ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΡΗΤΗΣ - ΤΜΗΜΑ ΕΦΑΡΜΟΣΜΕΝΩΝ ΜΑΘΗΜΑΤΙΚΩΝ
Archimedes Center for Modeling, Analysis & Computation
UNIVERSITY OF CRETE - DEPARTMENT OF APPLIED MATHEMATICS
Archimedes Center for Modeling, Analysis & Computation

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D. C. Antonopoulou and Georgia D Karali and M. Plexousakis and Georgios E. Zouraris

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CRANK-NICOLSON FINITE ELEMENT DISCRETIZATIONS FOR A 2D LINEAR SCHRÖDINGER-TYPE EQUATION POSED IN A NONCYLINDICAL DOMAIN

D. C. ANTONOPOULOU, G. D. KARALI, M. PLEXOUSAKIS, G. E. ZOURARIS

Abstract. Motivated by the paraxial narrow–angle approximation of the Helmholtz equation in domains of variable topography that appears as an important application in Underwater Acoustics, we analyze a general Schrödinger-type equation posed on two-dimensional variable domains with mixed boundary conditions. The resulting initial- and boundary-value problem is transformed into an equivalent one posed on a rectangular domain and is approximated by fully discrete, L^2-stable, finite element, Crank–Nicolson type schemes. We prove a global elliptic regularity theorem for complex elliptic boundary value problems with mixed conditions and derive L^2-error estimates of optimal order. Numerical experiments are presented which verify the optimal rate of convergence.

1. Introduction

1.1. **The physical problem.** The standard narrow-angle Parabolic Equation (PE) in three space dimensions is the following Schrödinger-type equation

$$\psi_r = \frac{\mathrm{i}}{2\,k_0}\,\left(\psi_{zz} + \frac{1}{r^2}\,\psi_{\theta\theta}\right) + \mathrm{i}\,\frac{k_0}{2}\,(n^2-1)\,\psi, \tag{1.1}$$

that models the long-range sound propagation in the sea, and is used in the context of underwater acoustics as the paraxial and far-field approximation of the Helmholtz equation in the presence of cylindrical symmetry, cf. [25, 10]. Here, $r_{\max} \geq r \geq r_{\min} > 0$ is the horizontal distance from a harmonic point source placed on the z axis and emitting at a frequency f_0. The function $\psi = \psi(r, z, \theta)$ depending on range, depth and azimuth measures the acoustic pressure in inhomogeneous, weakly range-dependent marine environments. The depth variable $z \geq 0$ is increasing downwards while the azimuth varies in the interval $[\theta_{\min}, \theta_{\max}]$; $k_0 = \frac{2\,\pi\,f_0}{c_0}$ is a reference wave number, the constant c_0 is a reference sound speed, $n(r, z, \theta) = c_0/c(r, z, \theta)$ is the refraction index and $c(r, z, \theta)$ is the sound speed in the water. The bottom topography, being variable, is identified in cylindrical coordinates by a positive surface $z = s(r, \theta)$.

For a fixed range $r \in [r_{\min}, r_{\max}]$, we define the r-dependent space domain:

$$\Omega(r) := \Big\{(z, \theta) \in \mathbb{R}^2 :\ \theta \in [\theta_{\min},\ \theta_{\max}],\ z \in [0, s(r, \theta)]\Big\},$$

where obviously, $\partial\Omega(r) = \cup_{i=1}^4 \omega_i(r)$ for $\omega_1(r) := \{(0, \theta) \in \mathbb{R}^2 :\ \theta \in [\theta_{\min}, \theta_{\max}]\}$, $\omega_2(r) := \{(z, \theta_{\min}) \in \mathbb{R}^2 :\ z \in [0, s(r, \theta_{\min})]\}$, $\omega_3(r) := \{(s(r, \theta), \theta) \in \mathbb{R}^2 :\ \theta \in [\theta_{\min}, \theta_{\max}]\}$, and $\omega_4(r) := \{(z, \theta_{\max}) \in \mathbb{R}^2 :\ z \in [0, s(r, \theta_{\max})]\}$ (cf. Figure 1).

The horizontal sea surface of the naval environment is assumed to be perfectly absorbing, so a free–release condition $\psi = 0$ is imposed on $\omega_1(r)$. We also set $\psi = 0$ on the minimum and maximum azimuthal values i.e. at $\omega_2(r) \cup \omega_4(r)$. We denote by $\omega_D(r) := \omega_1(r) \cup \omega_2(r) \cup \omega_4(r)$ the piecewise linear boundary segment where these homogeneous Dirichlet conditions are imposed. The acoustically rigid bottom is mathematically modeled by the Neumann boundary condition $\frac{\partial\psi}{\partial\eta_s} = 0$ along the bottom

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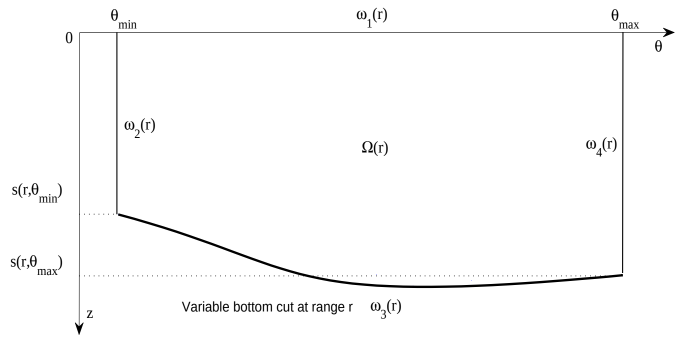

FIGURE 1. The range dependent domain $\Omega(r)$.

surface $z = s(r,\cdot)$, i.e., the variable boundary segment $\omega_3(r)$ of $\Omega(r)$. Even in one space dimension, the well-posedness of the standard narrow–angle Parabolic Equation (1.1) with Neumann condition was proved under the assumption that the bottom topography is strictly monotone, cf. [1]. Considering the same problem, in [2, 8] the authors verified numerically that significant instabilities develop even in strictly monotone downsloping bottom profiles.

Abrahamsson and Kreiss in [1, 2] proposed alternatively the use of a Robin-type condition as an approximation of the Neumann one that yields a well-posed initial and boundary value problem when the domain topography is variable. This approximate condition in two dimensions has the form (cf. [24]):

$$\psi_z - \frac{s_\theta}{r^2}\psi_\theta = \mathrm{i}k_0 s_r \psi \quad \text{at} \quad z = s(r,\theta). \tag{1.2}$$

We impose (1.2) at $z = s$, we set in (1.1) $a := \frac{1}{2k_0}$, $\beta_\psi(r,z,\theta) := \frac{k_0}{2}(n^2-1)$ and arrive at the following initial and boundary value problem (ibvp) of Schrödinger type:

$$\begin{aligned}
&\psi_r = \mathrm{i}\,\mathrm{div}(D_\mathrm{a}\nabla\psi) + \mathrm{i}\beta_\psi\psi && \text{in} \quad S,\\
&\psi = 0 && \text{on} \quad \omega_D(r) \quad \forall r \in [r_{\min}, r_{\max}],\\
&\eta_\mathrm{s}^t(D_\mathrm{a}\nabla\psi) = \frac{\mathrm{i}}{2}\frac{s_r}{\sqrt{1+s_\theta^2}}\psi && \text{on} \quad \omega_3(r) \quad \forall r \in [r_{\min}, r_{\max}],\\
&\psi(r_{\min},z,\theta) = \psi_0(z,\theta) && \text{on} \quad \Omega(r_{\min}),
\end{aligned} \tag{1.3}$$

posed on the non-cylindrical domain $S := \cup_{r\in[r_{\min},r_{\max}]}\Omega(r)$. Here, the gradient is with respect to the z,θ variables, $D_\mathrm{a} := \begin{pmatrix} a & 0 \\ 0 & a/r^2 \end{pmatrix}$, $\eta_\mathrm{s} = -\frac{(-1,s_\theta)^t}{\sqrt{1+s_\theta^2}}$ is the vector normal to the surface $z = s$ and the initial condition ψ_0 models the acoustic source.

Remark 1.1. In view of the ibvp (1.3), we observe that the same term $D_\mathrm{a}\nabla\psi$ appears at the equation as well as at the left-hand side of the Abrahamsson-Kreiss Robin condition.

1.2. **Change of variables.** The focus of our interest herein is to write the problem into an equivalent form posed on a cylindrical domain where simpler stable numerical schemes can be applied. This is achieved by a horizontal change of variables combined with an exponential transformation. Specifically, we let

$$\begin{aligned}
&y = z/s(r,\theta), \quad v(r,y,\theta) = e^{-q(r,\theta)}\psi(r,z,\theta),\\
&\Omega(r) \hookrightarrow \mathfrak{D} := (0,1)\times(\theta_{\min},\theta_{\max}),\\
&S \hookrightarrow [r_{\min}, r_{\max}] \times \mathfrak{D},
\end{aligned} \tag{1.4}$$

where $q(r,\theta) = -\frac{1}{2}\ln s(r,\theta)$ [24, 6]. With this choice of q, the initial- and boundary-value problem (1.3) takes the following form (see [6] for the details):

$$\begin{aligned} & v_r = \mathrm{i}\,\mathrm{div}(\widehat{D}_{\mathrm{a}}\nabla v) + y\frac{s_r}{s}v_y + \mathrm{i}\beta_v v && \text{in} \quad [r_{\min}, r_{\max}] \times \mathfrak{D}, \\ & v = 0 && \text{at} \quad y = 0 \quad \forall (r,\theta) \in [r_{\min}, r_{\max}] \times [\theta_{\min}, \theta_{\max}], \\ & \eta^t(\widehat{D}_{\mathrm{a}}\nabla v) = \mathrm{i}\gamma_{\mathrm{bc}} v && \text{at} \quad y = 1 \quad \forall (r,\theta) \in [r_{\min}, r_{\max}] \times [\theta_{\min}, \theta_{\max}], \\ & v(r_{\min}, y, \theta) = v_0(y,\theta) && \forall (y,\theta) \in \overline{\mathfrak{D}}, \end{aligned} \tag{1.5}$$

where $\eta := (1,0)^t$, $\gamma_{\mathrm{bc}}(r,\theta) := \frac{1}{2}\Big[\frac{s_r}{s} + \mathrm{i}\frac{a}{r^2}\Big(\frac{s_\theta}{s}\Big)^2\Big]$, and $\widehat{D}_{\mathrm{a}} := \begin{pmatrix} \widehat{a} & \widehat{\beta} \\ \widehat{\beta} & \widehat{\gamma} \end{pmatrix}$ with

$$\widehat{a}(r,y,\theta) = \frac{a}{s^2} + \frac{a}{r^2}y^2\Big(\frac{s_\theta}{s}\Big)^2, \;\; \widehat{\beta}(r,y,\theta) = -\frac{a}{r^2}y\Big(\frac{s_\theta}{s}\Big), \;\; \widehat{\gamma}(r) = \frac{a}{r^2},$$

$\beta_v := \beta_\psi + \frac{a}{r^2}\frac{3s_\theta^2 - 2ss_{\theta\theta}}{4s^2} - \mathrm{i}\frac{s_r}{2s}$, and $v_0(y,\theta) = \sqrt{s(r_{\min},\theta)}\,u_0(ys(r_{\min},\theta),\theta)$.

We note that $\widehat{D}_{\mathrm{a}}$ is a real, symmetric and positive definite matrix and therefore $\det(\widehat{D}_{\mathrm{a}}) > 0$, [6]. Furthermore, due to the definition of q the coefficient of v_y in the first equation is a real function, which at $y = 1$ equals to $2\mathrm{Re}\gamma_{\mathrm{bc}}$.

Certain three-dimensional effects have been observed to influence the acoustic transmission in variable domains mainly because the refraction index depends on r, z, θ and since significant reflections may occur between the bottom and the see surface (cf. [19, 14, 27, 12, 13]). In [24], F. Sturm considered the Narrow–angle parabolic equation with the Abrahamsson-Kreiss condition in three dimensions over a variable bottom in the case of a multilayered fluid medium.

The single layer case in the presence of azimuthal symmetry where the physical problem is posed on one-dimensional variable domains has been analyzed rigorously in [5, 8]. More specifically, in [5] the authors constructed finite difference schemes and proved optimal rate of convergence. In [8], error estimates of optimal order in the L^2- and H^1-norms have been proved for semidiscrete and fully discrete Crank-Nicolson-Galerkin finite element approximations. Discontinuous Galerkin methods for the linear Schrödinger equation Dirichlet problem in non-cylindrical domains of $\mathbb{R}^m$, $m \geq 1$, were analyzed in [9]. When $m = 1$ the resulting problem is the standard Narrow–angle parabolic approximation modeling an acoustically soft bottom; for this case the authors investigated theoretically and numerically the order of convergence using finite element spaces of piecewise polynomial functions. The Wide–angle parabolic equation consists an alternative approximation model of Helmholtz equation in underwater acoustics; for a rigorous numerical analysis and numerical experiments on this model cf. [3, 4, 7, 16].

