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A FINITE DIFFERENCE METHOD FOR THE WIDE-ANGLE 'PARABOLIC' EQUATION IN A WAVEGUIDE WITH DOWNSLOPING BOTTOM

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ABSTRACT. We consider the third-order wide-angle 'parabolic' equation of underwater acoustics in a cylindrically symmetric fluid medium over a bottom of range-dependent bathymetry. It is known that the initial-boundary-value problem for this equation may not be well posed in the case of (smooth) bottom profiles of arbitrary shape if it is just posed e.g. with a homogeneous Dirichlet bottom boundary condition. In this paper we concentrate on downsloping bottom profiles and propose an additional boundary condition that yields a well posed problem, in fact making it L^2-conservative in the case of appropriate real parameters. We solve the problem numerically by a Crank-Nicolson-type finite difference scheme, which is proved to be unconditionally stable and second-order accurate, and simulates accurately realistic underwater acoustic problems.

1. INTRODUCTION

We consider the third-order wide-angle 'parabolic' equation of underwater acoustics in a cylindrically symmetric fluid medium, [7, 9, 17, 3]

$$(1 + q\,\beta)\,v_r + \alpha^2\,q\,v_{zzr} = \mathrm{i}\,\lambda\,(\alpha^2\,v_{zz} + \beta\,v), \tag{WA}$$

posed for $(r, z) \in \mathcal{D} := \{(r, z) \in \mathbb{R}^2,\ 0 \le z \le s(r),\ 0 \le r \le R\}$ for a given $R > 0$ and a given bottom profile $s = s(r)$. Here $v = v(r, z)$ is a complex-valued function of the range r and the depth z, representing

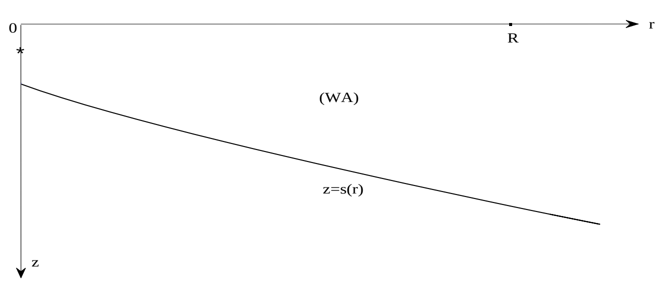

FIGURE 1. Domain of validity of (WA).

the acoustic field generated in the fluid medium ('water') $\mathcal{D}$ by a point time-harmonic source of frequency f placed on the z-axis, cf. Figure 1. In (WA) we have put $\alpha = \frac{1}{k_0}$, where $k_0 = \frac{2\pi \mathrm{f}}{c_0}$ is a reference wave number and c_0 a reference sound speed, and $\lambda = \frac{p-q}{\alpha}$, where p, q are complex constants. The function $\beta = \beta(r, z)$ is complex-valued and represents $n^2 - 1$, where n is the index of refraction of the medium.

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In practice β is real- or complex-valued with a small nonnegative imaginary part modeling attenuation in the water column. The coefficients p, q are such that the rational function $\frac{1+p\,x}{1+q\,x}$ is an approximation to $\sqrt{1+x}$ near $x=0$. The p.d.e. (WA) is obtained formally as a corresponding paraxial approximation of the outgoing pseudodifferential factor of the Helmholtz equation written in cylindrical coordinates in the presence of azimuthal symmetry, [7], [17]. The choice $p=\frac{1}{2}$, $q=0$ yields the standard, narrow-angle Parabolic Equation, [18], while the $(1,1)$-Padé approximant of $\sqrt{1+x}$, given by $p=\frac{3}{4}$, $q=\frac{1}{4}$, gives the Claerbout equation, [9]. In general, we shall take p and q complex; the choice $p=q+\frac{1}{2}$, $\mathrm{Im}(q)<0$, [10], has certain theoretical and numerical advantages as will be seen in the sequel. Let us also note that although in this paper we have in mind the application of the (WA) in underwater acoustics examples, our analytical and numerical methods can also be applied to wide-angle seismic wave, [9, 15], and aeroacoustic [8] wave propagation.

The p.d.e. (WA) is posed as an evolution equation with respect to the time-like variable r and the space-like variable z in the domain $\mathcal{D}$, that has a range-dependent bottom described by $z=s(r)$, where s is a smooth positive function on $[0,R]$. We shall supplement (WA) by an initial condition modelling the sound source at $r=0$, i.e. require that

$$v(0,z)=v_0(z),\quad 0\le z\le s(0), \tag{1.1}$$

where v_0 is a given complex-valued function defined in $[0,s(0)]$, and by the homogeneous Dirichlet boundary conditions

$$v(r,0)=0,\quad 0\le r\le R, \tag{1.2}$$

$$v(r,s(r))=0,\ \ 0\le r\le R, \tag{1.3}$$

corresponding to a pressure-release surface and an acoustically soft bottom, respectively.

There is considerable theoretical and numerical evidence to the effect that the initial-boundary-value problem (ibvp) consisting of (WA), and (1.1)-(1.3) is well posed if the bottom is horizontal or upsloping, i.e. when $\dot{s}(r)\le 0$ in $[0,R]$, and that it may be ill posed if the bottom is downsloping, i.e. if $\dot{s}(r)>0$ for $r\in[0,R]$, [14], [11], [12], [5]. In Refs. [11] and [12] an additional bottom boundary condition was proposed, that together with (WA) and (1.1)-(1.3) yields, under certain hypotheses, a well posed problem for any smooth profile s.

In [5] the authors of the paper at hand, in collaboration with F. Sturm, presented other types of additional bottom boundary conditions that render the resulting ibvp well posed and in addition, for real β and q, L^2-conservative, in the sense that

$$\int_0^{s(r)}|v(r,z)|^2\,dz=\int_0^{s(0)}|v_0(z)|^2\,dz \tag{1.4}$$

holds for $0\le r\le R$. Specifically, it was observed that the ibvp consisting of (WA) and (1.1)-(1.3) is L^2-conservative, for real β and q, if and only if the following boundary value condition holds

$$\mathrm{Im}\left[\,\mathcal{F}(v_z;r,s(r))\,\overline{\mathcal{F}(v;r,s(r))}\,\right]=0,\quad 0\le r\le R, \tag{1.5}$$

where for $(r,z)\in\mathcal{D}$

$$\mathcal{F}(v;r,z):=q\,v_r(r,z)-\mathrm{i}\,\lambda\,v(r,z), \tag{1.6}$$

provided that v satisfies (WA) and (1.1)-(1.3). Our main motivation for studying boundary conditions for which the 'energy' integral $\int_0^{s(r)}|v(r,z)|^2\,dz$ is conserved (for real β and q) in the case of the noncylindricral domain $\mathcal{D}$ is the fact that this happens for solutions of (WA), (1.1)-(1.3) in the case of a horizontal bottom and also for the standard PE ($p=\frac{1}{2}$, $q=0$) posed on $\mathcal{D}$ as an ibvp with (1.1)-(1.3), for a general profile s and real β.

Here, we consider the ibvp (WA), (1.1)-(1.3) for a downsloping bottom, with the additional boundary condition

$$v_r(r,s(r))=0,\quad 0\le r\le R, \tag{1.7}$$

which is equivalent to the condition $v_z(r, s(r)) = 0$, $r \in [0, R]$, (in the case of a differentiable bottom with $\dot{s}(r) \neq 0$), as it seen by differentiating both sides of (1.3) with respect to r. Obviously, (1.7), in the presence of (1.3), is also equivalent to $\mathcal{F}(v; r, s(r)) = 0$, i.e. satisfies (1.5) and yields an L^2-conservative problem for β, q real. In section 2 of the present paper, we prove that the resulting ibvp consisting of (WA), (1.1)-(1.3), (1.7) is stable in L^2, H^1 and H^2; also we show an H^2-stability result for the solution of the ibvp (WA), (1.1)-(1.3) with an upsloping bottom.

