\le\frac{1}{2\pi}\sum_{j=1}^{m-1}\Big(\int_{t^{j-1}}^{t^j}k_j^3\|\mu^j\partial^2U^j\|^2\Big)^{1/2}\max_{0\le t\le T}\|\rho(t)\|.\end{aligned}\tag{4.40}$$

For the remaining term in $\mathcal{I}_4$, upon using an argument similar to (4.36), we have

$$\int_{t^{m-1}}^{\tau}\langle\mu^m\partial^2U^m,\Pi^m\tilde{v}\rangle\le\Big(\int_{t^{m-1}}^{\tau}k_m^3\|\mu^m\partial^2U^m\|^2\Big)^{1/2}\max_{0\le t\le T}\|\rho(t)\|.\tag{4.41}$$

Recalling the definition of η_4 in §4.5 we conclude. □

4.12. **Concluding the proof of Theorem 4.6.** Starting from (4.20), integrating by parts the first term on the left-hand side, and using the properties of $\tilde{v}$, we arrive to

$$\int_0^\tau\frac{1}{2}\frac{d}{dt}\|\rho\|^2-\int_0^\tau\frac{1}{2}\frac{d}{dt}a(\tilde{v},\tilde{v})=\int_0^\tau\langle\epsilon_t,\rho\rangle+\langle e_t(0),\tilde{v}(0)\rangle+\sum_{i=1}^4\mathcal{I}_i(\tau),\tag{4.42}$$

which implies

$$\frac{1}{2}\|\rho(\tau)\|^2-\frac{1}{2}\|\rho(0)\|^2+\frac{1}{2}a(\tilde{v}(0),\tilde{v}(0))=\int_0^\tau\langle\epsilon_t,\rho\rangle+\langle e_t(0),\tilde{v}(0)\rangle+\sum_{i=1}^4\mathcal{I}_i(\tau).\tag{4.43}$$

Hence, we deduce

$$\begin{aligned}&\frac{1}{2}\|\rho(\tau)\|^2-\frac{1}{2}\|\rho(0)\|^2+\frac{1}{2}a(\tilde{v}(0),\tilde{v}(0))\\&\qquad\le\max_{0\le t\le T}\|\rho(t)\|\Big(\int_0^\tau\|\epsilon_t\|+\sum_{i=1}^4\eta_i(\tau)\Big)+\|e_t(0)\|\|\tilde{v}(0)\|.\end{aligned}\tag{4.44}$$

We select $\tau=\hat{\tau}$ such that $\|\rho(\hat{\tau})\|=\max_{0\le t\le t^N}\|\rho(t)\|$. First, observing that $\|\tilde{v}(0)\|\le\tau\|\rho(\hat{\tau})\|$, gives

$$\frac{1}{4}\|\rho(\tau)\|^2-\frac{1}{2}\|\rho(0)\|^2\le\Big(\int_0^\tau\|\epsilon_t\|+\sum_{i=1}^4\eta_i(\tau)+\tau\|e_t(0)\|\Big)^2.\tag{4.45}$$

Using the bound $\|\rho(0)\| \leq \|e(0)\| + \|\epsilon(0)\|$ and observing that $e(0) = \hat{U}(0) - u(0) = U^0 - u_0$ and that $e_t(0) = \hat{U}_t(0) - u_t(0) = V^0 - u_1$, we arrive to

$$\begin{aligned}\|e\|_{L^\infty(0,t^N;L^2(\Omega))} \leq &\|\epsilon\|_{L^\infty(0,t^N;L^2(\Omega))} + \sqrt{2}\Big(\|u_0 - U^0\| + \|\epsilon(0)\|\Big) \\ &+ 2\Big(\int_0^{t^N} \|\epsilon_t\| + \sum_{i=1}^4 \eta_i(t^N) + t^N\|u_1 - V^0\|\Big).\end{aligned} \tag{4.46}$$

The second way is completely analogous to the proof of the semidiscrete case.