1.3. **Generalization: The mathematical problem.** Motivated by the properties of the physical problem, for the sake of a more general mathematical setting, in our analysis we consider the following initial- and boundary-value problem of Schrödinger type with variable coefficients and mixed boundary conditions (Dirichlet-Robin)

$$\begin{aligned} & u_r = \mathrm{i}\,\mathrm{div}(D\nabla u) + b\nabla u + \mathrm{i}\beta u + F && \text{in} \quad [r_{\min}, r_{\max}] \times \mathfrak{D}, \\ & u = 0 && \text{at} \quad y = 0 \quad \forall (r,\theta) \in [r_{\min}, r_{\max}] \times [\theta_{\min}, \theta_{\max}], \\ & \eta^t(D\nabla u) = \mathrm{i}\lambda u && \text{at} \quad y = 1 \quad \forall (r,\theta) \in [r_{\min}, r_{\max}] \times [\theta_{\min}, \theta_{\max}], \\ & u(r_{\min}, y, \theta) = u_0(y,\theta) && \forall (y,\theta) \in \overline{\mathfrak{D}}. \end{aligned} \tag{1.6}$$

Here, $\mathfrak{D} = (0,1) \times (\theta_{\min}, \theta_{\max})$, $\eta := (1,0)^t$, while $\beta = \beta(r,y,\theta)$, $F = F(r,y,\theta)$ and $\lambda = \lambda(r,\theta)$ are complex-valued functions.

For the rest of this paper, we shall assume that the following conditions are satisfied:

$$D = D(r,y,\theta) \text{ is a } 2\times 2 \text{ real, symmetric matrix with } \det(D) > 0 \quad \forall\, r, y, \theta, \tag{1.7}$$

$$b=\Big(b_1(r,y,\theta),\ b_2(r,y,\theta)\Big)\ \text{ is real,} \tag{1.8}$$

and

$$b_1(r,1,\theta)-2\mathrm{Re}\lambda(r,\theta)\leq 0\ \ \forall\, r,\theta. \tag{1.9}$$

Remark 1.2. Since $D\in\mathbb{R}^{2\times 2}$, the condition (1.7) gives equivalently that D is either positive or negative definite for any r,y,θ, which in turn relates to the ellipticity of the operator $\mathrm{div}(D\nabla\cdot)$.

Remark 1.3. As we shall prove later, the conditions (1.8) and (1.9) are sufficient for L^2-stability, while when (1.9) holds as equality the problem is H^1-stable also, cf. Theorem 3.1 and Remark 3.2.

Remark 1.4. The form of the Robin boundary condition, considering only the first order terms, is related to the elliptic regularity of elliptic problems with mixed Dirichlet-Robin conditions in two dimensions proved in Theorem 4.3. The autonomous Section 4 of this paper presents a detailed proof of this argument.

Remark 1.5. The acoustic problem (1.5) is a specific case of the problem (1.6) for $u:=v$, $u_0:=v_0$, $D:=\widehat{D}_{\mathrm{a}}$, $b:=(y\frac{s_r}{s},0)$, $\beta:=\beta_v$, $F:=0$ and $\lambda:=\gamma_{\mathrm{bc}}$, satisfying (1.7), (1.8), and (1.9) as equality, [6].

1.4. **Main results.** The problem analyzed here is motivated by an important physical application. Nevertheless, the general mathematical setting encompasses the very interesting aspect of approximating numerically a multi-dimensional ibvp of Schrödinger-type with mixed conditions and coefficients depending on the evolutionary variable.

In this paper, we apply the Galerkin method on the general problem (1.6) using piecewise polynomial finite element spaces. We construct fully discrete Crank–Nicolson-type schemes in r for which we prove stability and optimal rate of accuracy in the L^2-norm. Numerical verification of the optimal rate of convergence is also presented.

The weak formulation of the problem is presented in Section 2. We define an appropriate r-dependent sesquilinear form which is, in general, non-Hermitian. As it is common, the rate of accuracy is investigated by using certain properties of the projection induced by this form. The projection being r-dependent and the fact that a two-dimensional r-dependent Robin boundary condition appears in (1.6) make the analysis difficult. We estimate the projection error and its r-derivative in the H^1- and L^2-norms (cf. paragraph 2.3, Propositions 2.3-2.5). The later is accomplished by applying an Elliptic Regularity Theorem for two-dimensional complex boundary value problems with mixed Dirichlet and Robin conditions, proved in Section 5. In the proof of Proposition 2.5, where the r-derivative of the projection error is estimated in the L^2-norm, we present a very refined argument when treating the boundary terms.

In Section 3, we write (1.6) in a weak form and prove L^2-stability, and H^1-stability in the case where (1.9) holds as equality, so that the sesquilinear form is Hermitian. We then construct a fully discrete Crank-Nicolson scheme in range r that is shown to be L^2-stable. Even though the evolutionary variable is discretized by a standard Crank-Nicolson method, the error analysis presented in this section is non-standard. This is due mainly to the fact that the form and the projection used are r-dependent and calculated at the mid-points of a uniform range partition. We define properly a test function split in two terms involving projections applied on second order derivatives (cf. Remark 3.6), use the projection estimates of Section 2, and derive an optimal error estimate in the L^2-norm.

A general complex elliptic boundary value problem posed on a two-dimensional rectangular domain with mixed boundary conditions is analyzed in Section 4. If Dirichlet or Neumann conditions hold along the boundary, then in the weak formulation of the boundary value problem the trace integral terms vanish. A general approach of proving global regularity, [18], is to prove this estimate for half-balls, and then by change of variables, stretch the compact boundary locally and cover it by a finite union of half-balls. In our case, we analyze a complex elliptic problem posed on a rectangular domain of $\mathbb{R}^2$. The boundary is compact and consists of four linear segments along which Dirichlet and Robin conditions are imposed. We apply directly on this domain the half-balls technique without change of variables as the boundary is already stretched locally. Further, we define appropriate test functions, in order to eliminate the trace terms from the weak formulation of the problem and prove the regularity estimate in Theorem 4.1. The result is extended in Theorem 4.3. Our proof covers a class of Robin conditions related to the coefficients of the pde of the boundary value problem, a special case of which is the Abrahamsson-Kreiss condition of underwater acoustics.

Finally, in Section 5 we report on the results of some numerical experiments performed with our method, verifying experimentally the optimal order of convergence.

2. An elliptic projection

2.1. Preliminaries. Let $\mathfrak{D}=(0,1)\times(\theta_1,\theta_2)$. For r in $[r_{\min},r_{\max}]$ fixed, we define

$$\partial_{y=y_0}:=\{(r,y_0,\theta)\in\mathbb{R}^3:\theta\in[\theta_1,\theta_2]\},\quad \partial_{\theta=\theta_0}:=\{(r,y,\theta_0)\in\mathbb{R}^3:y\in[0,1]\},$$

and denote by $H^1(\mathfrak{D})$ the associated usual (complex) Sobolev space. In order to deal with the Dirichlet boundary condition we shall make use of the space

$$\tilde{H}^1_0(\mathfrak{D})=\{u\in H^1(\mathfrak{D}):u|_{\partial\mathfrak{D}_D}=0\},$$

where $\partial\mathfrak{D}_D:=(\partial_{y=0})\cup(\partial_{\theta=\theta_1})\cup(\partial_{\theta=\theta_2})$. $\tilde{H}^1_0(\mathfrak{D})$ is the space of functions in $H^1(\mathfrak{D})$ which vanish on $y=0,\theta=\theta_1,\theta=\theta_2$. We denote the $L^2(\mathfrak{D})$ inner product by $(\cdot,\cdot):L^2(\mathfrak{D})\times L^2(\mathfrak{D})\to\mathbb{C}$. $\|u\|:=\Big(\int_{\mathfrak{D}}|u|^2\Big)^{\frac12}$ denotes the induced L^2-norm, while $\|u\|_1:=\Big(\|u\|^2+\|u_y\|^2+\|u_\theta\|^2\Big)^{\frac12}$ is the usual $H^1(\mathfrak{D})$ norm.

Let $\partial\mathfrak{D}_R:=\partial_{y=1}$ be the part of $\partial\mathfrak{D}$ where the Robin boundary condition of problem (1.6) is posed, and let

$$<u,v>:=\int_{\theta_{\max}}^{\theta_{\min}}u(1,\theta)\overline{v}(1,\theta)d\theta,$$

denote the inner product on $H^{\frac12}(\partial\mathfrak{D}_R)$. In addition, we shall make use of the norms:

$$|g|_{\frac12,\partial\mathfrak{D}_R}:=\inf_{v\in\tilde{H}^1_0(\mathfrak{D}):v|_{\partial\mathfrak{D}_R}=g}\|v\|_1,$$

$$|u|_{-\frac12,\partial\mathfrak{D}_R}:=\sup_{\tilde{v}\in H^1(\mathfrak{D}),\tilde{v}\neq0}\frac{|<u,\tilde{v}>|}{\|\tilde{v}\|_1}.$$

Let $\tau\in\mathbb{N}$ and S_h be a finite dimensional subspace of $\tilde{H}^1_0(\mathfrak{D})$ consisting of complex-valued functions that are polynomials of degree less than or equal to τ in each interval of a non-uniform partition of $\mathfrak{D}$ with maximum length $h\in(0,h_\star]$. It is well-known, [11], that the following approximation property holds:

$$\inf_{\chi\in S_h}\big\{\|v-\chi\|+h\|v-\chi\|_1\big\}\le C\,h^{s+1}\,\|v\|_{s+1},\quad\forall\,v\in H^{s+1}(\mathfrak{D}),\qquad \forall\,h\in(0,h_\star],\quad s=0,\dots,\tau. \tag{2.1}$$

Also, we assume that the following inverse inequality holds:

$$\|\phi\|_1\le C\,h^{-1}\,\|\phi\|\quad\forall\,\phi\in S_h,\quad\forall\,h\in(0,h_\star], \tag{2.2}$$

which is true when, for example, the partition of $\mathfrak{D}$ is quasi-uniform, [11].

2.2. Definition of a sesquilinear form. Without loss of generality we assume that D is positive definite. For any r in $[r_{\min},r_{\max}]$ we define the sesquilinear form $\mathcal{B}(r;\cdot,\cdot):\tilde{H}^1_0(\mathfrak{D})\times\tilde{H}^1_0(\mathfrak{D})\to\mathbb{C}$

$$\mathcal{B}(r;v,w):=(D(r)\nabla v,\nabla w)-\mathrm{i}\Big(\int_{\theta_{\min}}^{\theta_{\max}}\lambda(r,\theta)v(1,\theta)\overline{w}(1,\theta)d\theta-(b(r)\nabla v,w)\Big)+\delta(v,w), \tag{2.3}$$

for δ a sufficiently large positive constant. Obviously, it holds that

$$|\mathcal{B}(r;v,w)| \leq c\|v\|_1\|w\|_1, \tag{2.4}$$

for any $v, w \in \tilde{H}_0^1(\mathfrak{D})$, uniformly in r.

We observe that $(D\nabla v, \nabla v) \geq c_0\|\nabla v\|^2$ for a constant $c_0 > 0$ uniformly in r and v, since D is real symmetric and positive definite. Therefore, by the trace inequality we obtain for $v \in \tilde{H}_0^1(\mathfrak{D})$

$$\begin{aligned}\mathrm{Re}\mathcal{B}(r;v,v) =&(D(r)\nabla v, \nabla v) + \delta\|v\|^2\\ &+ \mathrm{Im}\Big(\int_{\theta_{\min}}^{\theta_{\max}} \lambda(r,\theta)v(1,\theta)\overline{v}(1,\theta)d\theta - (b(r)\nabla v, v)\Big)\\ \geq& c_0\|\nabla v\|^2 + \delta\|v\|^2 - c^2\|v\|\|v\|_1.\end{aligned}$$

Thus, by choosing δ sufficiently large, it follows that there exists positive constant C such that

$$\mathrm{Re}\mathcal{B}(r;v,v) \geq C\|v\|_1^2, \tag{2.5}$$

uniformly, for any r and any $v \in \tilde{H}_0^1(\mathfrak{D})$.

Remark 2.1. If D is negative definite we may use

$$\begin{aligned}\mathcal{B}_n(r;v,w) :=& - \Big[(D(r)\nabla v, \nabla w) - \mathrm{i}\Big(\int_{\theta_{\min}}^{\theta_{\max}} \lambda(r,\theta)v(1,\theta)\overline{w}(1,\theta)d\theta - (b(r)\nabla v, w)\Big)\Big]\\ &+ \delta(v,w),\end{aligned}$$

with δ a sufficiently large positive constant. In this case (2.4) and (2.5) also hold since now $-D$ is positive definite.

2.3. **Projection estimates.** Let $R_h(r) : \tilde{H}_0^1(\mathfrak{D}) \to S_h$ be a projection operator defined by

$$\mathcal{B}(r; R_h(r)v, \phi) = \mathcal{B}(r;v,\phi) \ \ \forall \phi \in S_h. \tag{2.6}$$

Obviously, since (2.4) and (2.5) hold true, then by Lax-Milgram Theorem the projection is well defined.