In order to develop a numerical method for ibvp (WA), (1.1)-(1.3), (1.7) when the bottom is downsloping and $q \neq 0$, we transform it, using the range-dependent change of the depth variable $y = \frac{z}{s(r)}$ that renders the bottom horizontal, into an equivalent problem on the strip $0 \leq y \leq 1$, $0 \leq r \leq R$. Letting $\Omega := [0, R] \times [0, 1]$, $I := [0, 1]$ and $u(r, y) := v(r, y\, s(r))$ for $(r, y) \in \Omega$, the ibvp consisting of (WA), (1.1)-(1.3), (1.7) takes the following form

$$\begin{aligned} &\Lambda(r) \left(u_r - \mathrm{i}\, \tfrac{\lambda}{q}\, u - \tfrac{\dot{s}(r)}{s(r)}\, y\, u_y \right) = \mathrm{i}\, \tfrac{\lambda\, s^2(r)}{\alpha^2\, q^2}\, u \quad \forall\, (r, y) \in \Omega, \\ &u(r, 0) = 0 \quad \forall\, r \in [0, R], \\ &u(r, 1) = u_y(r, 1) = 0 \quad \forall\, r \in [0, R], \\ &u(0, y) = u_0(y) := v_0(y\, s(0)) \quad \forall\, y \in I, \end{aligned} \tag{1.8}$$

where $\gamma(r, y) := b(r, ys(r))$ for $(r, y) \in \Omega$ and, for $r \in [0, R]$, $\Lambda(r) : H^2(I) \cap H^1_0(I) \to L^2(I)$ is an indefinite one-dimensional elliptic differential operator in the $y-$variable defined by

$$\Lambda(r)v := -v'' - \tfrac{(1 + q\, \gamma(r, \cdot))\, s^2(r)}{\alpha^2\, q}\, v, \quad \forall\, v \in H^2(I) \cap H^1_0(I), \tag{1.9}$$

keeping in mind that the term $u_r - \mathrm{i}\, \frac{\lambda}{q}\, u - \frac{\dot{s}(r)}{s(r)}\, y\, u_y$ vanish at the endpoint of I. In section 3 we provide some conditions on the data of the problem that ensure invertibility of $\Lambda(r)$ for $r \in [0, R]$ along with some regularity properties. We note that the p.d.e. in (1.8), which follows from (WA) after the aforementioned change of variable, is not a usual Sobolev-type equation (cf. [16]) like (WA) over a horizontal bottom (cf. [3], [1]). Due to the presence of the term $\frac{\dot{s}(t)}{s(t)}\, y\, u_y$, the differential equation is of third order with respect to the space variable y and this offers an explanation of why an additional boundary condition may be needed.

In section 4, for the approximation to the problem (1.8) we propose and analyze a numerical method that combines Crank-Nicolson time-stepping with a standard second-order finite difference method in space. We would like to stress that the convergence analysis of the proposed finite difference scheme is not a repetition of the corresponding analysis for the flat bottom case (cf. [1], [3]). This is due to the fact that the differential operator is third order with respect to y, which leads to a truncation error of $O(1)$ at the nodes adjacent to the endpoints of I. Building up a careful consistency argument is important in proving a second-order error estimate.

Finally, in section 5, we verify the accuracy and stability of the finite difference scheme by means of numerical experiments and apply it to solve an underwater acoustics problem in a downsloping benchmark domain comparing the results with those obtained from the model proposed in [12].

For rigorous error estimates on other finite difference and finite element methods for the PE and WA equation on domains with horizontal or variable bottom we refer the reader e.g. to the papers [1], [3], [4], [6], [2], and their references.

2. A priori estimates

Our aim in this section is to establish some *a priori* estimates for the solution v of the wide-angle equation (WA) on $\mathcal{D}$ under some or all of the auxiliary conditions considered in the previous section, and under several hypotheses on the coefficients and the bottom. In what follows we shall assume that the functions v, β, s, and v_0 are sufficiently smooth so that the various estimates are valid. Also, in our analysis we shall employ L^2, H^1 and H^2 range-dependent norms on $[0, s(r)]$ which are given, respectively, by $\|v\| := (\int_0^{s(r)} |v(r, z)|^2 dz)^{\frac{1}{2}}$, $\|v\|_1 := (\|v\|^2 + \|v_z\|^2)^{\frac{1}{2}}$ and $\|v\|_2 := (\|v\|_1^2 + \|v_{zz}\|^2)^{\frac{1}{2}}$.

We begin by establishing some basic identities.

Lemma 2.1. *Let $q \neq 0$ and $\mathcal{F}$ be defined by* (1.6)*. If v satisfies* (WA), (1.1) *and* (1.2)*, then, for $0 \le r \le R$, the following identities hold:*

$$
\begin{aligned}
\tfrac{d}{dr}\|v(r,\cdot)\|^2 &= \dot{s}(r)|v(r,s(r))|^2 + \tfrac{2}{\lambda}\operatorname{Im}(q)\,\|v_r(r,\cdot)\|^2 \\
&\quad - \tfrac{2}{\lambda}\int_0^{s(r)} \operatorname{Im}\left[\beta(r,z)\right]\,|\mathcal{F}(v;r,z)|^2\,dz - \tfrac{2\,\alpha^2}{\lambda}\operatorname{Im}\Big[\mathcal{F}(v_z;r,s(r))\,\overline{\mathcal{F}(v;r,s(r))}\Big],
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
\tfrac{d}{dr}\|v_z(r,\cdot)\|^2 &= \dot{s}(r)\,|v_z(r,s(r))|^2 + \tfrac{2}{\alpha^2}\operatorname{Re}\Big[\int_0^{s(r)} \tfrac{1+q\,\beta(r,z)}{q}\,v_r(r,z)\overline{v}(r,z)\,dz\Big] - 2\,\lambda\operatorname{Im}(\tfrac{1}{q})\|v_z(r,\cdot)\|^2 \\
&\quad + 2\operatorname{Re}\Big[\tfrac{1}{q}\,\mathcal{F}(v_z;r,s(r))\,\overline{v(r,s(r))}\Big] + \tfrac{2}{\alpha^2}\,\lambda\int_0^{s(r)} \operatorname{Im}\Big(\tfrac{\beta(r,z)}{q}\Big)\,|v(r,z)|^2\,dz,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
\frac{d}{dr}\|v_{zz}(r,\cdot)\|^2 =&\dot{s}(r)|v_{zz}(r,s(r))|^2 + \tfrac{2}{\alpha^2}\operatorname{Re}\Big[\frac{1}{q}\int_0^{s(r)} v_r(r,\cdot)\,\overline{v_{zz}}(r,z)\,dz\Big] \\
&- 2\,\lambda\operatorname{Im}\Big(\tfrac{1}{q}\Big)\,\|v_{zz}\|^2 - \tfrac{2}{\alpha^2}\operatorname{Re}\Big[\tfrac{1}{q}\int_0^{s(r)} \beta(r,z)\,\mathcal{F}(v;r,z)\,\overline{v_{zz}}(r,z)\,dz\Big],
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
\int_0^{s(r)} (1+q\,\beta(r,z))\,|v_r(r,z)|^2\,dz &= q\,\alpha^2\,\|v_{rz}(r,\cdot)\|^2 \\
&\quad + \mathrm{i}\,\alpha^2\,\lambda\int_0^{s(r)} \Big[\tfrac{\beta(r,z)}{\alpha^2}\,v(r,z) + v_{zz}(r,z)\Big]\,\overline{v_r(r,z)}\,dz \\
&\quad - q\,\alpha^2\,v_{rz}(r,s(r))\,\overline{v_r(r,s(r))}.
\end{aligned}
\tag{2.4}
$$

Proof. We multiply equation (WA) using (1.6) by $\overline{\mathcal{F}(v;r,\cdot)}$, integrate by parts with respect to z and take imaginary parts to get (2.1). We then multiply (WA) using (1.6) by $\overline{v(r,\cdot)}$, $\overline{v_{zz}(r,\cdot)}$, respectively, integrate by parts, and take real parts, obtaining (2.2), (2.3), respectively. The last equality (2.4) follows, by multiplying (WA) using (1.6) by $\overline{v_r(r,\cdot)}$ and integrating. ☐

From (2.1) it follows that if v satisfies (WA), (1.1), (1.2), and (1.3), and β and q are real, then v satisfies the L^2 conservation property (1.4) if and only if (1.5) holds.

Before embarking on our study of the downsloping bottom problem, we prove a H^2-stability result in the upsloping bottom case assuming only the homogeneous Dirichlet boundary condition (1.3), thus complementing the H^1-estimate of [12]. The proof requires that $\frac{1}{\alpha}\max\limits_{0\le r\le R} s(r)$ be sufficiently small, i.e. a 'small frequency-shallow water' assumption. In what follows, c or C will denote generic positive constants, not necessarily having the same values in any two places.

Proposition 2.1. *Suppose that $\dot{s}(r) \le 0$ for $r \in [0,R]$, $q \neq 0$, and $\frac{1}{\alpha}\max\limits_{0\le r\le R} s(r)$ is sufficiently small. Then, there exists a constant c such that any solution v of the ibvp* (WA), (1.1)-(1.3) *satisfies*

$$\|v(r,\cdot)\|_2 \le c\|v_0\|_2, \quad 0 \le r \le R. \tag{2.5}$$

Proof. We first prove that for $r \in [0,R]$ we have

$$\|v_r(r,\cdot)\| + \|\mathcal{F}(v;r,\cdot)\| \le\; c\,\|v(r,\cdot)\|_2. \tag{2.6}$$

To see this, note that (2.4) gives for $0 \le r \le R$

$$\|v_{rz}(r,\cdot)\|^2 \le\; \tfrac{c}{\alpha^2}\,\|v_r(r,\cdot)\|^2 + c\,\|v_r(r,\cdot)\|\,\|v(r,\cdot)\|_2 + |v_{rz}(r,s(r))|\,|v_r(r,s(r))|. \tag{2.7}$$

Since $v_r(r,0) = 0$, we have

$$\|v_r(r,\cdot)\| \le\; s(r)\,\|v_{rz}(r,\cdot)\|, \quad 0 \le r \le R. \tag{2.8}$$

By differentiating with respect to r the Dirichlet boundary condition $v(r,s(r)) = 0$, we obtain by the trace inequality

$$|v_r(r,s(r))| = |\dot{s}|\;|v_z(r,s(r))| \le\; c\,\|v(r,\cdot)\|_2.$$

Therefore, (2.7) gives for $0 \le r \le R$

$$\|v_{rz}(r,\cdot)\|^2 \le \tfrac{c}{\alpha^2}\,\|v_r(r,\cdot)\|^2 + c\,\|v_{rz}(r,\cdot)\|\,\|v(r,\cdot)\|_2 + c\,|v_{rz}(r,s(r))|\,\|v(r,\cdot)\|_2. \tag{2.9}$$