5. Fully-discrete a posteriori estimates of residual type

To arrive to a practical a posteriori bound for the fully-discrete scheme from Theorem 4.6, the quantities involving the elliptic error ϵ should be estimated in an a posteriori fashion: this is the content of Lemmas 5.1 and 5.3 below, when residual-type a posteriori estimates are used.

Lemma 5.1 (estimation of the elliptic error)**.** *With the notation introduced in Definition 4.1, we have*

$$\|\epsilon\|_{L^\infty(0,t^N;L^2(\Omega))} + \sqrt{2}\|\epsilon(0)\| \leq \delta_1(t^N) + \sqrt{2}C_{\mathrm{el}}\mathcal{E}^0, \tag{5.1}$$

where

$$\begin{aligned}\delta_1(t^N) := \max\Big\{ &\frac{8k_1}{27}C_{\mathrm{el}}\mathcal{E}(V^0, \partial g^0, \mathcal{T}^0), \\ &\Big(\frac{35}{27} + \frac{31}{27}\max_{1\leq j\leq N}\frac{k_j}{k_{j-1}}\Big)\max_{0\leq j\leq N}\big(C_{\mathrm{el}}\mathcal{E}^j + C_\Omega \alpha_{\min}^{-1}\|\bar{f}^j - f^j\|\big)\Big\},\end{aligned} \tag{5.2}$$

with $\mathcal{E}^j := \mathcal{E}(U^j, A^jU^j - \Pi^j f^j + f^j, \mathcal{T}^j)$*,* $j = 0, 1, \ldots, N$*.*

Proof. For $t \in (t^{j-1}, t^j]$, $j = 1, \ldots, N$, we have

$$\epsilon = \frac{t - t^{j-1}}{k_j}(w^j - U^j) + \frac{t^j - t}{k_j}(w^{j-1} - U^{j-1}) - \frac{(t - t^{j-1})(t^j - t)^2}{k_j}(\partial^2 w^j - \partial^2 U^j), \tag{5.3}$$

from which, we can deduce

$$\|\epsilon\| \leq \max\Big\{\Big(\frac{35}{27} + \frac{31}{27}\max_{1\leq j\leq N}\frac{k_j}{k_{j-1}}\Big)\max_{0\leq j\leq N}\|w^j - U^j\|, \frac{8k_1}{27}\|\partial w^0 - V^0\|\Big\}, \tag{5.4}$$

noting that

$$\max_{t\in(t^{j-1},t^j]}\frac{(t - t^{j-1})(t^j - t)^2}{k_j} = \frac{4k_j^2}{27}. \tag{5.5}$$

It remains to estimate the terms $\|w^j - U^j\|$ and $\|\partial w^0 - V^0\|$. To this end, recalling the notation of Definition 4.1, we define $w_*^j \in H_0^1(\Omega)$ to be the solution of the elliptic problem

$$a(w_*^j, v) = \langle A^jU^j - \Pi^j f^j + f^j, v\rangle \quad \forall v \in H_0^1(\Omega), \tag{5.6}$$

for $j = 0, 1, \ldots, N$. Note that, due to the fact that $\bar{f}^0 = f^0$, we have $w_*^0 = w^0$. By construction, we have $a(w_*^j, V) = \langle A^jU^j - \Pi^j f^j + f^j, V\rangle = a(U^j, V)$ for all $V \in V_h^j$,

$j = 0, 1, \dots, N$. Hence, U^j is the finite element solution (in V_h^j) of the elliptic boundary-value problem (5.6). In view of Theorem 3.7, this implies that

$$\|w_*^j - U^j\| \le C_{\mathrm{el}}\mathcal{E}^j, \tag{5.7}$$

for $j = 0, \dots, N$. Similarly, by construction, we have $a(\partial w^0, V) = \langle A^0V^0 - \Pi^0 f^0 + f^0, V\rangle = a(V^0, V)$ for all $V \in V_h^0$. Hence,

$$\|\partial w^0 - \partial U^0\| \le C_{\mathrm{el}}\mathcal{E}(V^0, \partial g^0, \mathcal{T}^0). \tag{5.8}$$