Let us now define the operator

$$\begin{aligned}&\mathcal{L}^*(r)w := -\mathrm{div}(D\nabla w) + \mathrm{i}b\nabla w + \mathrm{i}[b_{1y} + b_{2\theta}]w + \delta w \ \text{ in } \ \mathfrak{D},\\ &w = 0 \ \text{ on } \ \partial\mathfrak{D}_D,\\ &\eta^t(D\nabla w) = \mathrm{i}\lambda^* w \ \text{ on } \ \partial\mathfrak{D}_R,\end{aligned} \tag{2.7}$$

with λ^* a complex-valued function to be chosen appropriately in the sequel. For $\phi \in \tilde{H}_0^1(\mathfrak{D})$ we get

$$\begin{aligned}(\mathcal{L}^*(r)w, \phi) &= (D\nabla w, \nabla\phi) - \int_{\theta_{\min}}^{\theta_{\max}} \mathrm{i}\lambda^* w(1)\overline{\phi}(1)d\theta\\ &\quad + \mathrm{i}(b\nabla w, \phi) + \mathrm{i}([b_{1y} + b_{2\theta}]w, \phi) + \delta(w,\phi)\\ &= (D\nabla w, \nabla\phi) - \int_{\theta_{\min}}^{\theta_{\max}} \mathrm{i}\lambda^* w(1)\overline{\phi}(1)d\theta\\ &\quad - \mathrm{i}([b_{1y} + b_{2\theta}]w, \phi) - \mathrm{i}(bw, \nabla\phi) + \mathrm{i}\int_{\theta_{\min}}^{\theta_{\max}} b_1(1)w(1)\overline{\phi}(1)d\theta\\ &\quad + \mathrm{i}([b_{1y} + b_{2\theta}]w, \phi) + \delta(w,\phi)\\ &= (D\nabla w, \nabla\phi) + \mathrm{i}\int_{\theta_{\min}}^{\theta_{\max}} [b_1(1) - \lambda^*]w(1)\overline{\phi}(1)d\theta - \mathrm{i}(bw, \nabla\phi) + \delta(w,\phi).\end{aligned} \tag{2.8}$$

Since D, b and δ are real, then for any ϕ in $\tilde{H}_0^1(\mathfrak{D})$ it follows that

$$\overline{(\mathcal{L}^*(r)w,\phi)} = (D\nabla\phi,\nabla w) - \mathrm{i}\int_{\theta_{\min}}^{\theta_{\max}}[b_1(1)-\overline{\lambda^*}]\phi(1)\overline{w}(1)d\theta + \mathrm{i}(b\nabla\phi,w)+\delta(\phi,w). \tag{2.9}$$

Setting

$$\lambda^* := b_1(1)-\overline{\lambda}, \tag{2.10}$$

we obtain

$$\overline{(\mathcal{L}^*(r)w,\phi)} = \mathcal{B}(r;\phi,w), \tag{2.11}$$

and thus

$$(\mathcal{L}^*(r)w,\phi) = \overline{\mathcal{B}(r;\phi,w)}, \tag{2.12}$$

for any $\phi\in\tilde{H}_0^1(\mathfrak{D})$. Throughout the rest of this paper, we consider λ^* given by (2.10).

Remark 2.2. We observe that in the case of the specific problem (1.5), $b_1 = y\frac{s_r}{s}$, $b_2=0$, $\lambda = \frac{1}{2}\Big[\frac{s_r}{s} + \mathrm{i}\frac{a}{r^2}\Big(\frac{s_\theta}{s}\Big)^2\Big]$ and thus

$$\lambda^* = b_1(1)-\overline{\lambda} = \frac{s_r}{s} - \frac{1}{2}\Big[\frac{s_r}{s} - \mathrm{i}\frac{a}{r^2}\Big(\frac{s_\theta}{s}\Big)^2\Big] = \lambda.$$

Proposition 2.3. *There exists a positive constant c such that if $v\in\tilde{H}_0^1(\mathfrak{D})\cap H^s(\mathfrak{D})$ then*

$$\|R_h(r)v-v\|_1 \le ch^\tau\|v\|_{\tau+1}, \tag{2.13}$$

and

$$\|R_h(r)v-v\| \le ch^{\tau+1}\|v\|_{\tau+1}. \tag{2.14}$$

Proof. We set $e := R_h(r)v-v$, use (2.5), (2.4) and (2.1) to obtain for $\phi\in S_h$

$$\begin{aligned} c\|e\|_1^2 &\le \mathrm{Re}\mathcal{B}(r;e,e) = \mathrm{Re}\mathcal{B}(r;e,R_h(r)v-v) = \mathrm{Re}\mathcal{B}(r;e,\phi-v)\\ &\le c\|e\|_1\inf_{\phi\in S_h}\|\phi-v\|_1 \le c\|e\|_1h^\tau\|v\|_{\tau+1}, \end{aligned}$$

which establishes (2.13).

Let now w be the solution of the problem: $\mathcal{L}^*(r)w=e$. Then by using (2.12), the approximation property and elliptic regularity, proved in Theorem 4.3, we get for $\phi\in S_h$:

$$\begin{aligned} \|e\|^2 &= (\mathcal{L}^*(r)w,e) = \overline{\mathcal{B}(r;e,w)} = \overline{\mathcal{B}(r;e,w-\phi)}\\ &\le c\|e\|_1\inf_{\phi\in S_h}\|w-\phi\|_1 \le ch^\tau\|v\|_{\tau+1}h\|w\|_2 \le ch^{\tau+1}\|v\|_{\tau+1}\|e\|, \end{aligned}$$

which yields (2.14). □

Proposition 2.4. *Let $r\in C^1([r_{\min},r_{\max}],H^{\mathrm{r}}(\mathfrak{D}))$. Then it holds that*

$$\|\partial_r\left(R_h(r)v(r)-v(r)\right)\|_1 \le C\,h^\tau\left(\|v\|_{\tau+1}+\|\partial_r v\|_{\tau+1}\right). \tag{2.15}$$

Proof. We set $e := R_h(r)v(r)-v(r)$. Let $v:[r_{\min},r_{\max}]\to H^{\mathrm{r}}(\mathfrak{D})$ and $e(r)=R_h(r)v(r)-v(r)$ for $r\in[r_{\min},r_{\max}]$. Then, we have

$$\mathcal{B}(r;e(r),\phi)=0\ \ \forall\phi\in S_h.$$

Differentiating the above relation with respect to r we obtain

$$\mathcal{B}(r;\dot{e}(r),\phi)+\dot{\mathcal{B}}(r;e(r),\phi)=0\ \ \forall\phi\in S_h.$$

Now, for $\phi \in S_h$ we have

$$\begin{aligned}
c\|\dot e(r)\|_1^2 &\le \mathcal{B}(r;\dot e(r),\dot e(r)) = \mathcal{B}(r;\dot e(r),\dot e(r)+\phi) - \mathcal{B}(r;\dot e(r),\phi)\\
&= \mathcal{B}(r;\dot e(r),\dot e(r)+\phi) + \mathcal{B}(r;e(r),\phi)\\
&\le c\Big[\|\dot e(r)\|_1\|\dot e(r)+\phi\|_1 + \|e(r)\|_1\|\phi\|_1\Big]\\
&\le c\Big[\|\dot e(r)\|_1\|\dot e(r)+\phi\|_1 + \|e(r)\|_1(\|\dot e(r)+\phi\|_1 + \|\dot e(r)\|_1)\Big]\\
&= c\Big[(\|\dot e(r)\|_1 + \|e(r)\|_1)\|\dot e(r)+\phi\|_1 + \|e(r)\|_1\|\dot e(r)\|_1\Big]\\
&\le c\Big[(\|\dot e(r)\|_1 + \|e(r)\|_1)\inf_{\phi\in S_h}\|\partial_r(R_h v)(r) - \partial_r v + \phi\|_1 + \|e(r)\|_1\|\dot e(r)\|_1\Big]\\
&\le c\Big[(\|\dot e(r)\|_1 + \|e(r)\|_1)\inf_{\phi\in S_h}\|\partial_r v - \phi\|_1 + \|e(r)\|_1\|\dot e(r)\|_1\Big]\\
&= c\|\dot e(r)\|_1\Big[\|e(r)\|_1 + \inf_{\phi\in S_h}\|\partial_r v - \phi\|_1\Big] + c\|e(r)\|_1\inf_{\phi\in S_h}\|\partial_r v - \phi\|_1.
\end{aligned}$$

The claim of the proposition follows by using the approximation property 2.1 with $s = \tau + 1$. ☐

Using a technique introduced in [17], we are able to show the following optimal order approximation result for the time-derivative of the elliptic projection.

Proposition 2.5. *There exists a positive constant c such that*

$$\|\partial_r\,(R_h(r)v(r) - v(r))\| \le C\,h^{\tau+1}\,(\,\|v\|_{\tau+1} + \|\partial_r v\|_{\tau+1}\,)\,. \tag{2.16}$$

Proof. We set $e := R_h(r)v(r) - v(r)$. Let w be the solution of the problem: $\mathcal{L}^* w = \dot e$. For $\chi \in S_h$ we have

$$\begin{aligned}
\|\dot e(r)\|^2 &= (\mathcal{L}^* w, \dot e(r)) = \overline{\mathcal{B}(r;\dot e(r), w)}\\
&= \mathrm{Re}\Big[\overline{\mathcal{B}(r;\dot e(r), w-\chi)} - \overline{\dot{\mathcal{B}}(r;e,\chi)}\Big]\\
&\le c\Big[\|\dot e(r)\|_1\|w-\chi\|_1 + \|e(r)\|_1\|w-\chi\|_1\Big] - \mathrm{Re}\Big[\dot{\mathcal{B}}(r;e,w)\Big]\\
&\le c\Big(\|\dot e(r)\|_1 + \|e(r)\|\Big)\inf_{\chi\in S_h}\|w-\chi\|_1 - \mathrm{Re}\Big[\dot{\mathcal{B}}(r;e,w)\Big]\\
&\le ch^{\mathrm{r}}\Big(\|v\|_{\mathrm{r}+1} + \|\partial_r v\|_{\mathrm{r}+1}\Big)h\|w\|_2 - \mathrm{Re}\Big[\dot{\mathcal{B}}(r;e,w)\Big].
\end{aligned}$$

For convenience we set $I := \mathrm{Re}\Big[\dot{\mathcal{B}}(r;e,w)\Big]$. First, observe that

$$\dot{\mathcal{B}}(r;e,w) = (\partial_r D\nabla e, \nabla w) - \mathrm{i}\Big(\int_{\theta_{\min}}^{\theta_{\max}} \partial_r\lambda(r,\theta)e(r,1,\theta)\overline{w}(r,1,\theta)d\theta - (\partial_r b\nabla e, w)\Big),$$

so that

$$I = \mathrm{Re}\Big[(\partial_r D\nabla e, \nabla w) - \mathrm{i}\Big(\int_{\theta_{\min}}^{\theta_{\max}} \partial_r\lambda(r,\theta)e(r,1,\theta)\overline{w}(r,1,\theta)d\theta - (\partial_r b\nabla e, w)\Big)\Big].$$

By the definition of the inner product $< u, v >$ we have

$$\begin{aligned}
I_1 := \mathrm{Re}\Big[\mathrm{i}(\partial_r b\nabla e, w)\Big] &= \mathrm{Re}\Big[-\mathrm{i}(\partial_r[b_{1y} + b_{2\theta}]e, w) - \mathrm{i}(\partial_r b e, \nabla w)\\
&\qquad + \mathrm{i}\int_{\theta_{\min}}^{\theta_{\max}} (\partial_r b_1)(1)e(1)\overline{w}(1)d\theta\Big]\\
&\le c\|e\|\|w\|_1 + \mathrm{Re}\Big[\mathrm{i} < \partial_r b_1 e, w > \Big].
\end{aligned}$$

We set $I_2 := \mathrm{Re}\Big[-\mathrm{i}\int_{\theta_{\min}}^{\theta_{\max}} \partial_r \lambda e \overline{w} d\theta\Big]$. Using the estimates above we obtain

$$\begin{aligned} I_1 + I_2 &\le \mathrm{Re}\Big[-\mathrm{i} < [\partial_r\lambda - \partial_r b_1]e, w >\Big] + c\|e\|\|w\|_1 \\ &\le c|e|_{-\frac{1}{2},\partial\mathfrak{D}_R}\|w\|_1 + c\|e\|\|w\|_1 \le c\Big[\|e\| + |e|_{-\frac{1}{2},\partial\mathfrak{D}_R}\Big]\|w\|_1. \end{aligned}$$

In addition,

$$\begin{aligned} I_3 &:= \mathrm{Re}\Big[(\partial_r D\nabla e, \nabla w)\Big] = \mathrm{Re}\Big[\int_{\theta_{\min}}^{\theta_{\max}} \eta(\partial_r D\nabla w\overline{e})d\theta - (\mathrm{div}(\partial_r D\nabla w), e)\Big] \\ &\le c\|w\|_2\|e\| + \mathrm{Re}\Big[< \eta(\partial_r D\nabla w), e >\Big] \\ &\le c\|w\|_2\|e\| + |e|_{-\frac{1}{2},\partial\mathfrak{D}_R}\|w\|_2 \le c\Big[\|e\| + |e|_{-\frac{1}{2},\partial\mathfrak{D}_R}\Big]\|w\|_2, \end{aligned}$$

so that

$$I \le c\Big[\|e\| + |e|_{-\frac{1}{2},\partial\mathfrak{D}_R}\Big]\|w\|_2. \tag{2.17}$$

Now, for $g \in H^{\frac{1}{2}}(\partial\mathfrak{D}_R)$ we consider the elliptic problem

$$\begin{aligned} &-\mathrm{div}(D\nabla z) + \mathrm{i}b\nabla z + \mathrm{i}[b_{1y} + b_{2\theta}]z + \delta z = 0 \ \text{ in } \ \mathfrak{D}, \\ &z = 0 \ \text{ on } \ \partial\mathfrak{D}_D, \\ &\eta^t(D\nabla z) = \mathrm{i}\lambda^* z + g \ \text{ on } \ \partial\mathfrak{D}_R. \end{aligned}$$