The equation (WA) solved for v_{rzz} and (2.8) yields now

$$\begin{aligned}\|v_{rzz}(r,\cdot)\| &\le c\,(\,\|v_r(r,\cdot)\| + \|v(r,\cdot)\|_2\,)\\ &\le c\,(\,\|v_{rz}(r,\cdot)\| + \|v(r,\cdot)\|_2\,).\end{aligned}$$

Thus by Sobolev's inequality we obtain

$$\begin{aligned}|v_{rz}(r,s(r))| &\le c\,(\,\|v_{rz}(r,\cdot)\| + \|v_{rzz}(r,\cdot)\|\,)\\ &\le c\,(\,\|v_{rz}(r,\cdot)\| + \|v(r,\cdot)\|_2\,).\end{aligned}$$

Using this in (2.9) and using (2.8) we obtain, for any $\varepsilon > 0$ and $0 \le r \le R$

$$\|v_{rz}(r,\cdot)\|^2 \le \tfrac{c}{\alpha^2}\,s^2(r)\,\|v_{rz}(r,\cdot)\|^2 + \varepsilon\,\|v_{rz}(r,\cdot)\|^2 + c_\varepsilon\,\|v(r,\cdot)\|_2^2.$$

Hence, if $\frac{1}{\alpha}\max\limits_{0\le r\le R} s(r)$ is sufficiently small, we see, using again (2.8), that $\|v_r(r,\cdot)\| \le c\,\|v(r,\cdot)\|_2$ for $0 \le r \le R$. Hence, (2.6) follows, since

$$\begin{aligned}\|\mathcal{F}(v;r,\cdot)\| &\le c\,(\,\|v_r(r,\cdot)\| + \|v(r,\cdot)\|\,)\\ &\le c\,\|v(r,\cdot)\|_2.\end{aligned}$$

Adding now (2.2) and (2.3) and using the fact that $\dot{s}(r) \le 0$, estimates from the proof of (2.6), and the Poincaré-Friedrichs inequality we obtain for $0 \le r \le R$ that

$$\begin{aligned}\frac{d}{dr}\,(\,\|v_z(r,\cdot)\|^2 + \|v_{zz}(r,\cdot)\|^2\,) &\le c\,\big(\,\|v_r(r,\cdot)\|^2 + \|v_z(r,\cdot)\|^2 + \|v_{zz}(r,\cdot)\|^2 + \|\mathcal{F}(v;r,\cdot)\|^2 + \|v(r,\cdot)\|^2\,\big)\\ &\le c\,\big(\,\|v_z(r,\cdot)\|^2 + \|v_{zz}(r,\cdot)\|^2\,\big).\end{aligned}$$

Hence, from Grönwall's inequality we obtain

$$\|v_z(r,\cdot)\|^2 + \|v_{zz}(r,\cdot)\|^2 \le c\Big(\|v_z(0,\cdot)\|^2 + \|v_{zz}(0,\cdot)\|^2\Big),$$

from which our conclusion follows. ☐

The H^2-stability estimate (2.5) implies of course the uniqueness of solution of the ibvp (WA), (1.1)-(1.3) under the hypotheses of Proposition 2.1, and forms the basis for a well-posedness theory for this problem.

We turn now to the downsloping bottom case, with which we shall be concerned for the sequel of this paper. We first establish the following basic *a priori* estimates.

Theorem 2.2. *Supose that* $\dot{s}(r) > 0$ *for* $r \in [0,R]$, $q \ne 0$, *and that either* $\mathrm{Im}(q^{-1} + \beta(r,z))$ *is of one sign in* $\mathcal{D}$ *or that* $\frac{1}{\alpha}\max\limits_{0\le r\le R} s(r)$ *is sufficiently small. Then the ibvp (WA),* (1.1)-(1.3), (1.7) *is* L^2-, H^1-, *and* H^2-*stable.*

Proof. Let v be the solution of the ibvp (WA), (1.1)-(1.3), (1.7). Then it follows from (1.6) that $f(r,z) := \mathcal{F}(v;r,z)$ satisfies

$$\begin{aligned}&\Big(\tfrac{1}{q} + \beta\Big)\;f + \alpha^2\,f_{zz} = -\mathrm{i}\,\tfrac{\lambda}{q}\,v, \quad (r,z) \in \mathcal{D},\\ &f(r,0) = f(r,s(r)) = 0, \quad 0 \le r \le R.\end{aligned} \tag{2.10}$$

For each $r \in [0,R]$ consider the operator $\mathcal{L}(r) : H^2 \cap H_0^1 \to L^2$ defined by

$$\mathcal{L}(r)u := \Big(\tfrac{1}{q} + \beta(r,\cdot)\Big)\;u + \alpha^2\,\partial_z^2 u, \quad \forall\, u \in H^2 \cap H_0^1.$$

Then, under our hypotheses, $\mathcal{L}(r)$ is invertible in $H^2 \cap H_0^1$, in the sense that $\mathcal{L}(r)u = 0$ implies $u = 0$. To see this, note that from $\mathcal{L}u = 0$ for $u \in H^2 \cap H_0^1$, it follows that $\int_0^{s(r)} \mathcal{L}u\,\overline{u}dz = 0$, and by integration by parts, that

$$\int_0^{s(r)} \Big[(q^{-1} + \beta)|u|^2 - \alpha^2|u_z|^2\Big]dz = 0. \tag{2.11}$$

Taking real parts in (2.11) and using the fact that $\|u\| \le s(r)\|u_z\|$, it follows that

$$\|u_z(r,\cdot)\|^2 \left(1 - \tfrac{1}{\alpha^2} \max_{0\le r\le R} s^2(r) \max_{z\in\mathcal{D}} |\mathrm{Re}(q^{-1}+\beta)|\right) \le 0.$$

Hence, if $\frac{1}{\alpha}\max_{0\le r\le R} s(r)$ is sufficiently small we have that $u=0$. Alternatively, taking imaginary parts in (2.11) we obtain that $\int_0^{s(r)} \mathrm{Im}(q^{-1}+\beta)|u|^2 dz = 0$, from which if $\mathrm{Im}(q^{-1}+\beta)$ is nonzero and of one sign in $\mathcal{D}$, then $u=0$.

We conclude, from standard elliptic p.d.e. theory and the Fredholm alternative, [13], that given $w \in L^2$, then $\mathcal{L}^{-1}w \in H^2 \cap H_0^1$ and $\|\mathcal{L}^{-1}w\|_2 \le C\|w\|$, for some positive $C = C(r)$, independent of w. Since the coefficients of $\mathcal{L}$ are smooth, C may be taken as a continuous function on $[0,R]$. Applying this result to the bvp (2.10) we see that for $0 \le r \le R$,

$$\|f(r,\cdot)\|_2 \le C\,\|v(r,\cdot)\|. \tag{2.12}$$

By the definition of f, it follows that on $[0,R]$

$$\|v_r(r,\cdot)\|_\ell \le C\,\|v(r,\cdot)\|_\ell, \quad \ell = 0,1,2. \tag{2.13}$$

Hence, by (2.1), L^2-stability of the ibvp under consideration follows. To get the H^1-estimate, we use (2.2), (2.13) and the fact that $v_z(z,s(r)) = 0$ for $0 \le r \le R$, as remarked in the Introduction. It follows that for $0 \le r \le R$

$$\tfrac{d}{dr}\|v_z(r,\cdot)\|^2 \le C\,\left(\|v(r,\cdot)\|^2 + \|v_z(r,\cdot)\|^2\right),$$

from which H^1-stability follows by the Poincaré-Friedrichs inequality. Finally, to get H^2-stability, note that (2.3), (2.12), (2.13) yield for $0 \le r \le R$

$$\tfrac{d}{dr}\|v_{zz}\|^2 \le \dot{s}(r)|v_{zz}(r,s(r))|^2 + C\,\left(\|v(r,\cdot)\|^2 + \|v_{zz}(r,\cdot)\|^2\right). \tag{2.14}$$

From $v_z(r,s(r)) = 0$ for $0 \le r \le R$, it follows that $v_{zr}(r,s(r)) + \dot{s}(r)v_{zz}(r,s(r)) = 0$. Since, $v_{zr}(r,s(r)) = \frac{1}{q}\,f(r,s(r))$, it follows by the trace inequality and (2.12) that for $0 \le r \le R$

$$\begin{aligned}\dot{s}(r)\,|v_{zz}(r,s(r))|^2 &= \tfrac{1}{|\dot{s}(r)|}\,|v_{zr}(r,s(r))|^2\\ &\le C\,\|f(r,\cdot)\|_2^2\\ &\le C\,\|v(r,\cdot)\|^2.\end{aligned}$$

From (2.14) the H^2-stability estimate follows now. □

Remark 2.1. *If $\mathrm{Im}(\beta) \ge 0$ and $\mathrm{Im}(q) < 0$ or if $\mathrm{Im}(\beta) > 0$ and q is a real, nonzero constant, then $\mathrm{Im}\left(\frac{1}{q}+\beta\right) > 0$ follows. The ibvp* (WA), *(1.1)-(1.3), (1.7) is of course L^2-conservative if q and β are real.*