Moreover, since $w^j - w_*^j$ is the solution of an elliptic problem with right hand-side $\bar f^j - f^j$, standard elliptic stability results yield

$$\|w^j - w_*^j\| \le C_\Omega \alpha_{\min}^{-1}\|\bar f^j - f^j\|, \tag{5.9}$$

for $j = 1, \dots, N$. Finally, using the triangle inequality

$$\|w^j - U^j\| \le \|w^j - w_*^j\| + \|w_*^j - U^j\|, \tag{5.10}$$

along with the bounds (5.9), (5.8) and (5.7), already implies the result. □

Remark 5.2. *The bound (5.1) contains both the* elliptic estimators $\mathcal{E}(\cdot,\cdot,\cdot)$ *and the data-oscillation terms* $\|\bar f^j - f^j\|$ *which are, in general, of first order with respect to the time-step. The data-oscillation terms are expected to dominate the data error indicator* η_3 *(cf. Remark 4.10). On the other hand, if the numerical scheme (2.8) is altered so that* $f^j = \bar f^j$ *(as done, e.g., in* [5]*), then the data-oscillation terms in (5.1) vanish. Similar remarks apply to the result of Lemma 4.12 below.*

For each $n = 1, \dots, N$, we denote by $\hat{\mathcal{T}}^n$ the finest common coarsening of $\mathcal{T}^n$ and $\mathcal{T}^{n-1}$, and by $\hat V_h^n := V_h^n \cap V_h^{n-1}$, the corresponding finite element space, along with the orthogonal L^2-projection operator $\hat\Pi^n : L^2(\Omega) \to \hat V_h^n$.

Lemma 5.3 (estimation of the time derivative of the elliptic error)**.** *With the notation introduced in §4.1 we have*

$$\int_0^{t^N} \|\epsilon_t\| \le \delta_2(t^N), \tag{5.11}$$

where

$$\delta_2(t^N) := \frac{2}{3}\sum_{j=0}^{N}(2k_j + k_{j+1})\Big(C_{\mathrm{el}}\mathcal{E}_\partial^j + C_\Omega\alpha_{\min}^{-1}\|\partial f^j - \partial\bar f^j\|\Big), \tag{5.12}$$

with

$$\mathcal{E}_\partial^j := \mathcal{E}(\partial U^j, \partial(A^jU^j) - \partial(\Pi^j f^j) + \partial f^j, \hat{\mathcal{T}}^j), \quad j = 0, 1, \dots, N. \tag{5.13}$$

Proof. For $t \in (t^{j-1}, t^j]$, $j = 1, \dots, N$, we have

$$\epsilon_t = \partial w^j - \partial U^j - k_j^{-1}(t^j - t)(t^j - 2t^{j-1} + t)(\partial^2 w^j - \partial^2 U^j), \tag{5.14}$$

from which, we deduce

$$\int_{t^{j-1}}^{t^j} \|\epsilon_t\| \le \frac{4k_j}{3}\|\partial w^j - \partial U^j\| + \frac{2k_j}{3}\|\partial w^{j-1} - \partial U^{j-1}\|, \tag{5.15}$$

noting that

$$\int_{t^{j-1}}^{t^j} k_j^{-2}(t^j - t)(t^j - 2t^{j-1} + t) = \frac{2k_j}{3}. \tag{5.16}$$

Combining (5.15) for $j = 1, \dots, N$, we arrive to

$$\int_0^{t^N} \|\epsilon_t\| \le \frac{2}{3} \sum_{j=0}^{N} (2k_j + k_{j+1}) \|\partial w^j - \partial U^j\|, \tag{5.17}$$

with $k_0 = 0$ and $k_{N+1} = 0$.