Then we have $0 = \overline{\mathcal{B}(r; e(r), z)} - < g, e >$ and thus,

$$< g, e > = \overline{\mathcal{B}(r; e(r), z)} = \overline{\mathcal{B}(r; e(r), z - \phi)} \ \ \forall \phi \in S_h.$$

It follows then that

$$| < e, g > | \le c\|e\|_1 \inf_{\phi\in S_h} \|z - \phi\|_1,$$

and therefore,

$$| < e, g > | \le ch^\tau\|v\|_{\tau+1}h\|z\|_2.$$

The elliptic regularity result (cf. Theorem 4.3 and Remark 4.4) for the solution z of the elliptic problem above, reads

$$\|z\|_2 \le c|g|_{\frac{1}{2},\partial\mathfrak{D}_R}.$$

Thus we have

$$|e|_{-\frac{1}{2},\partial\mathfrak{D}_R} := \sup_{\tilde{v}\in H^1(\mathfrak{D}),\tilde{v}\ne 0} \frac{| < e, \tilde{v} > |}{\|\tilde{v}\|_1} \le ch^{\tau+1}\|v\|_{\tau+1},$$

and subsequently, using the elliptic regularity of w, cf. again Theorem 4.3, we arrive at

$$\begin{aligned} \|\dot{e}(r)\|^2 &\le ch^{\tau+1}\Big(\|v\|_{\tau+1} + \|\partial_r v\|_{\tau+1}\Big)\|w\|_2 + c\Big[\|e\| + |e|_{-1/2,\partial\mathfrak{D}_R}\Big]\|w\|_2 \\ &\le c\|w\|_2 h^{\tau+1}\Big(\|v\|_{\tau+1} + \|\partial_r v\|_{\tau+1}\Big) \le c\|\dot{e}(r)\|h^{\tau+1}\Big(\|v\|_{\tau+1} + \|\partial_r v\|_{\tau+1}\Big), \end{aligned}$$

which completes the proof of the proposition. □

3. A Crank–Nikolson-type fully discrete scheme

3.1. **Weak Formulation.** Let $\phi \in \tilde{H}_0^1(\mathfrak{D})$. Multiplying the partial differential equation of (1.6) by $\bar{\phi}$ and integrating by parts we have

$$
\begin{aligned}
\Big(u_r(r),\phi\Big) &= \mathrm{i}\Big[-\Big(D(r)\nabla u(r),\nabla\phi\Big)+\int_{\partial\mathfrak{D}}\eta^t D(r)\nabla u(r)\overline{\phi}ds\Big]+\Big(b(r)\nabla u(r),\phi\Big)\\
&+\mathrm{i}\Big(\beta(r)u(r),\phi\Big)+(F(r),\phi)\\
&=-\mathrm{i}\Big[\Big(D(r)\nabla u(r),\nabla\phi\Big)-\mathrm{i}\Big\{\int_{\theta_{\min}}^{\theta_{\max}}\lambda(r,\theta)u(r,1,\theta)\overline{\phi}(1,\theta)d\theta-\Big(b(r)\nabla u(r),\phi\Big)\Big\}\Big]\\
&+\mathrm{i}\Big(\beta(r)u(r),\phi\Big)+(F(r),\phi)\\
&=-\mathrm{i}\mathcal{B}\Big(r;u(r),\phi\Big)+\mathrm{i}\Big((\beta(r)+\delta)u(r),\phi\Big)+(F(r),\phi),
\end{aligned}
\tag{3.1}
$$

for any $\phi \in \tilde{H}_0^1(\mathfrak{D})$. In the following theorem we prove that (3.1) defines u in $\tilde{H}_0^1(\mathfrak{D})$ uniquely.

Theorem 3.1. *The weak problem* (3.1) *has at most one solution in* $\tilde{H}_0^1(\mathfrak{D})$.

Proof. Let $u \in \tilde{H}_0^1(\mathfrak{D})$ be a solution of (3.1). We set $\phi = u$ in (3.1), integrate by parts, use the facts that D is a real, symmetric matrix, that b is real, and take real parts. More specifically, we obtain first

$$
\begin{aligned}
(u_r,u) = &-\mathrm{i}\Big[(D\nabla u,\nabla u)-\mathrm{i}\Big(\int_{\theta_{\min}}^{\theta_{\max}}\lambda(r,\theta)u(r,1,\theta)\overline{u}(1,\theta)d\theta-(b\nabla u,u)\Big)\Big]\\
&+\mathrm{i}(\beta u,u)+(F,u).
\end{aligned}
\tag{3.2}
$$

Observe that

$$
\mathrm{Re}(b\nabla u,u) = -\frac{1}{2}(b_{1y}u,u)-\frac{1}{2}(b_{2\theta}u,u)+\frac{1}{2}\int_{\theta_{\min}}^{\theta_{\max}}b_1(r,1,\theta)|u(r,1,\theta)|^2d\theta,
$$

since b is real and $u = 0$ at $y = 0$, $\theta = \theta_{\min}, \theta_{\max}$. Since D is real, then using this observation in (3.2) we arrive at

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dr}\|u\|^2 = &\int_{\theta_{\min}}^{\theta_{\max}}[-\mathrm{Re}\lambda(r,\theta)+\frac{1}{2}b_1(r,1,\theta)]|u(r,1,\theta)|^2d\theta\\
&-\frac{1}{2}([b_{1y}+b_{2\theta}]u,u)-(\mathrm{Im}(\beta)u,u)+\mathrm{Re}(F,u).
\end{aligned}
$$

Using the condition (1.9) and Grönwall's inequality we obtain the stability estimate

$$
\|u\| \le c\|u_0\| + c\textstyle\int_{r_{\min}}^{r_{\max}}\|F\|\,dr. \tag{3.3}
$$

Uniqueness of the solution u follows readily from the estimate above. □

Remark 3.2. If (1.9) holds as equality then the sesquilinear form is Hermitian. Therefore, if $F = 0$, using (3.1), setting $\phi = u_r - \mathrm{i}(\beta+\delta)u$ and taking imaginary parts we obtain

$$
\frac{1}{2}\frac{d}{dt}\mathrm{Re}\mathcal{B}(r;u(r),u(r)) \le c_1\|u\|_1^2 \le c\mathrm{Re}\mathcal{B}(r;u(r),u(r)),
$$

so that $c\|u\|_1^2 \le \mathrm{Re}\mathcal{B}(r;u(r),u(r)) \le c\|u_0\|_1^2$, i.e. we also obtain an H^1 stability estimate.

Remark 3.3. Note that for the specific case of problem (1.5), if β_ψ is real, we have $F = 0$, $\mathrm{Im}(\beta) := \mathrm{Im}(\beta_v) = -\frac{s_r}{2s}$, $b_1 = y\frac{s_r}{s}$, $b_2 = 0$ and (1.9) holds as equality, therefore (cf. the proof of the previous theorem) we obtain the conservation property

$$
\|v\| = \|v_0\|
$$

for any r, while the problem is also H^1–stable.

3.2. **The numerical scheme.** For $N > 1$ integer, we consider a uniform partition in range $r_{\min} = r^0 < r^1 < \cdots < r^N = r_{\max}$, $0 \le n \le N$, $k := r^{n+1} - r^n = \frac{1}{N}$ for any $n \le N-1$, and set $r^{n+\frac12} := \frac{r^n + r^{n+1}}{2}$. If u is the solution of the continuous problem (1.6), we approximate $u(r^{n+1})$ by $U^{n+1} \in S_h$ as follows: for U^n known we seek $U^{n+1} \in S_h$ such that

$$\begin{aligned}(3.4)\qquad \Big(\frac{U^{n+1}-U^n}{k},\phi\Big) = &-\mathrm{i}\mathcal{B}\Big(r^{n+\frac12};\frac{U^{n+1}+U^n}{2},\phi\Big) + \mathrm{i}\Big((\beta(r^{n+\frac12})+\delta)\frac{U^{n+1}+U^n}{2},\phi\Big)\\ &+ (F(r^{n+\frac12}),\phi),\end{aligned}$$

for any $\phi \in S_h$, and any $0 \le n \le N-1$. In order to obtain an optimal order approximation we shall take $U^0 := R_h(r^0)u_0 \in S_h$.

Remark 3.4. Let $w \in \tilde{H}^1_0(\mathfrak{D})$. The need of the condition (1.9) appears once again (recall that (1.9) was used for the L^2 stability of the continuous problem). More specifically, since $w \in \tilde{H}^1_0(\mathfrak{D})$ then by (1.9) the following inequality holds

$$(3.5)\qquad \mathrm{Re}\Big\{-\mathrm{i}\mathcal{B}\Big(r;w,w\Big)\Big\} \le c\|w\|^2.$$

Theorem 3.5. *The fully discrete scheme (3.4) is L^2-stable.*

Proof. In (3.4) we set $\phi = U^{n+1} + U^n \in S_h \subset \tilde{H}^1_0(\mathfrak{D})$, take real parts and use the estimate of Remark 3.4 to arrive at

$$(1-ck)\|U^{n+1}\| \le (1+ck)\|U^n\| + ck\|F(r^{n+\frac12})\|.$$

Choosing k sufficiently small we get a stability result for the scheme (3.4)

$$\|U^n\| \le c\|U^0\| + c\max_{n\le N}\|F\|,$$

for any $1 \le n \le N$. Consequently, uniqueness of solution in S_h is also established. □

3.3. **Error estimates for the fully discrete scheme.**

3.3.1. *Preliminaries.* We define in $S_h \subset \tilde{H}^1_0(\mathfrak{D})$ the quantities

$$\theta_1^n := U^n - R_h(r^{n+\frac12})u(r^n) + \frac{k^2}{8}R_h(r^{n+\frac12})u_{rr}(r^{n+\frac12}),\quad 0 \le n \le N-1,$$

$$\theta_2^{n+1} := U^{n+1} - R_h(r^{n+\frac12})u(r^{n+1}) + \frac{k^2}{8}R_h(r^{n+\frac12})u_{rr}(r^{n+\frac12}),\quad 0 \le n \le N-1,$$

where U^n is the solution of the fully discrete scheme (3.4).

Remark 3.6. The main idea is to mimic the continuous problem. In the fully discrete scheme, we set $\phi := \theta_2^{n+1} + \theta_1^n$ as test function. The choice of θ_1^n, θ_2^{n+1} is not standard and is made in order to treat efficiently the r-dependent sesquilinear form at the midpoints of the partition and since the projection is range-dependent. Therefore, in ϕ, the projections are computed in $r^{n+\frac12}$. The introduction of the specific additive term

$$(3.6)\qquad \frac{k^2}{8}R_h(r^{n+\frac12})u_{rr}(r^{n+\frac12})$$

is motivated by the approximation

$$\frac{u(r^n)+u(r^{n+1})}{2} - u(r^{n+\frac12}) = \frac{k^2}{8}u_{rr}(r^{n+\frac12}) + \mathcal{O}(k^4),$$

used in (3.10). The residual being of order $\mathcal{O}(k^4)$ permits us to apply then in (3.11) the inverse inequality without loss of optimality in space, and avoid thus any integration by parts (denote that in this case suboptimal trace integral terms would appear, as the problem is posed in $S_h \subset \tilde{H}_0^1(\mathfrak{D}) \neq H_0^1(\mathfrak{D})$). Furthermore, the term (3.6) is related to the approximations

$$\begin{aligned} u(r^{n+1}) - \frac{k^2}{8}u_{rr}(r^{n+\frac{1}{2}}) &= u(r^{n+\frac{1}{2}}) + \frac{k}{2}u_r(r^{n+\frac{1}{2}}) + \mathcal{O}(k^3),\\ u(r^{n}) - \frac{k^2}{8}u_{rr}(r^{n-\frac{1}{2}}) &= u(r^{n-\frac{1}{2}}) + \frac{k}{2}u_r(r^{n-\frac{1}{2}}) + \mathcal{O}(k^3) \end{aligned}$$

used in the proof of Lemma 3.8 when treating the r-derivative of the projection error.