3. Invertibility conditions for Λ

Assuming that the operator $\Lambda(r)$ defined by (1.9) is invertible for all $r \in [0,R]$ and that $u_0 \in H_0^1(I)$, we may write the problem (1.8) equivalently as:

$$\begin{aligned}&u_r - \tfrac{\dot{s}(r)}{s(r)}\,y\,u_y = \mathrm{i}\,\tfrac{\lambda\,s^2(r)}{\alpha^2\,q^2}\;T(r)u + \mathrm{i}\,\tfrac{\lambda}{q}\,u \quad \forall\,(r,y)\in\Omega,\\ &u(r,1) = 0 \quad \forall\,r\in[0,R],\\ &u(0,y) = u_0(y) \quad \forall\,y\in I,\end{aligned} \tag{3.1}$$

where $T(r) := \Lambda^{-1}(r)$ for $r \in [0,R]$. To simplify the notation we set

$$\delta(r) := \tfrac{\dot{s}(r)}{s(r)}, \quad \zeta(r,y) := \tfrac{(1+q\,\gamma(r,y))\,s^2(r)}{\alpha^2\,q}, \quad \xi(r) := \tfrac{\lambda\,s^2(r)}{\alpha^2\,q^2}$$

for $r \in [0,R]$ and $y \in I$.

Remark 3.1. *Let $g(r,y) := u_r(r,y) - \delta(r)\,y\,u_y(r,y)$. Assuming that the solution of* (3.1) *is smooth on Ω, we obtain the compatibility conditions $g(r,1) = g(r,0) = 0$, which yield that $u_r(r,0) = 0$ and $u_y(r,1) = 0$ for $r \in [0,R]$. Finally, since $u(0,0) = u_0(0) = 0$, we get the surface pressure release condition $u(r,0) = 0$ for $r \in [0,R]$.*

Remark 3.2. *Whereas the ibvp (WA), (1.1)-(1.3), (1.7) in the r, z variables is L^2-conservative, in the sense that its solution satisfies (1.4) for q, β real, the solution of the transformed ibvp (1.8) in the r, y variables conserves the quantity $s(r)\|u(r,\cdot)\|_{0,I}^2$ for q, γ real.*

In the rest of this section we present some conditions on the data that ensure invertibility of the operator Λ. The conditions are similar in nature to those in the hypothesis of Theorem 2.2, but it is useful to present them here because they are expressed in terms of the y variable and motivate analogous sufficient conditions for the invertibility of the discrete operators in the next section. We use the notation $\|v\|_{j,I}$, $j \geq 0$, for the norm in the Sobolev space $H^j(I)$ and put $|v|_{1,I} = \|v_y\|_{0,I}$.

Lemma 3.1. *Assume that*

$$C_{EB} := \inf_{\Omega} \left[\tfrac{1}{(C_{PF})^2} - \tfrac{s^2}{\alpha^2\,|q|^2}\, \left(\mathrm{Re}(q) + |q|^2\,\mathrm{Re}(\gamma)\right)\right] > 0, \tag{3.2}$$

where $C_{PF} > 0$ is the constant of the Poincaré-Friedrichs inequality on I, i.e., $\|v\|_{0,I} \leq C_{PF}\,|v|_{1,I}$ for $v \in H_0^1(I)$, or that there exists $\delta_\star \in \{1, -1\}$ such that

$$C_{BB} := \inf_{\Omega} \left[\tfrac{\delta_\star\, s^2}{\alpha^2\,|q|^2}\, \left(\mathrm{Im}(q) - |q|^2\,\mathrm{Im}(\gamma)\right)\right] > 0. \tag{3.3}$$

Then, there exists a constant $C > 0$ such that

$$\max_{r\in[0,R]} \|T(r)\psi\|_{1,I} \leq C\,\|\psi\|_{0,I}, \quad \forall\,\psi \in L^2(I). \tag{3.4}$$

Proof. Let $r \in [0, R]$, $v \in H^2(I) \cap H_0^1(I)$ and $\psi \in L^2(I)$. First, observe that

$$\mathrm{Re}(\Lambda(r)v, v)_{0,I} = \|v'\|_{0,I}^2 - \int_I \tfrac{\mathrm{Re}(\overline{q}+|q|^2\,\gamma)\,s^2(r)}{\alpha^2\,|q|^2}\,|v|^2\,dy. \tag{3.5}$$

When (3.2) holds, use of the Poincaré-Friedrichs inequality and (3.5) gives

$$\begin{aligned} \mathrm{Re}(\Lambda(r)v, v)_{0,I} &\geq \int_I \left[\tfrac{1}{(C_{PF})^2} - \tfrac{\mathrm{Re}(q+|q|^2\,\gamma)\,s^2(r)}{\alpha^2\,|q|^2}\right]|v|^2\,dy \\ &\geq C_{EB}\,\|v\|_{0,I}^2. \end{aligned} \tag{3.6}$$

When (3.3) holds, we have

$$\begin{aligned} \left|\,\mathrm{Im}(\Lambda(r)v, v)_{0,I}\,\right| &\geq \delta_\star\,\mathrm{Im}(\Lambda(r)v, v)_{0,I} \\ &\geq \delta_\star \int_I \tfrac{\mathrm{Im}(q+|q|^2\,\overline{\gamma})\,s^2(r)}{\alpha^2\,|q|^2}\,|v|^2\,dx \\ &\geq C_{BB}\,\|v\|_{0,I}^2. \end{aligned} \tag{3.7}$$

Since, by definition, $T(r)\psi = \Lambda^{-1}(r)\psi \in H^2(I) \cap H_0^1(I)$, we set $v = T(r)\psi$ in (3.6) or (3.7) and apply the Cauchy-Schwarz inequality to conclude for $r \in [0, R]$

$$\|T(r)\psi\|_{0,I} \leq C\,\|\psi\|_{0,I}. \tag{3.8}$$

Then, we combine (3.5) for $v = T(r)\psi$, and (3.8) to get

$$\begin{aligned} |T(r)\psi|_{1,I}^2 &= \mathrm{Re}(\psi, T(r)\psi)_{0,I} + \int_I \tfrac{\mathrm{Re}(\overline{q}+|q|^2\,\gamma)\,s^2(r)}{\alpha^2\,|q|^2}\,|T(r)\psi|^2\,dy \\ &\leq \|\psi\|_{0,I}\,\|T(r)\psi\|_{0,I} + C\,\|T(r)\psi\|_{0,I}^2 \\ &\leq C\,\|\psi\|_{0,I}^2. \end{aligned} \tag{3.9}$$

Now (3.4) follows easily from (3.9) and (3.8). ☐

Assuming that (3.2) or (3.3) holds and using a induction argument, it is easy to establish that for $m \in \mathbb{N}_0$, there exists $C_m > 0$ such that

$$\max_{r\in[0,R]} |T(r)f|_{m+2,\infty,I} \leq C_m\,|f|_{m,\infty,I}, \quad \forall\,f \in C^m(I;\mathbb{C}), \tag{3.10}$$

where on C^m, $|f|_{m,\infty,I} = \max_{y\in I} |f^{(m)}(y)|$. In addition, for $m \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$, there exists $C_{\ell,m} > 0$ such that

$$\max_{r\in[0,R]} |\partial_r^\ell (T(r)f)|_{m+2,\infty,I} \leq C_{\ell,m}\, |f|_{\max\{m-2,0\},\infty,I}, \quad \forall\, f \in C^{\max\{m-2,0\}}(I;\mathbb{C}). \tag{3.11}$$

Remark 3.3. *Differentiating both sides of the p.d.e. in* (3.1) *once with respect to* y*, taking* $y = 1$ *and using the boundary conditions we get*

$$u_{yy}(r,1) = -\mathrm{i}\, \tfrac{\lambda\, s^3(r)}{\alpha^2\, q^2\, \dot{s}(r)}\, (T(r)u)_y(r,1), \quad \forall\, r \in [0,R].$$

This reminds us in some sense of the boundary condition proposed in [12] *for downsloping bottoms, which has the form*

$$u_{yy}(r,1) = \mathrm{i}\, \tfrac{2}{\alpha}\, s(r)\, \dot{s}(r)\, u_y(r,1), \quad \forall\, r \in [0,R].$$

Remark 3.4. *Condition* (3.2) *follows from the hypothesis that* $\frac{s}{\alpha}$ *is sufficiently small for* $0 \leq r \leq R$*. If* $\mathrm{Im}(\gamma) \geq 0$ *in* Ω *and* $\mathrm{Im}(q) < 0$ *or* $\mathrm{Im}(\gamma) > 0$ *in* Ω *and* q *is real and nonzero, condition* (3.3) *is valid for* $\delta_* = -1$*.*

4. A Finite difference method

In this section we construct and analyze a finite difference method for approximating the solution of the ibvp (3.1).