It remains to estimate the terms $\|\partial w^j - \partial U^j\|$. To this end, recalling the definition of the functions $w_*^j \in H_0^1(\Omega)$ from the proof of Lemma 5.1 and, since $\hat{V}_h^j := V_h^j \cap V_h^{j-1}$, we have $a(w_*^j, V) = a(U^j, V)$ for all $V \in \hat{V}_h^j$ and $a(w_*^{j-1}, V) = a(U^{j-1}, V)$ for all $V \in \hat{V}_h^j$, for $j = 1, \dots, N$. Therefore, we deduce

$$a(\partial w_*^j, V) = a(\partial U^j, V) \quad \text{for all} \quad V \in \hat{V}_h^j, \tag{5.18}$$

for $j = 1, \dots, N$, i.e., ∂U^j is the finite element solution in $\hat{V}_h^j$ of the boundary-value problem

$$a(\partial w_*^j, V) = \langle \partial(A^j U^j) - \partial(\Pi^j f^j) + \partial f^j, v\rangle \quad \forall v \in H_0^1(\Omega). \tag{5.19}$$

In view of Theorem 3.7, this implies that

$$\|\partial w_*^j - \partial U^j\| \le C_{\mathrm{el}} \mathcal{E}_\partial^j, \tag{5.20}$$

for $j = 1, \dots, N$. We also recall that, by construction, we have $a(\partial w^0, V) = a(V^0, V)$ for all $V \in V_h^0$. Hence, (5.8) also holds.

Moreover, since

$$a(\partial w^j, V) = \langle \partial(A^j U^j) - \partial(\Pi^j f^j) + \partial \bar{f}^j, v\rangle \quad \forall v \in H_0^1(\Omega), \tag{5.21}$$

$j = 1, \dots, N$, (cf. Definition 4.1). As in (5.9), elliptic stability implies

$$\|\partial w^j - \partial w_*^j\| \le C_\Omega \alpha_{\min}^{-1} \|\partial \bar{f}^j - \partial f^j\|, \tag{5.22}$$

for $j = 1, \dots, N$ and, using the triangle inequality

$$\|\partial w^j - \partial U^j\| \le \|\partial w^j - \partial w_*^j\| + \|\partial w_*^j - \partial U^j\|, \tag{5.23}$$

along with the bounds (5.22), (5.8) and (5.20), already implies the result. □

Theorem 5.4 (fully-discrete residual-type a posteriori bound)**.** *With the same hypotheses and notation as in Theorems 4.6 and 3.7, we have the bound*

$$\begin{aligned} \|e\|_{L^\infty(0,t^N;L^2(\Omega))} \le & \delta_1(t^N) + \sqrt{2} C_{\mathrm{el}} \mathcal{E}^0 + \sqrt{2}\|u_0 - U(0)\| \\ & + 2\delta_2(t^N) + 2\sum_{i=1}^{4} \eta_i(t^N) + C_{a,N}\|u_1 - V^0\|, \end{aligned} \tag{5.24}$$

where $\delta_1, \mathcal{E}^0$ are defined in Lemma 5.1, δ_2 is defined in Lemma 5.3, and η_i, $i = 1, 2, 3, 4$ after (41) respectively.

Proof. Combining Theorem 4.6 with the bounds derived for ϵ in Lemma 5.1, and ϵ_t in Lemma 5.3, we arrive to an a posteriori error bound. □

6. Final remarks

The design and implementation of adaptive algorithms for the wave equation based or rigorous a posteriori error estimators is a largely unexplored subject, despite the importance of these problems in the modeling of a number of physical phenomena. To this end, this work presents rigorous a posteriori error bounds in the $L^\infty(L^2)$-norm for second order linear hyperbolic initial/boundary value problems. The derived bounds are formally of optimal order. The numerical implementation of the proposed bounds in the context of adaptive algorithm design for second order hyperbolic problems remains a challenge that deserves special attention and will be considered elsewhere.

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Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom.
E-mail address: `Emmanuil.Georgoulis@mcs.le.ac.uk`

Department of Mathematics, University of Sussex, Falmer near Brighton, GB-BN1 9RF, England. http://www.maths.sussex.ac.uk/Staff/OL
E-mail address: `O.Lakkis@sussex.ac.uk`

Department of Applied Mathematics, University of Crete, GR-71409 Heraklion, Greece; and Institute for Applied and Computational Mathematics, Foundation for Research and Technology-Hellas, Vasilika Vouton P.O. Box 1527, GR-71110 Heraklion, Greece.
E-mail address: `makr@tem.uoc.gr`