We notice that

$$\begin{aligned} \frac{U^{n+1}-U^n}{k} =& \frac{\theta_2^{n+1}-\theta_1^n}{k} + \frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1}) - R_h(r^{n+\frac{1}{2}})u(r^n)}{k},\\ \frac{U^{n+1}+U^n}{2} =& \frac{\theta_2^{n+1}+\theta_1^n}{2} + \frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1}) + R_h(r^{n+\frac{1}{2}})u(r^n)}{2}\\ & - \frac{k^2}{8}R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}). \end{aligned}$$

Replacing these identities in the fully discrete scheme we obtain

$$\begin{aligned} \Big(\frac{\theta_2^{n+1}-\theta_1^n}{k},\phi\Big) =& -\Big(\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1}) - R_h(r^{n+\frac{1}{2}})u(r^n)}{k},\phi\Big)\\ & - \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{\theta_2^{n+1}+\theta_1^n}{2},\phi\Big)\\ & - \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1}) + R_h(r^{n+\frac{1}{2}})u(r^n)}{2},\phi\Big)\\ & + \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{k^2}{8}R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\phi\Big)\\ & + \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\frac{\theta_2^{n+1}+\theta_1^n}{2},\phi\Big)\\ & + \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1}) + R_h(r^{n+\frac{1}{2}})u(r^n)}{2},\phi\Big)\\ & - \mathrm{i}\frac{k^2}{8}\Big((\beta(r^{n+\frac{1}{2}})+\delta)R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\phi\Big) + (F(r^{n+\frac{1}{2}}),\phi). \end{aligned} \tag{3.7}$$

From the continuous problem we have that

$$\begin{aligned} (\partial_r u(r^{n+\frac{1}{2}}),\phi) =& -\mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};u(r^{n+\frac{1}{2}}),\phi\Big)\\ & + \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)u(r^{n+\frac{1}{2}}),\phi\Big) + (F(r^{n+\frac{1}{2}}),\phi). \end{aligned} \tag{3.8}$$

We now solve (3.8) for $(F(r^{n+\frac{1}{2}}),\phi)$, replace in (3.7), and use the definition of the elliptic projection R_h to arrive at

$$
\begin{aligned}
\Big(\frac{\theta_2^{n+1}-\theta_1^n}{k},\phi\Big) &= -\Big(\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1})-R_h(r^{n+\frac{1}{2}})u(r^n)}{k}-u_r(r^{n+\frac{1}{2}}),\phi\Big)\\
&- \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{\theta_2^{n+1}+\theta_1^n}{2},\phi\Big) - \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{u(r^{n+1})+u(r^n)}{2}-u(r^{n+\frac{1}{2}}),\phi\Big)\\
&+ \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{k^2}{8}R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\phi\Big)\\
&+ \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\frac{\theta_2^{n+1}+\theta_1^n}{2},\phi\Big)\\
&+ \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\Big[\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1})+R_h(r^{n+\frac{1}{2}})u(r^n)}{2}-u(r^{n+\frac{1}{2}})\Big],\phi\Big)\\
&- \mathrm{i}\frac{k^2}{8}\Big((\beta(r^{n+\frac{1}{2}})+\delta)R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\phi\Big).
\end{aligned}
\tag{3.9}
$$

By Taylor's formula the following identity holds for $r_1, r_2 \in (r^n, r^{n+1})$

$$
\frac{u(r^n)+u(r^{n+1})}{2}-u(r^{n+\frac{1}{2}}) = \frac{k^2}{8}u_{rr}(r^{n+\frac{1}{2}})+\frac{k^4}{2\cdot 16\cdot 4!}[u_{rrrr}(r_1)+u_{rrrr}(r_2)].
$$

Using the above in (3.9) we obtain

$$
\begin{aligned}
&- \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{u(r^{n+1})+u(r^n)}{2}-u(r^{n+\frac{1}{2}}),\phi\Big) + \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{k^2}{8}R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\phi\Big)\\
&= -\mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};-\frac{k^2}{8}\Big[R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}})-u_{rr}(r^{n+\frac{1}{2}})\Big],\phi\Big)\\
&- \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{k^4}{2\cdot 16\cdot 4!}[u_{rrrr}(r_1)+u_{rrrr}(r_2)],\phi\Big)\\
&= 0 - \mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{k^4}{2\cdot 16\cdot 4!}[u_{rrrr}(r_1)+u_{rrrr}(r_2)],\phi\Big).
\end{aligned}
\tag{3.10}
$$

Therefore, applying an inverse inequality we obtain

$$
\begin{aligned}
&\mathrm{Re}\Big[-\mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{u(r^{n+1})+u(r^n)}{2}-u(r^{n+\frac{1}{2}})-\frac{k^2}{8}R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\phi\Big)\Big]\\
&\le ck^4\max_r\|u_{rrrr}\|_1\|\phi\|_1 \le ck^4h^{-1}\max_r\|u_{rrrr}\|_1\|\phi\|.
\end{aligned}
\tag{3.11}
$$

In addition, the Taylor formula gives

$$
\begin{aligned}
&\mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\Big[\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1})+R_h(r^{n+\frac{1}{2}})u(r^n)}{2}-u(r^{n+\frac{1}{2}})\Big],\phi\Big)\\
&= \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\Big[R_h(r^{n+\frac{1}{2}})u(r^{n+\frac{1}{2}})-u(r^{n+\frac{1}{2}})\Big],\phi\Big)\\
&+ \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)R_h(r^{n+\frac{1}{2}})\Big[\frac{k^2}{8}u_{rr}(r^{n+\frac{1}{2}})+\frac{k^4}{2\cdot 16\cdot 4!}[u_{rrrr}(r_1)+u_{rrrr}(r_2)]\Big],\phi\Big)\\
&= \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\Big[R_h(r^{n+\frac{1}{2}})u(r^{n+\frac{1}{2}})-u(r^{n+\frac{1}{2}})\Big],\phi\Big)\\
&+ \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)(R_h(r^{n+\frac{1}{2}})-I)\Big[\frac{k^2}{8}u_{rr}(r^{n+\frac{1}{2}})+\frac{k^4}{2\cdot 16\cdot 4!}[u_{rrrr}(r_1)+u_{rrrr}(r_2)]\Big],\phi\Big)\\
&+ \mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\Big[\frac{k^2}{8}u_{rr}(r^{n+\frac{1}{2}})+\frac{k^4}{2\cdot 16\cdot 4!}[u_{rrrr}(r_1)+u_{rrrr}(r_2)]\Big],\phi\Big).
\end{aligned}
$$

Thus, we obtain

$$\begin{aligned}&\mathrm{Re}\Big[\mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\Big[\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1})+R_h(r^{n+\frac{1}{2}})u(r^n)}{2}-u(r^{n+\frac{1}{2}})\Big],\phi\Big)\Big]\\&\leq c\Big\{h^{\tau+1}+k^2\Big\}\|\phi\|.\end{aligned}\tag{3.12}$$

Also Taylor gives for $r_3, r_4 \in (r^n, r^{n+1})$

$$\frac{u(r^{n+1})-u(r^n)}{k}=\frac{k^2}{8\cdot 3!}[u_{rrr}(r_3)+u_{rrr}(r_4)]+u_r(r^{n+\frac{1}{2}}),$$

therefore,

$$\begin{aligned}&-\Big(\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1})-R_h(r^{n+\frac{1}{2}})u(r^n)}{k}-u_r(r^{n+\frac{1}{2}}),\phi\Big)\\&=-\Big([R_h(r^{n+\frac{1}{2}})-I]u_r(r^{n+1}),\phi\Big)-\frac{k^2}{8\cdot 3!}\Big([R_h(r^{n+\frac{1}{2}})-I][u_{rrr}(r_3)+u_{rrr}(r_4)],\phi\Big)\\&-\frac{k^2}{8\cdot 3!}\Big(u_{rrr}(r_3)+u_{rrr}(r_4),\phi\Big).\end{aligned}$$

The above yields

$$\mathrm{Re}\Big[-\Big(\frac{R_h(r^{n+\frac{1}{2}})u(r^{n+1})-R_h(r^{n+\frac{1}{2}})u(r^n)}{k}-u_r(r^{n+\frac{1}{2}}),\phi\Big)\Big]\leq c\Big\{h^{\tau+1}+k^2\Big\}\|\phi\|.\tag{3.13}$$

Let us now assume that $k<1$ and $k\leq ch^{\frac{1}{2}}$. In (3.9), we take real parts and use (3.11), (3.12), and (3.13) to obtain

$$\begin{aligned}&\mathrm{Re}\Big[\Big(\frac{\theta_2^{n+1}-\theta_1^n}{k},\phi\Big)\Big]\leq c\Big\{h^{\tau+1}+k^2\Big\}\|\phi\|\\&+\mathrm{Re}\Big[-\mathrm{i}\mathcal{B}\Big(r^{n+\frac{1}{2}};\frac{\theta_2^{n+1}+\theta_1^n}{2},\phi\Big)+\mathrm{i}\Big((\beta(r^{n+\frac{1}{2}})+\delta)\frac{\theta_2^{n+1}+\theta_1^n}{2},\phi\Big)\Big].\end{aligned}\tag{3.14}$$

In the above estimate, we set $\phi:=\theta_2^{n+1}+\theta_1^n\in S_h\subset\tilde{H}_0^1(\mathfrak{D})$, and use the estimate of Remark 3.4 to obtain for k sufficiently small

$$\|\theta_2^{n+1}\|\leq\Big(\frac{1+ck}{1-ck}\Big)\|\theta_1^n\|+\mathcal{A},\ \ 0\leq n\leq N-1,\tag{3.15}$$

where $\mathcal{A}\leq\frac{ck(h^{\tau+1}+k^2)}{1-ck}$.

Let us now define

$$\theta^n:=U^n-R_h(r^n)u(r^n),\ \ 0\leq n\leq N.\tag{3.16}$$

and

$$\begin{aligned}B_2^{(n+1)}&:=-R_h(r^{n+1})u(r^{n+1})+R_h(r^{n+\frac{1}{2}})u(r^{n+1})-\frac{k^2}{8}R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\ \ 0\leq n\leq N-1,\\B_1^{(n)}&:=-R_h(r^n)u(r^n)+R_h(r^{n+\frac{1}{2}})u(r^n)-\frac{k^2}{8}R_h(r^{n+\frac{1}{2}})u_{rr}(r^{n+\frac{1}{2}}),\ \ 0\leq n\leq N-1.\end{aligned}\tag{3.17}$$

So, we obtain

$$\begin{aligned}\theta_2^{n+1}&=\theta^{n+1}-B_2^{(n+1)},\ \ 0\leq n\leq N-1,\\\theta_1^n&=\theta^n-B_1^{(n)},\ \ 0\leq n\leq N-1.\end{aligned}\tag{3.18}$$

We replace in (3.15) so that for any $1 \le n \le N-1$ we arrive at

$$\begin{aligned}\|\theta^{n+1} - B_2^{(n+1)}\| \le & \Big(\frac{1+ck}{1-ck}\Big)\|\theta^n - B_1^{(n)}\| + \mathcal{A} \\ \le & \Big(\frac{1+ck}{1-ck}\Big)\|\theta^n - B_2^{(n)}\| + \mathcal{A} \\ & + \Big(\frac{1+ck}{1-ck}\Big)\|B_2^{(n)} - B_2^{(n+1)}\| + \Big(\frac{1+ck}{1-ck}\Big)\|B_2^{(n+1)} - B_1^{(n)}\|.\end{aligned} \tag{3.19}$$

3.3.2. *The estimates.* We prove first the following lemmas.

Lemma 3.7. *For any* $0 \le n \le N-1$ *it holds that*

$$\|B_2^{(n+1)} - B_1^{(n)}\| \le ckh^{\tau+1}. \tag{3.20}$$

Proof. By the definition of $B_2^{(n+1)}, B_1^{(n)}$ we obtain

$$B_2^{(n+1)} - B_1^{(n)} = [R_h(r^{n+\frac12}) - I][u(r^{n+1}) - u(r^n)] - \int_{r^n}^{r^{n+1}} \partial_r[R_h(r)u(r) - u(r)]dr.$$

Using Taylor's theorem, we obtain for $r_5, r_6 \in (r^n, r^{n+1})$

$$u(r^{n+1}) - u(r^n) = ku_r(r^{n+\frac12}) + \frac{k^3}{8\cdot 3!}[u_{rrr}(r_5) - u_{rrr}(r_6)],$$

The result now follows from the estimates of $R_h(r)v(s) - v(s)$ and $\partial_r(R_h(r)v(r) - v(r))$. □

Lemma 3.8. *For any* $1 \le n \le N-1$ *it holds that*

$$\|B_2^{(n+1)} - B_2^{(n)}\| \le ck\Big\{h^{\tau+1} + k^2\Big\}. \tag{3.21}$$

Proof. By the definition of $B_2^{(n+1)}, B_1^{(n)}$ we have that

$$\begin{aligned}B_2^{(n)} - B_2^{(n+1)} = & R_h(r^{n+\frac12})\Big[\frac{k^2}{8}u_{rr}(r^{n+\frac12}) - u(r^{n+1})\Big] \\ & - R_h(r^{n-\frac12})\Big[\frac{k^2}{8}u_{rr}(r^{n-\frac12}) - u(r^n)\Big] + R_h(r^{n+1})u(r^{n+1}) - R_h(r^n)u(r^n).\end{aligned} \tag{3.22}$$

Using Taylor's theorem we have that for $r_7 \in (r^{n+\frac12}, r^{n+1})$, $r_8 \in (r^{n-\frac12}, r^n)$

$$\begin{aligned}u(r^{n+1}) - \frac{k^2}{8}u_{rr}(r^{n+\frac12}) &= u(r^{n+\frac12}) + \frac{k}{2}u_r(r^{n+\frac12}) + \frac{k^3}{8\cdot 3!}u_{rrr}(r_7), \\ u(r^n) - \frac{k^2}{8}u_{rr}(r^{n-\frac12}) &= u(r^{n-\frac12}) + \frac{k}{2}u_r(r^{n-\frac12}) + \frac{k^3}{8\cdot 3!}u_{rrr}(r_8).\end{aligned}$$

Replacing these expansions in (3.22) it follows that

$$\begin{aligned}B_2^{(n)} - B_2^{(n+1)} = & -R_h(r^{n+\frac12})\Big[u(r^{n+\frac12}) + \frac{k}{2}u_r(r^{n+\frac12})\Big] \\ & + R_h(r^{n-\frac12})\Big[u(r^{n-\frac12}) + \frac{k}{2}u_r(r^{n-\frac12})\Big] + R_h(r^{n+1})u(r^{n+1}) - R_h(r^n)u(r^n) + \mathcal{B}_1 \\ = & \int_{r^n}^{r^{n+1}} [\partial_r R_h(r)u(r) - u_r(r)]dr \\ & - \int_{r^{n-\frac12}}^{r^{n+\frac12}} \Big(\partial_r R_h(r)[u(r) + \frac{k}{2}u_r(r)] - [u_r(r) + \frac{k}{2}u_{rr}(r)]\Big)dr \\ & + u(r^{n+1}) - u(r^n) - u(r^{n+\frac12}) + u(r^{n-\frac12}) - \frac{k}{2}u_r(r^{n+\frac12}) + \frac{k}{2}u_r(r^{n-\frac12}) + \mathcal{B}_1,\end{aligned} \tag{3.23}$$

where $|\mathcal{B}_1| \leq ck^3$ for $h < 1$. Expanding in Taylor series around r^n, r^{n+1} we finally have that

$$|u(r^{n+1}) - u(r^n) - u(r^{n+\frac{1}{2}}) + u(r^{n-\frac{1}{2}}) - \frac{k}{2}u_r(r^{n+\frac{1}{2}}) + \frac{k}{2}u_r(r^{n-\frac{1}{2}})| \leq ck^3,$$

and the result follows from (3.23).