4.1. **Notation and preliminaries.** Let $J \in \mathbb{N}$ with $J \geq 3$. We introduce a partition of $[0,1]$ with width $h := \frac{1}{J}$ and nodes $y_j := j\,h$ for $j = 0,\ldots,J$. Taking into account the homogeneous Dirichlet boundary conditions at the endpoints of I we define the space X_h of the finite difference approximations by

$$X_h := \{(v_j)_{j=0}^{J} \in \mathbb{C}^{J+1} : \ v_0 = v_J = 0\}.$$

On X_h we define a discrete $L^2(I)$ norm $\|\cdot\|_{0,h}$ given by $\|v\|_{0,h} := \Big(h\sum_{j=1}^{J-1} |v_j|^2\Big)^{\frac{1}{2}}$ for $v \in X_h$, which is produced by the inner product $(\cdot,\cdot)_{0,h}$ defined by $(v,w)_{0,h} := h\sum_{j=1}^{J-1} v_j\, \overline{w_j}$ for v, $w \in X_h$. Also, on X_h we define a discrete $H^1(I)$ seminorm $|\cdot|_{1,h}$ by $|v|_{1,h} := \Big(h\sum_{j=0}^{J-1} \Big|\frac{v_{j+1}-v_j}{h}\Big|^2\Big)^{\frac{1}{2}}$ for $v \in X_h$, a discrete $H^1(I)$ norm $\|\cdot\|_{1,h}$ by $\|v\|_{1,h} := \Big[\|v\|_{0,h}^2 + |v|_{1,h}^2\Big]^{\frac{1}{2}}$ for $v \in X_h$, and a discrete $L^\infty(I)$ norm $|\cdot|_{\infty,h}$ by $|v|_{\infty,h} := \max_{1\leq j\leq J-1} |v_j|$ for $v \in X_h$.

We define further the second-order difference operator $\Delta_h : X_h \to X_h$ by

$$(\Delta_h v)_j = \tfrac{v_{j-1} - 2\,v_j + v_{j+1}}{h^2}, \quad j = 1,\ldots,J-1, \quad \forall\, v \in X_h,$$

and for $r \in [0,R]$, the discrete elliptic operator $\Lambda_h(r) : X_h \to X_h$ by

$$(\Lambda_h(r)v)_j := -(\Delta_h v)_j - s^2(r)\, \tfrac{1+q\,\gamma(r,y_j)}{\alpha^2\, q}\, v_j, \quad j = 1,\ldots,J-1, \quad \forall\, v \in X_h.$$

Also, we define the first-order difference operator $\partial_h : X_h \to X_h$ by

$$(\partial_h v)_j := \tfrac{v_{j+1} - v_{j-1}}{2h}, \quad j = 1,\ldots,J-1, \quad \forall\, v \in X_h,$$

and introduce the auxiliary operators $I_h : X_h \to X_h$ given by $(I_h v)_j := \frac{v_{j+1}+v_{j-1}}{2}$ for $j = 1,\ldots,J-1$ and $v \in X_h$, and $\otimes : X_h^2 \to X_h$ defined by $(v \otimes w)_j = v_j\, w_j$ for $j = 1,\ldots,J-1$ and v, $w \in X_h$. Also, we let $\omega \in X_h$ be such that $\omega_j := y_j$ for $j = 0,\ldots,J-1$. For $f : [0,1] \to \mathbb{C}$ we define $P_h f \in X_h$ by $(P_h f)_j := f(y_j)$ for $j = 1,\ldots,J-1$.

It is easy to show that

$$\|v\|_{0,h} \leq \tfrac{\sqrt{2}}{2}\, |v|_{1,h} \quad \forall\, v \in X_h, \tag{4.1}$$

$$|v|_{\infty,h} \leq |v|_{1,h} \quad \forall\, v \in X_h, \tag{4.2}$$

$$|v|_{\infty,h} \leq h^{-\frac{1}{2}}\, \|v\|_{0,h} \quad \forall\, v \in X_h, \tag{4.3}$$

and

$$(\Delta_h v, v)_{0,h} = -|v|^2_{1,h} \quad \forall\, v \in X_h. \tag{4.4}$$

Let $N \in \mathbb{N}$. We define a partition of the range interval $[0,R]$ with nodes $(r^n)_{n=0}^N$ given by $r^n := n\,k$ for $n = 0,\ldots,N$, and let $(u^n)_{n=0}^N \subset X_h$ be such that $(u^n)_j := u(r^n, y_j)$ for $j = 1,\ldots,J-1$ and $n = 0,\ldots,N$, where u is the solution of (1.8). Also, we set $r^{n+\frac{1}{2}} := \frac{r^{n+1}+r^n}{2}$ for $n = 0,\ldots,N-1$.

Lemma 4.1. *For $v \in X_h$ we have*

$$\mathrm{Re}(\omega \otimes \partial_h v, v)_{0,h} = -\tfrac{1}{2}\,(v, I_h v)_{0,h}. \tag{4.5}$$

Proof. For $v \in X_h$, we have

$$\begin{aligned}
(\omega \otimes \partial_h v, v)_{0,h} &= \tfrac{1}{2}\,\sum_{j=1}^{J-1} y_{j-1}\,\overline{v_{j-1}}\,v_j - \tfrac{1}{2}\,\sum_{j=1}^{J-1} y_{j+1}\,\overline{v_{j+1}}\,v_j\\
&= \tfrac{1}{2}\,\sum_{j=1}^{J-1}(y_{j-1}-y_j)\,v_j\,\overline{v_{j-1}} - \tfrac{1}{2}\,\sum_{j=1}^{J-1} y_j\,v_j\,\overline{(v_{j+1}-v_{j-1})} - \tfrac{1}{2}\,\sum_{j=1}^{J-1}(y_{j+1}-y_j)\,v_j\,\overline{v_{j+1}}\\
&= -\,\overline{(\omega \otimes \partial_h v, v)_{0,h}} - h\,\sum_{j=1}^{J-1} v_j\,\overline{\tfrac{v_{j+1}+v_{j-1}}{2}},
\end{aligned}$$

which easily yields (4.5). ∎

Lemma 4.2. *For $v \in X_h$ we have*

$$\|I_h v\|_{0,h} \le \|v\|_{0,h}. \tag{4.6}$$

Proof. For $v \in X_h$, we have

$$\begin{aligned}
\|I_h v\|_{0,h} &= \tfrac{1}{2}\,\left(h\sum_{j=1}^{J-1}|v_{j+1}+v_{j-1}|^2\right)^{\frac{1}{2}}\\
&\le \tfrac{1}{2}\,\left(h\sum_{j=1}^{J-1}|v_{j-1}|^2\right)^{\frac{1}{2}} + \tfrac{1}{2}\,\left(h\sum_{j=1}^{J-1}|v_{j+1}|^2\right)^{\frac{1}{2}}\\
&\le \tfrac{1}{2}\,\left(h\sum_{j=0}^{J-2}|v_j|^2\right)^{\frac{1}{2}} + \tfrac{1}{2}\,\left(h\sum_{j=2}^{J}|v_j|^2\right)^{\frac{1}{2}},
\end{aligned}$$

which easily yields (4.6). ∎

Lemma 4.3. *For $v \in X_h$ we have*

$$|v|_{\infty,h} \le \sqrt{2}\,\left(\|v\|_{0,h} + \|\partial_h v\|_{0,h}\right). \tag{4.7}$$

Proof. Let $v \in X_h$ with $v \neq 0$. Then, there exists $i_0 \in \{1,\ldots,J-1\}$ such that $|v|_{\infty,h} = |v_{i_0}|$. If i_0 is even, i.e. $i_0 = 2\,m_0$ for some $m_0 \in \mathbb{N}$, then we have

$$\begin{aligned}
|v_{i_0}| &= |v_{2m_0}|\\
&\le 2\,\sum_{\ell=0}^{m_0-1} h\,\left|\tfrac{v_{2(\ell+1)}-v_{2\ell}}{2h}\right|\\
&\le 2\,\left(h\sum_{\ell=0}^{m_0-1} 1\right)^{\frac{1}{2}}\,\left(h\sum_{\ell=0}^{m_0-1}\left|\tfrac{v_{2(\ell+1)}-v_{2\ell}}{2h}\right|^2\right)^{\frac{1}{2}}\\
&\le \sqrt{2}\,\|\partial_h v\|_{0,h}.
\end{aligned} \tag{4.8}$$

Let us now assume that i_0 is odd, i.e. $i_0 = 2\,m_0 - 1$ for some $m_0 \in \mathbb{N}$. If J is odd, i.e. $J = 2\,J_\star + 1$ for some $J_\star \in \mathbb{N}$, then

$$
\begin{aligned}
|v_{i_0}| &= |v_{2m_0-1}| \\
&\leq 2\,h \sum_{\ell=m_0}^{J_\star} \left| \tfrac{v_{2\ell+1}-v_{2\ell-1}}{2h} \right| \\
&\leq 2 \left(h \sum_{\ell=m_0}^{J_\star} 1 \right)^{\frac{1}{2}} \left(h \sum_{\ell=m_0}^{J_\star} \left| \tfrac{v_{2(\ell+1)}-v_{2\ell}}{2h} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\, \|\partial_h v\|_{0,h}.
\end{aligned}
\tag{4.9}
$$

Now, we assume that J is even, i.e. $J = 2\,J_\star$ for some $J_\star \in \mathbb{N}$. We define $(w_\ell)_{\ell=1}^{J_\star} \subset \mathbb{C}$, by $w_\ell = v_{2\ell-1}$ for $\ell = 1, \dots, J_\star$. Also, let $\ell_\star \in \{1, \dots, J_\star\}$ such that $|w_{\ell_\star}| = \min_{1\leq \ell \leq J_\star} |w_\ell|$. Then, we have

$$
\begin{aligned}
|w_{m_0}| &\leq |w_{\ell_\star}| + \sum_{\ell=1}^{J_\star-1} |w_{\ell+1} - w_\ell| \\
&\leq 2\,h \sum_{\ell=1}^{J_\star} |w_\ell| + 2\,h \sum_{\ell=1}^{J_\star-1} \left| \tfrac{w_{\ell+1}-w_\ell}{2h} \right| \\
&\leq \sqrt{2}\, \|v\|_{0,h} + \sqrt{2}\, \|\partial_h v\|_{0,h}.
\end{aligned}
\tag{4.10}
$$

Thus, (4.7) follows from (4.8), (4.9) and (4.10). ☐

4.2. **Properties of the discrete elliptic operator Λ_h and its inverse.**

Lemma 4.4. *We assume that*

$$
C_{\scriptscriptstyle DEB} := \inf_{\Omega} \left[\tfrac{1}{(C_{\scriptscriptstyle DPF})^2} - \tfrac{s^2}{\alpha^2\,|q|^2} \, \big(\mathrm{Re}(q) + |q|^2\, \mathrm{Re}(\gamma) \big) \right] > 0,
\tag{4.11}
$$

where $C_{\scriptscriptstyle DPF} \in (0, \frac{\sqrt{2}}{2}]$ is the optimal constant in (4.1), or, that there exists $\delta_\star \in \{1, -1\}$ such that

$$
C_{\scriptscriptstyle DBB} := \inf_{\Omega} \left[\tfrac{\delta_\star\, s^2}{\alpha^2\,|q|^2} \, \big(\mathrm{Im}(q) - |q|^2\, \mathrm{Im}(\gamma) \big) \right] > 0.
\tag{4.12}
$$

Then, we have

$$
\max_{r\in[0,R]} \|T_h(r) v\|_{1,h} \leq \, C\, \|v\|_{0,h} \quad \forall\, v \in X_h,
\tag{4.13}
$$

where $T_h(r) := \Lambda_h^{-1}(r)$ for $r \in [0, R]$.

Proof. Let $r \in [0, R]$ and $v \in X_h$. Then, we have

$$
\mathrm{Re}(\Lambda_h(r)v, v)_{0,h} = |v|_{1,h}^2 - \tfrac{s^2(r)}{\alpha^2\,|q|^2}\, h \sum_{j=1}^{J-1} \big[\mathrm{Re}(q) + |q|^2\, \mathrm{Re}(\gamma(r, y_j)) \big] \, |v_j|^2.
\tag{4.14}
$$

When (4.11) holds, then (4.14) and (4.1) yield

$$
\begin{aligned}
\mathrm{Re}(\Lambda_h(r)v, v)_{0,h} &\geq h \sum_{j=1}^{J-1} \left[\tfrac{1}{(C_{\scriptscriptstyle DPF})^2} - \tfrac{s^2(r)}{\alpha^2\,|q|^2}\, (\mathrm{Re}(q) + |q|^2\, \mathrm{Re}(\gamma(r, y_j))) \right] \, |v_j|^2 \\
&\geq C_{\scriptscriptstyle DEB}\, \|v\|_{0,h}^2 \quad \forall\, v \in X_h, \quad \forall\, r \in [0, R].
\end{aligned}
\tag{4.15}
$$

When (4.12) holds, then, we have

$$
\begin{aligned}
\big| \mathrm{Im}(\Lambda_h(r)v, v)_{0,h} \big| &\geq \delta_\star\, \mathrm{Im}(\Lambda_h(r)v, v)_{0,h} \\
&\geq \delta_\star\, \tfrac{s^2(r)}{\alpha^2\,|q|^2}\, h \sum_{j=1}^{J-1} \big[\mathrm{Im}(q) - |q|^2\, \mathrm{Im}(\gamma(r, y_j)) \big] \, |v_j|^2 \\
&\geq C_{\scriptscriptstyle DBB}\, \|v\|_{0,h}^2 \quad \forall\, v \in X_h, \quad \forall\, r \in [0, R].
\end{aligned}
\tag{4.16}
$$

Now, using (4.15) or (4.16), and the Cauchy-Schwarz inequality we arrive at

$$\|T_h(r)v\|_{0,h} \le C\,\|v\|_{0,h}. \tag{4.17}$$

Next, we use (4.14) and (4.17) to get

$$\begin{aligned} |T_h(r)v|_{1,h}^2 &\le C\,\left(\,\|T_h(r)v\|_{0,h}^2 + \|v\|_{0,h}\,\|T_h(r)v\|_{0,h}\,\right) \\ &\le C\,\|v\|_{0,h}^2. \end{aligned} \tag{4.18}$$

Thus, (4.13) follows easily from (4.17) and (4.18). □

Proposition 4.1. *We assume that (4.11) or (4.12) hold. Then, there exist positive constants C_A and C_B such that*

$$\|T_h(r)P_h\phi - P_hT(r)\phi\|_{1,h} \le C_A\,h^2\,|\phi|_{2,\infty,I}, \quad \forall\,\phi \in C^2(I;\mathbb{C}), \quad \forall\,r\in[0,R], \tag{4.19}$$

and

$$\begin{aligned} \|\,[T_h(r)P_h\phi - T_h(\tau)P_h\phi] - [P_hT(r)\phi - P_hT(\tau)\phi]\,\|_{1,h} \le C_B\left[h^3\,\,|\phi|_{3,\infty,I} + h^2\,|r-\tau|\,|\phi|_{2,\infty,I}\,\right] \\ \forall\,\phi\in C^3(I;\mathbb{C}), \quad \forall\,r,\tau\in[0,R]. \end{aligned} \tag{4.20}$$

Proof. Let r, $\tau\in[0,R]$, $\phi\in C^2(I;\mathbb{C})$, $\psi(r,y) := (T(r)\phi)(y)$ and $E(r)\in X_h$ defined by

$$E(r) := T_h(r)P_h\phi - P_hT(r)\phi.$$

Then, we have that

$$\Lambda_h(r)E(r) = \eta(r) \tag{4.21}$$

where $\eta(r)\in X_h$ with

$$\begin{aligned} (\eta(r))_j &= -\left[\,\psi_{yy}(r,y_j) - \tfrac{\psi(r,y_{j-1})-2\psi(r,y_j)+\psi(r,y_{j+1})}{h^2}\right] \\ &= \tfrac{h^2}{12}\,\psi_{yyyy}(r,\xi_j(r)), \quad j=1,\dots,J-1, \end{aligned}$$

with $\xi_j(r)\in(y_{j-1},y_{j+1})$ for $j=1,\dots,J-1$. Thus, along with (3.10), we obtain that

$$\begin{aligned} |\eta(r)|_{\infty,h} &\le \tfrac{h^2}{12}\,\max_I|\psi_{yyyy}(r,\cdot)| \\ &\le C\,h^2\,|\phi|_{2,\infty,I}. \end{aligned} \tag{4.22}$$

We use (4.21) and (4.13) to have

$$\begin{aligned} \|E(r)\|_{1,h} &= \|T_h(r)\eta(r)\|_{1,h} \\ &\le C\,\|\eta\|_{0,h} \\ &\le C\,|\eta|_{\infty,h}. \end{aligned} \tag{4.23}$$

If we combine (4.23) and (4.22), (4.19) easily follows.

Now, assuming that $r\le\tau$, we have

$$\begin{aligned} (\eta(r))_j - (\eta(\tau))_j &= \tfrac{h^2}{12}\,\{\,[\,\psi_{yyyy}(r,\xi_j(r)) - \psi_{yyyy}(\tau,\xi_j(r))\,] + [\,\psi_{yyyy}(\tau,\xi_j(r)) - \psi_{yyyy}(\tau,\xi_j(\tau))]\,\} \\ &= \tfrac{h^2}{12}\,\left[\,(r-\tau)\,\partial_r\partial_y^4\psi(\mu_j(r,\tau),\xi_j(r)) + (\xi_j(r)-\xi_j(\tau))\,\partial_y^5\psi(\tau,\widetilde{\xi}_j(r,\tau))\,\right], \quad j=1,\dots,J-1, \end{aligned}$$

with $\mu_j(r,\tau)\in(r,\tau)$ and $\widetilde{\xi}_j(r,\tau)\in(y_{j-1},y_{j+1})$ for $j=1,\dots,J-1$. Thus, using (3.10) and (3.11), we obtain that

$$\begin{aligned} |\eta(r)-\eta(\tau)|_{\infty,h} &\le \tfrac{h^2}{12}\,\left[\,|r-\tau|\,\max_\Omega|\psi_{yyyyr}| + 2\,h\,\max_\Omega|\psi_{yyyyy}|\,\right] \\ &\le C\,h^2\,\,(\,|r-\tau|\,|\phi|_{\infty,I} + h\,|\phi|_{3,\infty,I}\,)\,. \end{aligned} \tag{4.24}$$