Remark 3.9. Obviously we assumed $n \geq 1$, since we used the nodal point $r^{n-\frac{1}{2}}$.

□

Lemma 3.10. *We have*

$$\|B_1^{(0)}\| \leq ckh^{\tau+1} + ck^2, \tag{3.24}$$

$$\|B_2^{(1)}\| \leq ckh^{\tau+1} + ck^2, \tag{3.25}$$

$$\|B_2^{(n+1)}\| \leq ch^{\tau+1} + ck^2, \quad 0 \leq n \leq N-1. \tag{3.26}$$

Proof. We use the fact that for $r_9 \in (r^0, r^{\frac{1}{2}})$

$$u(r^0) = u(r^{\frac{1}{2}}) - \frac{k}{2}u_r(r^{\frac{1}{2}}) + \frac{k^2}{8}u_{rr}(r^{\frac{1}{2}}) + \frac{k^3}{8 \cdot 3!}u_{rrr}(r_9),$$

and obtain

$$B_1^{(0)} = \int_{r^0}^{r^{\frac{1}{2}}} [\partial_r R_h(r)u(r) - u_r(r)]dr - \frac{k}{2}[R_h(r^{\frac{1}{2}})u_r(r^{\frac{1}{2}}) - u_r(r^{\frac{1}{2}})] + u(r^{\frac{1}{2}}) - u(r^0) - \frac{k}{2}u_r(r^{\frac{1}{2}}) + \mathcal{B}_2,$$

for $|\mathcal{B}_2| \leq ck^2$. Finally, using

$$|u(r^{\frac{1}{2}}) - u(r^0) - \frac{k}{2}u_r(r^{\frac{1}{2}})| \leq ck^2,$$

we arrive at (3.24).

By Lemma 3.7 applied for $n = 0$, we get

$$\|B_2^{(1)} - B_1^{(0)}\| \leq ckh^{\tau+1},$$

so using (3.24) the estimate (3.25) follows.

Using that

$$\|B_2^{(n+1)}\| \leq \|B_2^{(n+1)} - B_2^{(n)}\| + \|B_2^{(n)} - B_2^{(n-1)}\| + \cdots + \|B_2^{(2)} - B_2^{(1)}\| + \|B_2^{(1)}\|,$$

and Lemma 3.8 we obtain

$$\|B_2^{(n+1)}\| \leq \|B_2^{(1)}\| + ch^{\tau+1} + ck^2,$$

and by (3.25) we arrive at (3.26). □

We now estimate θ^1.

Lemma 3.11. *If $k = O(h)$ then*

$$\|\theta^1\| \leq ch^{\tau+1} + ck^2. \tag{3.27}$$

Proof. We use the continuous problem and the fact that $\theta^0 = 0$, set $\phi = \theta^1$ in the fully discrete scheme, take real parts and use the inverse inequality to obtain

$$\begin{aligned}
&\Big(\frac{\theta^{n+1}-\theta^n}{k},\phi\Big) = -\Big(\frac{R_h(r^{n+1})u(r^{n+1}) - R_h(r^n)u(r^n)}{k} - u_r(r^{n+1/2}),\phi\Big)\\
&-\mathrm{i}\mathcal{B}\Big(r^{n+1/2};\frac{\theta^{n+1}+\theta^n}{2},\phi\Big) + \mathrm{i}\Big([\beta(r^{n+1/2})+\delta]\frac{\theta^{n+1}+\theta^n}{2},\phi\Big)\\
&-\mathrm{i}\mathcal{B}\Big(r^{n+1/2};\frac{R_h(r^{n+1})u(r^{n+1}) + R_h(r^n)u(r^n)}{2} - u(r^{n+1/2}),\phi\Big)\\
&+\mathrm{i}\Big([\beta(r^{n+1/2})+\delta]\frac{R_h(r^{n+1})u(r^{n+1}) + R_h(r^n)u(r^n)}{2} - u(r^{n+1/2}),\phi\Big).
\end{aligned}$$

Obviously, if $\mathcal{B}$ has smooth coefficients and g, a, b are smooth, it follows that

$$\begin{aligned}
&\Big|\mathcal{B}(r^{n+1/2};a,b)\Big| \le \frac{1}{2}\Big|\mathcal{B}(r^{n+1};a,b)+\mathcal{B}(r^n;a,b)\Big| + ck^2\|a\|_1\|b\|_1,\\
&\Big|\mathcal{B}(r^{n+1};a,b)-\mathcal{B}(r^n;a,b)\Big| \le ck\|a\|_1\|b\|_1,\\
&\Big|\mathcal{B}(r;\frac{g(r^{n+1})+g(r^n)}{2},b)-\mathcal{B}(r;g(r^{n+1/2}),b)\Big| \le ck^2\|b\|_1.
\end{aligned}$$

So, we get

$$\begin{aligned}
&\mathcal{B}\Big(r^{n+1/2};\frac{R_h(r^{n+1})u(r^{n+1}) + R_h(r^n)u(r^n)}{2} - u(r^{n+1/2}),\phi\Big) =\\
&\mathcal{B}\Big(r^{n+1/2};\frac{R_h(r^{n+1})u(r^{n+1}) + R_h(r^n)u(r^n)}{2} - \frac{u(r^{n+1})+u(r^n)}{2},\phi\Big) + \mathcal{A}_1 =\\
&\frac{1}{2}\mathcal{B}\Big(r^{n+1};\frac{R_h(r^{n+1})u(r^{n+1}) + R_h(r^n)u(r^n)}{2} - \frac{u(r^{n+1})+u(r^n)}{2},\phi\Big)\\
&+\frac{1}{2}\mathcal{B}\Big(r^{n};\frac{R_h(r^{n})u(r^{n+1}) + R_h(r^n)u(r^n)}{2} - \frac{u(r^{n+1})+u(r^n)}{2},\phi\Big) + \mathcal{A}_2 + \mathcal{A}_1 =\\
&\frac{1}{2}\mathcal{B}\Big(r^{n+1};\frac{R_h(r^n)u(r^n)}{2} - \frac{u(r^n)}{2},\phi\Big) + \frac{1}{2}\mathcal{B}\Big(r^{n};\frac{R_h(r^{n+1})u(r^{n+1})}{2} - \frac{u(r^{n+1})}{2},\phi\Big)\\
&+\mathcal{A}_2 + \mathcal{A}_1 =\\
&\frac{1}{2}\mathcal{B}\Big(r^{n};\frac{R_h(r^n)u(r^n)}{2} - \frac{u(r^n)}{2},\phi\Big) + \mathcal{A}_3\\
&+\frac{1}{2}\mathcal{B}\Big(r^{n+1};\frac{R_h(r^{n+1})u(r^{n+1})}{2} - \frac{u(r^{n+1})}{2},\phi\Big) + \mathcal{A}_4 + \mathcal{A}_2 + \mathcal{A}_1 =\\
&\mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_2 + \mathcal{A}_1,
\end{aligned}$$

where

$$\begin{aligned}
&|\mathcal{A}_1| \le ck^2\|\phi\|_1,\\
&|\mathcal{A}_2| \le ck^2\|\phi\|_1 \quad (\text{since } \|R_h(r)u(r)\|_1 \le ch^\tau + c),\\
&|\mathcal{A}_3|, |\mathcal{A}_4| \le ckh^\tau\|\phi\|_1 \quad (\text{since } \|R_h(r)u(r) - u(r)\|_1 \le ch^\tau).
\end{aligned}$$

Therefore, we obtain setting $n = 0$, $\phi = \theta^1$ and using the inverse inequality

$$\begin{aligned}
\|\theta^1\|^2 &\le ck[h^{\tau+1} + kh^\tau + k^2]\|\theta^1\|_1 + ck\|\theta^1\|^2\\
&\le ckh^{-1}[h^{\tau+1} + kh^\tau + k^2]\|\theta^1\| + ck\|\theta^1\|^2.
\end{aligned}$$

So for $\mathcal{O}(k) = \mathcal{O}(h)$ the result follows. □

Lemma 3.12. *If $\mathcal{O}(k)=\mathcal{O}(h)$ then for any $n\geq 0$*

$$\|\theta^{n+1}\|\leq ch^{\tau+1}+ck^2. \tag{3.28}$$

Proof. Since $k\leq ch^{\frac{1}{2}}$ then the inequality (3.19) holds. So, using (3.19) and Lemmas 3.7-3.8 we arrive at

$$\|\theta^{n+1}-B_2^{(n+1)}\|\leq c\|\theta^n-B_2^{(n)}\|+ckh^{\tau+1}+ck^3,$$

therefore, setting $\mathcal{E}^n:=\theta^n-B_2^{(n)}$ for $n\geq 1$, we obtain

$$\begin{aligned}
\|\mathcal{E}^{n+1}\| &\leq \Big(\frac{1+ck}{1-ck}\Big)\|\mathcal{E}^n\|+ckh^{\tau+1}+ck^3\\
&\leq \Big(\frac{1+ck}{1-ck}\Big)^2\|\mathcal{E}^{n-1}\|+\Big(\frac{1+ck}{1-ck}\Big)(ckh^{\tau+1}+ck^3)+ckh^{\tau+1}+ck^3\\
&\leq \cdots\\
&\leq \Big(\frac{1+ck}{1-ck}\Big)^n\|\mathcal{E}^1\|+\Big(\frac{1+ck}{1-ck}\Big)^n\sum_{i=1}^{n}(ckh^{\tau+1}+ck^3)\\
&\leq c\|\mathcal{E}^1\|+ch^{\tau+1}+ck^2,
\end{aligned}$$

(since $k=\frac{1}{N}$ then $\Big(\frac{1+ck}{1-ck}\Big)^N\to e^c$ as $N\to\infty$ and thus $\Big(\frac{1+ck}{1-ck}\Big)^n$ is bounded).

Thus, we get replacing $\mathcal{E}^{n+1}$, $\mathcal{E}^1$

$$\|\theta^{n+1}-B_2^{(n+1)}\|\leq c\|\theta^1-B_2^{(1)}\|+ch^{\tau+1}+ck^2.$$

So we take

$$\|\theta^{n+1}\|-\|B_2^{(n+1)}\|\leq c\|\theta^1-B_2^{(1)}\|+ch^{\tau+1}+ck^2. \tag{3.29}$$

By (3.26) $\|B_2^{(n+1)}\|\leq ch^{\tau+1}+ck^2$, thus (3.29) together with (3.25), Lemma 3.7 for $n=0$, and Lemma 3.11, gives using the estimates of $\|\theta^1\|$ and $\|B_1^{(0)}\|$

$$\begin{aligned}
\|\theta^{n+1}\| &\leq c\|\theta^1-B_2^{(1)}\|+ch^{\tau+1}+ck^2\\
&\leq c\|\theta^1-B_1^{(0)}\|+c\|B_1^{(0)}-B_2^{(1)}\|+ch^{\tau+1}+ck^2\\
&\leq c\|\theta^1\|+\|B_1^{(0)}\|+ch^{\tau+1}+ck^2\leq ch^{\tau+1}+ck^2.
\end{aligned} \tag{3.30}$$

□

We are now ready to prove the main error estimate of this section:

Theorem 3.13. *If $\mathcal{O}(k)=\mathcal{O}(h)$ then*

$$\|U^n-u(r^n)\|\leq ch^{\tau+1}+ck^2,\ \ 0\leq n\leq N. \tag{3.31}$$

Proof. Obviously, using Lemma 3.12 and the fact that $\theta^0=0$, it follows that

$$\|U^n-u(r^n)\|\leq\|\theta^n\|+ch^{\tau+1}\leq ch^{\tau+1}+ck^2.$$

□

4. Global Elliptic Regularity

In this section, we present a general Global Elliptic Regularity Theorem for complex elliptic operators with mixed Dirichlet-Robin boundary conditions, in rectangles of $\mathbb{R}^2$. Our proof follows that of [18] which deals with the Dirichlet problem for real operators. In our approach, the main idea is that if the trace terms in the weak formulation of the problem vanish due to the boundary conditions, for suitably chosen test functions, then a Global Elliptic Regularity result is proved in Theorem 4.1. Note that the Robin condition in this Theorem does not involve any zero order term, while the first order terms are related to the coefficients of the boundary problem so that indeed in the weak formulation, after integration by parts, the trace integrals vanish. Our result is established by using the fact that the closure of a rectangle can be covered by using a finite union of half-balls together with an open smooth domain in the interior. We then apply an exponential transformation and extent our result, in Theorem 4.3, where an arbitrary zero order term is introduced at the Robin condition of Theorem 4.1.