Using (4.21) we see that

$$\Lambda_h(r)(E(r)-E(\tau)) = [\eta(r)-\eta(\tau)] + Z(r,\tau)\otimes E(\tau) \tag{4.25}$$

where $Z(r,\tau)\in X_h$ is defined by

$$(Z(r,\tau))_j := -\left[-s^2(r)\,\tfrac{1+q\,\gamma(r,y_j)}{\alpha^2\,q}+s^2(\tau)\,\tfrac{1+q\,\gamma(\tau,y_j)}{\alpha^2\,q}\right],\quad j=1,\dots,J-1. \tag{4.26}$$

Then, from (4.26) we obtain

$$|Z(r,\tau)|_{\infty,h}\le C\,|r-\tau|. \tag{4.27}$$

We use (4.25) and (4.13) to get

$$\begin{aligned}\|E(r)-E(\tau)\|_{1,h} &\le C\,\left(\,\|\eta(r)-\eta(\tau)\|_{0,h}+\|Z(r,\tau)\otimes E(\tau)\|_{0,h}\,\right)\\ &\le C\,\left(\,|\eta(r)-\eta(\tau)|_{\infty,h}+|Z(r,\tau)|_{\infty,h}\,\|E(\tau)\|_{0,h}\,\right).\end{aligned} \tag{4.28}$$

Then, (4.20) follows easily from (4.28), (4.27) and (4.24). □

4.3. **The Crank-Nicolson-type finite difference method** CNFD**.**

4.3.1. *Formulation of the method* CNFD. For $m=0,\cdots,N$, the method CNFD constructs an approximation U^m of u^m following the steps below:

`Step A1`: Define $U^0\in X_h$ by

$$U^0 := P_h u_0. \tag{4.29}$$

`Step A2`: For $n=1,\dots,N$, find $U^n\in X_h$ such that

$$\tfrac{U^n-U^{n-1}}{k}-\delta(r^{n-\frac12})\,\omega\otimes\partial_h\left(\tfrac{U^n+U^{n-1}}{2}\right)=\mathrm{i}\,\xi(r^{n-\frac12})\,T_h(r^{n-\frac12})\left(\tfrac{U^n+U^{n-1}}{2}\right)+\mathrm{i}\,\tfrac{\lambda}{q}\,\left(\tfrac{U^n+U^{n-1}}{2}\right), \tag{4.30}$$

where $\delta(r):=\frac{\dot s(r)}{s(r)}$ for $r\in[0,R]$, and $\xi(r):=\frac{\lambda\,s^2(r)}{\alpha^2\,q^2}$ for $r\in[0,R]$.

4.3.2. *Consistency of* CNFD.

Lemma 4.5. *We assume that* (4.11) *or* (4.12) *hold. Also, for* $n=1,\dots,N$, *we define* $\eta^n\in X_h$ *by*

$$\begin{aligned}\tfrac{u^n-u^{n-1}}{k}-\delta(r^{n-\frac12})\,\omega\otimes\partial_h\left(\tfrac{u^n+u^{n-1}}{2}\right) &=\mathrm{i}\,\xi(r^{n-\frac12})\,T_h(r^{n-\frac12})\left(\tfrac{u^n+u^{n-1}}{2}\right)\\ &\quad+\mathrm{i}\,\tfrac{\lambda}{q}\,\left(\tfrac{u^n+u^{n-1}}{2}\right)+\eta^n.\end{aligned} \tag{4.31}$$

Then, there exists a constant $C>0$ *depending on the data of the problem* (1.8) *which is such that*

$$\max_{1\le m\le N}|\eta^m|_{\infty,h}\le C\,\left[\,k^2\,\mathcal{B}_1(u)+h^2\,\mathcal{B}_2(u)\,\right] \tag{4.32}$$

and

$$\begin{aligned}\max_{2\le n\le N}|\eta^n-\eta^{n-1}|_{\infty,h} \le C\Big[\,&k^3\left(|\partial_r^4u|_{\infty,\Omega}+|\partial_r^3u|_{\infty,\Omega}+|\partial_r^2\partial_yu|_{\infty,\Omega}+|\partial_r^3\partial_yu|_{\infty,\Omega}\right)\\ &+h^3\,\left(\,|\partial_y^4u|_{\infty,\Omega}+\max_{[0,R]}|u|_{3,\infty,I}\right)\\ &+k\,h^2\left(|\partial_y^3u|_{\infty,\Omega}+|\partial_r\partial_y^3u|_{\infty,\Omega}+\max_{[0,R]}|u|_{2,\infty,I}+\max_{[0,R]}|u_r|_{2,\infty,I}\right)\Big].\end{aligned} \tag{4.33}$$

where

$$\mathcal{B}_1(u):=|u_{rrr}|_{\infty,\Omega}+|u_{yrr}|_{\infty,\Omega}+|u_{rr}|_{\infty,\Omega},\qquad \mathcal{B}_2(u):=\max_{[0,R]}|u|_{3,\infty,I}.$$

Proof. Let $n\in\{1,\dots,N\}$. Using Taylor's formula we conclude that

$$\begin{aligned}\eta_j^n=&\tfrac{k^2}{24}\,u_{rrr}(\tau_j^n,y_j)-\delta(r^{n-\frac12})\,y_j\,\left[\tfrac{k^2}{8}\,u_{yrr}(\widetilde{\tau}_j^n,y_j)+\tfrac{h^2}{6}\,u_{yyy}(r^n,z_j^n)+\tfrac{h^2}{6}\,u_{yyy}(r^{n-1},\widetilde{z}_j^n)\right]\\ &-\mathrm{i}\,\xi(r^{n-\frac12})\,\left[T_h(r^{n-\frac12})P_h\left(\tfrac{u(r^{n-1},\cdot)+u(r^n,\cdot)}{2}\right)-P_hT(r^{n-\frac12})\left(\tfrac{u(r^n,\cdot)+u(r^{n-1},\cdot)}{2}\right)\right]_j\\ &-\mathrm{i}\,\xi(r^{n-\frac12})\,\left[P_hT(r^{n-\frac12})\left(\tfrac{u(r^n,\cdot)+u(r^{n-1},\cdot)}{2}-u(r^{n-\frac12},\cdot)\right)\right]_j\\ &+\mathrm{i}\,\tfrac{\lambda}{q}\,\tfrac{k^2}{8}\,u_{rr}(t_j^n,y_j),\quad j=1,\dots,J-1,\end{aligned} \tag{4.34}$$

where t_j^n, τ_j^n, $\widetilde{\tau}_j^n \in (r^{n-1}, r^n)$ and $\widetilde{z}_j^n \in (y_{j-1}, y_{j+1})$.

Now, combine (4.34), (4.2), (4.19), (3.10) and (3.11) to obtain

$$
\begin{aligned}
|\eta^n|_{\infty,h} \le & C \Big[k^2 \left(|u_{rrr}|_{\infty,\Omega} + |u_{rr}|_{\infty,\Omega} + |u_{yrr}|_{\infty,\Omega} \right) + h^2 \, |u_{yyy}|_{\infty,\Omega}\Big] + C\, h^2 \max_{[0,R]} |u|_{2,\infty,I} \\
& + C \left| \tfrac{u(r^n,\cdot)+u(r^{n-1},\cdot)}{2} - u(r^{n-\frac12},\cdot) \right|_{\infty,I} \\
\le & C \left[k^2 \left(|u_{rrr}|_{\infty,\Omega} + |u_{rr}|_{\infty,\Omega} + |u_{yrr}|_{\infty,\Omega} \right) + h^2 \max_{[0,R]} |u|_{3,\infty,I} \right],
\end{aligned}
\tag{4.35}
$$

and thus arrive at (4.32).