Theorem 4.1. *Let $\mathcal{W} = (0,1) \times (\theta_1, \theta_2)$ be a rectangular domain in cartesian coordinates. We consider the following boundary value problem: We seek a complex-valued function u such that*

$$\begin{aligned} &Au_{zz} + Bu_{z\theta} + Cu_{\theta\theta} + Du_z + Eu_\theta + Fu = f \quad \text{in} \quad \mathcal{W}, \\ &u(0,\theta) = 0, \\ &u(z,\theta_1) = u(z,\theta_2) = 0, \\ &a(\theta)u_z + b(\theta)u_\theta = 0 \quad \text{at} \quad z = 1, \end{aligned} \tag{4.1}$$

where $A, B, C \in C^1(\overline{\mathcal{W}})$, $D, E, F \in L^\infty(\mathcal{W})$, $f \in L^2(\mathcal{W})$ and $a, b : [\theta_1, \theta_2] \to \mathbb{C}^*$. We also assume that A, B, C take imaginary values and $\frac{A}{\mathrm{i}}$, $\frac{C}{\mathrm{i}}$ are always positive (or always negative). Moreover, we assume that

$$|AC| > \frac{|B|^2}{4}, \qquad \text{for any } (z,\theta) \in \mathcal{W}, \tag{4.2}$$

$$\frac{A(1,\theta)}{a(\theta)} = \frac{B(1,\theta)}{2b(\theta)}, \qquad \text{for any } \theta \in [\theta_1, \theta_2]. \tag{4.3}$$

If $u \in H^1(\mathcal{W})$ is a weak solution of (4.1) then the following elliptic regularity estimate holds

$$u \in H^2(\mathcal{W}) \quad \text{and} \quad \|u\|_{H^2(\mathcal{W})} \le c\,\|f\|_{L^2(\mathcal{W})}. \tag{4.4}$$

Proof. We consider the rectangle $\mathcal{W}$. Obviously its boundary is the union of four linear segments and we write $\partial\mathcal{W} = \cup_{i=1}^4 \partial\mathcal{W}_i$ (cf. Figure 3). Let $\mathcal{U}_i = B^o(k_i, r_i) \cap \overline{\mathcal{W}}$, be a half-ball in $\mathbb{R}^2$ in $\overline{\mathcal{W}}$ laying at $\partial\mathcal{W}$ of range r_i and of diameter in $\partial\mathcal{W}_i$. We define its boundary by $\partial\mathcal{U}_i := \partial\mathcal{U}_{ih} \cup \partial\mathcal{U}_{ic}$, where $\partial\mathcal{U}_{ih}$ is the diameter such that $\partial\mathcal{U}_{ih} \subseteq \partial\mathcal{W}_i$, and $\partial\mathcal{U}_{ic}$ is the semicircle of range r_i such that $\mathcal{U}_i \subset \overline{\mathcal{W}}$, we also consider $\mathcal{V}_i = B^o(k_i, r_i/2) \cap \overline{\mathcal{W}}$, the half-ball being of the same center k_i as $\mathcal{U}_i$ and of range $r_i/2$ (cf. Figure 2). Obviously, $\partial\mathcal{W}$ is compact, thus $\partial\mathcal{W}$ may be covered by using a finite union of sets of the form $\mathcal{V}_i$, while the same union together with a suitably chosen smooth domain in $\mathcal{W}$ covers $\overline{\mathcal{W}}$. By [18] an interior regularity estimate holds. Our aim is to prove the regularity estimate

$$\|u\|_{H^2(\mathcal{V}_i)} \le c\,\|f\|_{L^2(\mathcal{U}_i)}, \quad i = 1, \ldots, 4. \tag{4.5}$$

Interior regularity combined with the estimate (4.5) gives the desired result (4.4) (cf. [18], pg. 322).

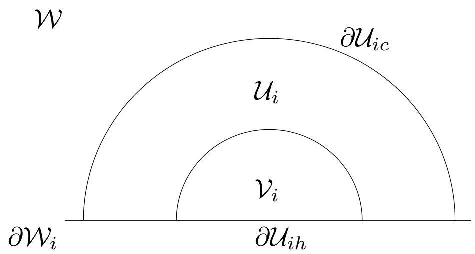

FIGURE 2. Half-balls, curved boundary, horizontal boundary.

We consider $\phi_i \in H^1(\mathcal{U}_i)$ and let u be the weak solution of (4.1). If $(u,v)_{\mathcal{U}_i} := \int_{\mathcal{U}_i} u\bar{v}\,ds$ then we have

$$\begin{aligned}(f,\phi_i)_{\mathcal{U}_i} =& -(Au_z,\partial_z\phi_i)_{\mathcal{U}_i} - \Big\{\Big(\frac{B}{2}u_z,\partial_\theta\phi_i\Big)_{\mathcal{U}_i} + \Big(\frac{B}{2}u_\theta,\partial_z\phi_i\Big)_{\mathcal{U}_i}\Big\}\\ &-(Cu_\theta,\partial_\theta\phi_i)_{\mathcal{U}_i} + (\widetilde{D}u_z,\phi_i)_{\mathcal{U}_i} + (\widetilde{E}u_\theta,\phi_i)_{\mathcal{U}_i} + (Fu,\phi_i)_{\mathcal{U}_i}\\ &+\int_{\partial\mathcal{U}_i}\Big[u_z\Big(A,\frac{B}{2}\Big) + u_\theta\Big(\frac{B}{2},C\Big)\Big]\bar{\phi}_i\overrightarrow{\eta_i}ds,\end{aligned} \tag{4.6}$$

where $\widetilde{D}$, $\widetilde{E}$ are the resulting terms after integration by parts, and $\overrightarrow{\eta_i}$ is the outward unit normal to $\partial\mathcal{U}_i$. We let $\Omega_i(u,\phi_i) := \int_{\partial\mathcal{U}_i}[u_z(A,\frac{B}{2})+u_\theta(\frac{B}{2},C)]\bar{\phi}_i\overrightarrow{\eta_i}ds$, and define the vector $K_i := [u_z(A,\frac{B}{2})+u_\theta(\frac{B}{2},C)]\bar{\phi}_i$; here $(\cdot,\cdot)$ denotes a vector of $\mathbb{R}^2$. Then for $\partial\mathcal{U}_i = \partial\mathcal{U}_{ih}\cup\partial\mathcal{U}_{ic}$ it holds that $\Omega_i(u,\phi_i) = \int_{\partial\mathcal{U}_{ih}} K_i\overrightarrow{\eta_i}ds + \int_{\partial\mathcal{U}_{ic}} K_i\overrightarrow{\eta_i}ds$. Using the boundary conditions of $u \in H^1(\mathcal{W})$ we obtain

$$\begin{aligned}&\Omega_1(u,\phi_1) = -\int_{\partial\mathcal{U}_{1h}} Au_z\bar{\phi}_1 ds + \int_{\partial\mathcal{U}_{1c}} K_1\overrightarrow{\eta_1}ds, \quad \Omega_2(u,\phi_2) = \int_{\partial\mathcal{U}_{2c}} K_2\overrightarrow{\eta_2}ds,\\ &\Omega_3(u,\phi_3) = -\int_{\partial\mathcal{U}_{3h}} Cu_\theta\bar{\phi}_3 ds + \int_{\partial\mathcal{U}_{3c}} K_3\overrightarrow{\eta_3}ds,\\ &\Omega_4(u,\phi_4) = \int_{\partial\mathcal{U}_{4h}} Cu_\theta\bar{\phi}_4 ds + \int_{\partial\mathcal{U}_{4c}} K_4\overrightarrow{\eta_4}ds.\end{aligned} \tag{4.7}$$

Our aim now is to find test functions ϕ_i such that in the weak formulation the trace terms vanish.

Assumption 1. We assume that there exist functions ϕ_i that satisfy the following requirements:

- The test functions are smooth and along the curved boundary $\mathcal{U}_{ic}$ of $\mathcal{U}_i$ vanish: $\phi_i \in H^1(\mathcal{U}_i)$, and $\phi_i = 0|\partial\mathcal{U}_{ic},\ i = 1,\ldots,4$.
- For $i = 1,3,4$, the test functions vanish also along the horizontal boundary $\mathcal{U}_{ih}$ of $\mathcal{U}_i$: $\phi_1 = 0$ at $z = 0$, ϕ_2 is arbitrary, $\phi_3 = 0$ at $\theta = \theta_1$, $\phi_4 = 0$ at $\theta = \theta_2$.

Under this assumption, the sum of trace integrals in the weak formulation equals zero because $\Omega_i(u,\phi_i) = 0$ for any $i = 1,\ldots,4$. The weak formulation (4.6) for $\mathcal{B}(u,\phi_i)_{\mathcal{U}_i} := (f,\phi_i)_{\mathcal{U}_i}$ becomes

$$\begin{aligned}\mathcal{B}(u,\phi_i)_{\mathcal{U}_i} =& -(Au_z,\partial_z\phi_i)_{\mathcal{U}_i} - \Big\{\Big(\frac{B}{2}u_z,\partial_\theta\phi_i\Big)_{\mathcal{U}_i} + \Big(\frac{B}{2}u_\theta,\partial_z\phi_i\Big)_{\mathcal{U}_i}\Big\}\\ &-(Cu_\theta,\partial_\theta\phi_i)_{\mathcal{U}_i} + (\widetilde{D}u_z,\phi_i)_{\mathcal{U}_i} + (\widetilde{E}u_\theta,\phi_i)_{\mathcal{U}_i} + (Fu,\phi_i)_{\mathcal{U}_i}.\end{aligned} \tag{4.8}$$

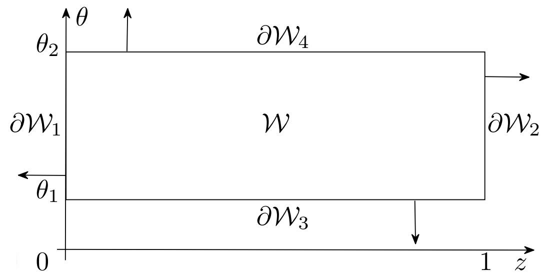

FIGURE 3. The rectangular domain $\mathcal{W}$.

The next step is to define, properly, for any $i = 1, \cdots, 4$, test functions ϕ_i satisfying this assumption. We define the following general cut-off function ([18])

$$J = \begin{cases} 0 \text{ in } \mathbb{R}^2 - B(\tilde{l}, r), \\ 1 \text{ in } B(\tilde{l}, r/2), \\ 0 \leq J \leq 1 \text{ elsewhere (with } J = 0 \text{ near } \partial\mathcal{U}_c). \end{cases} \tag{4.9}$$

Here $\mathcal{U} := B^o(\tilde{l}, r) \cap \overline{\mathcal{W}}$ is a half-ball in $\mathbb{R}^2$ of radius r and of center $\tilde{l}$ such that $\partial\mathcal{U}_h \subseteq \vartheta\mathcal{W}$. Let $\mathcal{V}$ be the half-ball in $\mathbb{R}^2$ of center $\tilde{l}$ and of range $r/2$ with diameter in $\partial\mathcal{U}_h$. Obviously the cut off function J in $\mathcal{V}$ equals 1, and near $\partial\mathcal{U}_c$ is 0. Let $\tilde{u}$ be a function in $H^1(\mathcal{W})$ that satisfies the boundary conditions of problem (4.1), we define the function ([18])

$$\tilde{v} := -D^{-h}(J^2 D^h \tilde{u}), \text{ with } D^h\tilde{u}(x) := \tfrac{\tilde{u}(x+he)-\tilde{u}(x)}{h}, \ x \in \mathcal{U}, \tag{4.10}$$

where h is a positive number and e is a unitary vector (direction) in $\mathbb{R}^2$ parallel to the diameter of the half-ball $\mathcal{U}$.

In this way for every boundary line ($i = 1, \cdots, 4$) of the rectangular domain $\mathcal{W}$ we define a cut-off function J_i and denote by e_i the unitary direction of the specific boundary line $\partial\mathcal{W}_i$. We then prove first that $\tilde{v}_i$ defined by these J_i in (4.10) for the directions e_i are test functions that satisfy the Assumption 1, and in the sequel we set $\phi_i := \tilde{v}_i$.

More specifically, for every $i = 1, \ldots, 4$ we consider $\mathcal{U}_i = B^o(k_i, r_i) \cap \overline{\mathcal{W}}, \ \mathcal{V}_i = B^0(k_i, \frac{r_i}{2}) \cap \overline{\mathcal{W}}, \ k_i, \ r_i$ such that $\mathcal{U}_i \subseteq \mathcal{W}, \ \partial\mathcal{U}_{ih} \subseteq \partial\mathcal{W}_i$ and define the cut-off function

$$J_i := \begin{cases} J_i = 0 \text{ in } \mathbb{R}^2 - B(k_i, r_i), \\ J_i = 1 \text{ in } B(k_i, \frac{r_i}{2}), \\ 0 \leq J_i \leq 1 \text{ elsewhere (with } J_i = 0 \text{ near } \partial\mathcal{U}_{ic}). \end{cases}$$

Let $\tilde{u}$ be a function in $H^1(\mathcal{W})$ that satisfies the boundary conditions of problem (4.1), we define as previously the function

$$\tilde{v}_i := -D_i^{-h}(J_i^2 D_i^h \tilde{u}), \text{ with } D_i^h\tilde{u}(x) := \tfrac{\tilde{u}(x+he_i)-\tilde{u}(x)}{h}, \ x \in \mathcal{U}_i. \tag{4.11}$$

By [18], for any $x \in \mathcal{U}_i$, the following identity holds

$$\tilde{v}_i(x) = -\tfrac{1}{h^2}(J_i^2(x-he_i)[\tilde{u}(x) - \tilde{u}(x-he_i)] - J_i^2(x)[\tilde{u}(x+he_i) - \tilde{u}(x)]). \tag{4.12}$$

Using the boundary conditions of the elliptic problem and the identity (4.12), we will prove that $\tilde{v}_i$ satisfy Assumption 1 for any $i = 1, \ldots, 4$.

<u>If $i = 1$</u>, then obviously $\tilde{v}_1$ is in $H^1(\mathcal{U}_1)$. We notice that if x is in $\partial\mathcal{U}_{1c}$ then $J_1(x) = 0$ and for h

small enough $J_1(x-he_1)=0$ so by (4.12) $\tilde{v}_1(x)=0|\partial\mathcal{U}_{1c}$. Along the boundary line $\partial\mathcal{U}_{1h}$ holds that $z=0$ and $e_1=(0,1)$. If $x=(0,\theta)$ then $\tilde{u}(x)=0$ and $\tilde{u}(x\pm he_1)=\tilde{u}(0,\theta\pm h)=0$, thus by (4.12) follows that $\tilde{v}_1(0,\theta)=0$.

If $i=2$, then $\tilde{v}_2(x)\in H^1(\mathcal{U}_2)$, and $e_2=(0,1)$. If $x\in\partial\mathcal{U}_{2c}$ then for h small $J_2(x)=J_2(x-he_2)=0$, thus by (4.12) $\tilde{v}_2(x)=0|\partial\mathcal{U}_{2c}$.

If $i=3$, then $\tilde{v}_3(x)\in H^1(\mathcal{U}_3)$ and $e_3=(1,0)$, for h small. If $x\in\mathcal{U}_{3c}$ then $J_3(x)=J_3(x-he_3)=0$ thus $\tilde{v}_3(x)=0|\partial\mathcal{U}_{3c}$. For $x=(z,\theta_1)$ then $\tilde{u}(x)=\tilde{u}(z,\theta_1)=0$ and $\tilde{u}(x\pm he_3)=\tilde{u}(z\pm h,\theta_1)=0$. By (4.12) follows that $\tilde{v}_3(z,\theta_1)=0$.

If $i=4$, then $\tilde{v}_4(x)\in H^1(\mathcal{U}_4)$ and $e_4=(1,0)$, if x is in $\partial\mathcal{U}_{4c}$ then $J_4(x)=0$ and for h small enough $J_4(x-he_4)=0$, thus by (4.12) $\tilde{v}_4(x)=0|\partial\mathcal{U}_{4c}$. If $x=(z,\theta_2)$ then $\tilde{u}(x)=\tilde{u}(z,\theta_2)=0$ and $\tilde{u}(x\pm he_4)=\tilde{u}(z\pm h,\theta_2)=0$, thus $\tilde{v}_4(z,\theta_2)=0$.

Therefore, in all cases Assumption 1 holds and the trace terms vanish from the weak formulation of the elliptic problem. If we set $\tilde{u}:=u$, where u is the weak solution of the elliptic problem satisfying the boundary conditions, then it can be easily proved (for details see [6] and [18]) by use of ellipticity, the weak formulation and the boundary conditions at $z=0$, $\theta=\theta_1$, $\theta=\theta_2$, that for every half-ball $\mathcal{V}_i$ it holds

$$\|u\|_{H^2(\mathcal{V}_i)}\le c[\|f\|_{L^2(\mathcal{U}_i)}+\|u\|_{H^1(\mathcal{U}_i)}]. \tag{4.13}$$

Finite summation of (4.13) over any $\mathcal{V}_i$ (of type $i=1,\cdots,4$) and the interior regularity give ([18])

$$\|u\|_{H^2(\mathcal{W})}\le c[\|f\|_{L^2(\mathcal{W})}+\|u\|_{H^1(\mathcal{W})}]. \tag{4.14}$$

Combining (4.14) with ellipticity we obtain the elliptic regularity result

$$\|u\|_{H^2(\mathcal{W})}\le c\|f\|_{L^2(\mathcal{W})}. \tag{4.15}$$

□

Remark 4.2. We note that an analogous result is also valid if in the assumptions of Theorem 4.1, the homogeneous condition at $z=1$ is replaced by the non-homogeneous condition $a(\theta)u_z+b(\theta)u_\theta=g$ at $z=1$, for any $g\in H^{\frac{1}{2}}(\partial\mathcal{W}_R)$, where $\partial\mathcal{W}_R=\{1\}\times(\theta_1,\theta_2)$. In this case, in the weak formulation the trace integral term containing g is hidden due to ellipticity, leaving at the right-hand side of (4.15) the extra term $c|g|_{\frac{1}{2},\partial\mathcal{W}_R}$ where $|g|_{\frac{1}{2},\partial\mathcal{W}_R}:=\inf_{v\in H_{\mathcal{W}}:v|_{z=1}=g}\|v\|_1$, for $H_{\mathcal{W}}:=\{u\in H^1(\mathcal{W}):u|_{z=0}=0\}$. More specifically, the following elliptic regularity estimate holds

$$\|u\|_{H^2(\mathcal{W})}\le c\|f\|_{L^2(\mathcal{W})}+c|g|_{\frac{1}{2},\partial\mathcal{W}_R}. \tag{4.16}$$

The following theorem extends Theorem 4.1 in the sense that we can add at the boundary condition along $z=1$ a zero order term multiplied by an arbitrary smooth function $c(\theta)$.

Theorem 4.3. *Under the assumptions of Theorem 4.1, if the boundary condition of (4.1) at $z=1$ has the form*

$$a(\theta)u_z+b(\theta)u_\theta+c(\theta)u(\theta)=0 \ \text{at}\ z=1,\ \theta\in[\theta_1,\theta_2], \tag{4.17}$$

with c a smooth complex function of θ, then the results of Theorem 4.1 hold (elliptic regularity).

Proof. We set $q = q(z,\theta)$ and consider the elliptic operator of (4.1), we apply the transformation $u := \exp(q)w$ and get the following equivalent problem

$$(4.18)\qquad \begin{aligned} &Aw_{zz} + Bw_{z\theta} + Cw_{\theta\theta} + D_w w_z + E_w w_\theta + F_w w = f_w \quad \text{in} \quad \mathcal{W},\\ &w(0,\theta) = 0,\\ &w(z,\theta_1) = w(z,\theta_2) = 0,\\ &a(\theta)w_z + b(\theta)w_\theta + c_w(\theta)w(\theta) = 0 \quad \text{at} \quad z = 1,\end{aligned}$$

where $D_w = 2Aq_z + Bq_\theta + D$, $E_w = Bq_z + 2Cq_\theta + E$, $f_w = \exp(-q)f$, $F_w = F + A(q_{zz} + q_z^2) + B(q_{z\theta} + q_z q_\theta) + C(q_{\theta\theta} + q_\theta^2) + Dq_z + Eq_\theta$, and $c_w(\theta) = a(\theta)q_z + b(\theta)q_\theta + c(\theta)$. We chose $q(z,\theta)$ such that $c_w(\theta) = 0$ or equivalently

$$(4.19)\qquad a(\theta)q_z(1,\theta) + b(\theta)q_\theta(1,\theta) + c(\theta) = 0 \;\; \text{for any } \theta \in [\theta_1,\theta_2].$$

The relation (4.19) can be achieved as $\frac{a}{b} = \frac{2A}{B}$ is real, for $\frac{a(\theta)}{b(\theta)}$ smooth and $a(\theta), b(\theta)$ in $\mathbb{C}^*$, [22]. Thus by (4.18) and (4.19) the problem is of the form covered by Theorem 4.1, and consequently

$$w \in H^2(\mathcal{W}) \text{ and } \|w\|_{H^2(\mathcal{W})} \le c\|f_w\|_{L^2(\mathcal{W})}.$$

Obviously $u = \exp(q)w$; therefore, $u \in H^2(\mathcal{W})$ and $\|u\|_{H^2(\mathcal{W})} \le c\|f\|_{L^2(\mathcal{W})}$. □

Remark 4.4. By using Remark 4.2, under the assumptions of Theorem 4.3 and if we impose the non-homogeneous condition $a(\theta)u_z + b(\theta)u_\theta + c(\theta)u(\theta) = g$ at $z = 1$, for $g \in H^{\frac{1}{2}}(\partial\mathcal{W}_R)$ in place of the homogeneous one, estimate (4.16) follows (the proof is the same as in Theorem 4.3).

Remark 4.5. Theorem 4.1 and 4.3 or the results of Remarks 4.2, 4.4 can be applied to cylindrical coordinates for r fixed when $\mathcal{W} = \{(z,r,\theta) \in \mathbb{R}^3\}$, by use of the change of variables $u(z,\theta) = \hat{u}(z,\hat{\theta})$ with $\hat{\theta} := \frac{2\pi r\theta}{360} = c_0\theta$; then the equivalent problem in cartesian coordinates is defined in a rectangular domain and satisfies the assumptions of Theorems 4.1 and 4.3 or those of Remarks 4.2, 4.4.

5. Numerical experiments

In this section we report on the outcome of some numerical experiments performed with the fully discrete scheme (3.4) to solve the initial- and boundary-value problem (1.6). In the notation established in Section 1, cf. (1.6), we took $\mathfrak{D} = (0,1)^2$, $r_{\min} = 0$, $r_{\max} = 1$, $b = 0$, $\beta = 1$, D the identity matrix, $\lambda = (0,1)$ and right-hand side F so that the exact solution is

$$(5.1)\qquad u(r,y,\theta) = e^{2r}y(e^{-y} - 1)\theta(1-\theta)^3.$$

Our first set of experiments concerns the experimental verification of the convergence rate of the scheme in the spatial variable. The measure of the error was the $E(r) = \|u - U\|$ for $r = nk$, $n = 1,2,\ldots$, whereas for other values of r E was defined by linear interpolation. To determine experimentally the spatial order of convergence the approximate solution was computed for $0 \le r \le 1$ using a rectangular partition of $\mathfrak{D}$ using $N = h^{-1}$ ranging from 20 to 160. The finite element space S_h consisted of piecewise polynomial functions of degree one. For these runs, very small r-steps were taken to ensure that the error due to the discretization in time-like variable r is negligible. The observed error was recorded at $r = 0.1, 0.5$ and 1. As usual, the convergence rate corresponding to two different runs with mesh sizes h_1, h_2 and corresponding errors E_1 and E_2 is defined to be $\log(E_1/E_2)/\log(h_1/h_2)$. The results are shown in Table 1. It is evident that the convergence rate of the spatial component of the error is indeed two.

The determination of the accuracy in the time-like variable r is more delicate. We took $h^{-1} = 20$ and computed the solution of our problem up to $r = 1$ for various values of k. For this fixed value of h we made a reference calculation with a small value of $k = k_{\text{ref}} = h/30$. The corresponding approximate solution, denoted by $U_{h,\text{ref}}$ differs from the exact solution by a factor which is almost entirely due to the

Table 1. Errors $E(r)$ and spatial convergence rate for $k^{-1} = 400$

	$r = 0.1$		$r = 0.5$		$r = 1.0$	
h^{-1}	$E(r)$	Rate	$E(r)$	Rate	$E(r)$	Rate
10	3.5162(-2)		4.9653(-2)		7.8266(-2)	
20	7.5323(-3)	2.22	1.0734(-2)	2.21	1.6921(-2)	2.21
40	1.7219(-3)	2.13	2.4518(-3)	2.13	3.8920(-3)	2.12
80	4.0438(-4)	2.09	5.7998(-4)	2.08	9.2042(-4)	2.08
160	9.7655(-5)	2.05	1.4100(-4)	2.04	2.2381(-4)	2.04

spatial discretization. We then define a modified measure of the error $E^*(r)$ as above but with the exact solution replaced by the reference solution $U_{h,\mathrm{ref}}$. The results are shown in Table 2.

Table 2. Errors $E(r)$ and r-convergence rate for $h^{-1} = 20$

k^{-1}	$E(r)$	$E^*(r)$	Rate
144	3.8104(-1)	3.9217(-1)	
192	7.1839(-1)	1.7832(-1)	2.74
240	1.1771(-2)	1.0652(-1)	2.31
288	1.7638(-2)	7.1442(-2)	2.19
600	8.9952(-3)		

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Department of Applied Mathematics, University of Crete, 714 09 Heraklion, Greece, and Institute of Applied and Computational Mathematics, FORTH, Greece.
E-mail address: danton@tem.uoc.gr
E-mail address: gkarali@tem.uoc.gr
E-mail address: plex@tem.uoc.gr

Department of Mathematics, University of Crete, 714 09 Heraklion, Greece.
E-mail address: zouraris@math.uoc.gr