Since it holds that

$$
\begin{aligned}
\eta_j^n - \eta_j^{n-1} = & \tfrac{k^2}{24} \left[u_{rrr}(\tau_j^n, y_j) - u_{rrr}(\tau_j^{n-1}, y_j) \right] + \mathrm{i}\, \tfrac{\lambda}{q}\, \tfrac{k^2}{8} \left[u_{rr}(t_j^n, y_j) - u_{rr}(t_j^{n-1}, y_j) \right] \\
& - \left[\delta(r^{n-\frac12}) - \delta(r^{n-\frac32}) \right] y_j \left[\tfrac{k^2}{8}\, u_{yrr}(\widetilde{\tau}_j^n, y_j) + \tfrac{h^2}{6}\, u_{yyy}(r^n, z_j^n) + \tfrac{h^2}{6}\, u_{yyy}(r^{n-1}, \widetilde{z}_j^n) \right] \\
& - \delta(r^{n-\frac32})\, y_j \Bigg\{ \left[\tfrac{k^2}{8}\, u_{yrr}(\widetilde{\tau}_j^n, y_j) + \tfrac{h^2}{6}\, u_{yyy}(r^n, z_j^n) + \tfrac{h^2}{6}\, u_{yyy}(r^{n-1}, \widetilde{z}_j^n) \right] \\
& - \left[\tfrac{k^2}{8}\, u_{yrr}(\widetilde{\tau}_j^{n-1}, y_j) + \tfrac{h^2}{6}\, u_{yyy}(r^{n-1}, z_j^{n-1}) + \tfrac{h^2}{6}\, u_{yyy}(r^{n-2}, \widetilde{z}_{n-1,j}) \right] \Bigg\} \\
& - \mathrm{i} \left[\xi(r^{n-\frac12}) - \xi(r^{n-\frac32}) \right] \left[T_h(r^{n-\frac12}) P_h \left(\tfrac{u(r^n,\cdot)+u(r^{n-1},\cdot)}{2} \right) - P_h T(r^{n-\frac12}) \left(\tfrac{u(r^n,\cdot)+u(r^{n-1},\cdot)}{2} \right) \right]_j \\
& - \mathrm{i}\, \xi(r^{n-\frac32}) \Bigg\{ \left[T_h(r^{n-\frac12}) P_h \left(\tfrac{u(r^n,\cdot)+u(r^{n-1},\cdot)}{2} \right) - P_h T(r^{n-\frac12}) \left(\tfrac{u(r^n,\cdot)+u(r^{n-1},\cdot)}{2} \right) \right]_j \\
& \quad - \left[T_h(r^{n-\frac32}) P_h \left(\tfrac{u(r^{n-1},\cdot)+u(r^n,\cdot)}{2} \right) - P_h T(r^{n-\frac32}) \left(\tfrac{u(r^{n-1},\cdot)+u(r^n,\cdot)}{2} \right) \right]_j \Bigg\} \\
& - \mathrm{i}\, \xi(r^{n-\frac32}) \Bigg\{ \left[T_h(r^{n-\frac32}) P_h \left(\tfrac{u(r^{n-1},\cdot)+u(r^n,\cdot)}{2} \right) - P_h T(r^{n-\frac32}) \left(\tfrac{u(r^{n-1},\cdot)+u(r^n,\cdot)}{2} \right) \right]_j \\
& \quad - \left[T_h(r^{n-\frac32}) P_h \left(\tfrac{u(r^{n-1},\cdot)+u(r^{n-2},\cdot)}{2} \right) - P_h T(r^{n-\frac32}) \left(\tfrac{u(r^{n-1},\cdot)+u(r^{n-2},\cdot)}{2} \right) \right]_j \Bigg\},
\end{aligned}
$$

for $j = 1, \ldots, J-1$ and $n = 2, \ldots, N$, (4.33) follows easily by the mean value theorem, (4.19) and (4.20). □

4.3.3. *Convergence of* CNFD.

Theorem 4.2. *Let $(U^m)_{m=0}^J \subset X_h$ be the finite difference approximation to the solution of (1.8) defined as in Section 4.3.1. Then, there exist constants $C_1 > 0$, $C_2 > 0$ and $C_3 \ge 0$ independent of k and h such that: if $C_3\, k \le 3$, then*

$$
\max_{0\le n\le N} \|U^n - u^n\|_{0,h} \le C_1 \max_{1\le n\le N} \|\eta^n\|_{0,h}
\tag{4.36}
$$

and

$$
\max_{2\le n\le N} \left| \omega \otimes \left(\tfrac{u^n+u^{n-1}}{2} - \tfrac{U^n+U^{n-1}}{2} \right) \right|_{\infty,h} \le C_2 \left[k^{-1} \max_{2\le n\le N} \|\eta^n - \eta^{n-1}\|_{0,h} + \max_{1\le n\le N} \|\eta^n\|_{0,h} \right].
\tag{4.37}
$$

Proof. Let $e^n := u^n - U^m$ for $n = 0, \ldots, N$, and observe that due to (4.29) there holds that $e^0 = 0$. Next, subtract (4.30) from (4.31) to obtain

$$
\begin{aligned}
\tfrac{e^n - e^{n-1}}{k} - \delta(r^{n-\frac12})\, \omega \otimes \partial_h \left(\tfrac{e^n+e^{n-1}}{2} \right) = & \, \mathrm{i}\, \xi(r^{n-\frac12})\, T_h(r^{n-\frac12}) \left(\tfrac{e^n+e^{n-1}}{2} \right) \\
& + \mathrm{i}\, \tfrac{\lambda}{q} \left(\tfrac{e^n+e^{n-1}}{2} \right) + \eta^n, \quad n = 1, \ldots, N.
\end{aligned}
\tag{4.38}
$$

Now, take the $(\cdot,\cdot)_{0,h}$−inner product of both sides of (4.38) with e^n+e^{n-1}, and then real parts to get

$$
\begin{aligned}
\|e^n\|_{0,h}^2-\|e^{n-1}\|_{0,h}^2 =&\ \tfrac{k}{2}\,\delta(r^{n-\frac12})\,\mathrm{Re}\left(\omega\otimes\partial_h(e^n+e^{n-1}),e^n+e^{n-1}\right)_{0,h}\\
&-\tfrac{k}{2}\,\mathrm{Im}\left[\xi(r^{n-\frac12})\,\left(T_h(r^{n-\frac12})(e^n+e^{n-1}),e^n+e^{n-1}\right)_{0,h}\right]\\
&-\tfrac{k}{2}\,\mathrm{Im}\left[\tfrac{\lambda}{q}\,\left(e^n+e^{n-1},e^n+e^{n-1}\right)_{0,h}\right]\\
&+k\,\mathrm{Re}\left(\eta^n,e^n+e^{n-1}\right)_{0,h},\quad n=1,\dots,N.
\end{aligned}\tag{4.39}
$$

Combining (4.39), (4.5), (4.6) and (4.13) we obtain

$$
\|e^n\|_{0,h}-\|e^{n-1}\|_{0,h}\le\ k\,c_\star\,\left(\ \|e^n\|_{0,h}+\|e^{n-1}\|_{0,h}\ \right)+k\,\|\eta^n\|_{0,h},\quad n=1,\dots,N.\tag{4.40}
$$

Assuming that $k\,c_\star\le\frac13$, from (4.40) we conclude that

$$
\|e^n\|_{0,h}\le\ \tfrac{1+c_\star\,k}{1-c_\star\,k}\,\|e^{n-1}\|_{0,h}+\tfrac{k}{1-c_\star\,k}\,\|\eta^n\|_{0,h},\quad n=1,\dots,N,
$$

which yields

$$
\|e^n\|_{0,h}\le\ e^{4c_* k}\,\|e^{n-1}\|_{0,h}+3\,k\,\|\eta^n\|_{0,h},\quad n=1,\dots,N.\tag{4.41}
$$

Applying a standard discrete Grönwall argument on (4.41) and using the fact that $\|e^0\|_{0,h}=0$, we obtain

$$
\begin{aligned}
\max_{0\le n\le N}\|e^n\|_{0,h}&\le C\left(\,\|e^0\|_{0,h}+\max_{1\le n\le N}\|\eta^n\|_{0,h}\,\right)\\
&\le C\,\max_{1\le n\le N}\|\eta^n\|_{0,h},
\end{aligned}\tag{4.42}
$$

which establishes the estimate (4.36).

Let $\nu^m:=e^m-e^{m-1}$ for $m=1,\dots,N$. Then, (4.38) yields

$$
\begin{aligned}
\tfrac{\nu^n-\nu^{n-1}}{k}-\delta(r^{n-\frac12})\,\omega\otimes\partial_h\left(\tfrac{\nu^n+\nu^{n-1}}{2}\right)=&\left[\delta(r^{n-\frac12})-\delta(r^{n-\frac32})\right]\,\omega\otimes\partial_h\left(\tfrac{e^{n-1}+e^{n-2}}{2}\right)\\
&+\,\mathrm{i}\,\xi(r^{n-\frac32})\,T_h(r^{n-\frac12})\left(\tfrac{\nu^n+\nu^{n-1}}{2}\right)\\
&+\,\mathrm{i}\,\left[\xi(r^{n-\frac12})-\xi(r^{n-\frac32})\right]\,T_h(r^{n-\frac12})\left(\tfrac{e^n+e^{n-1}}{2}\right)\\
&+\,\mathrm{i}\,\xi(r^{n-\frac32})\,\left[T_h(r^{n-\frac12})\left(\tfrac{e^n+e^{n-1}}{2}\right)-T_h(r^{n-\frac32})\left(\tfrac{e^n+e^{n-1}}{2}\right)\right]\\
&+\,\mathrm{i}\,\tfrac{\lambda}{q}\,\left(\tfrac{\nu^n+\nu^{n-1}}{2}\right)\\
&+(\eta^n-\eta^{n-1}),\quad n=2,\dots,N.
\end{aligned}\tag{4.43}
$$

Now, take the $(\cdot,\cdot)_{0,h}$−inner product of both sides of (4.43) with $\nu^n+\nu^{n-1}$ and then real parts to get

$$
\begin{aligned}
\|\nu^n\|_{0,h}^2-\|\nu^{n-1}\|_{0,h}^2\le C\,k\,\Big[&\,\|\nu^n+\nu^{n-1}\|_{0,h}+k\,\|e^n+e^{n-1}\|_{0,h}+\|\eta^n-\eta^{n-1}\|_{0,h}\\
&+\left\|T_h(r^{n-\frac12})(e^n+e^{n-1})-T_h(r^{n-\frac32})(e^n+e^{n-1})\right\|_{0,h}\\
&+k\,\left\|\omega\otimes\partial_h\left(\tfrac{e^{n-1}+e^{n-2}}{2}\right)\right\|_{0,h}\,\Big]\,\|\nu^n+\nu^{n-1}\|_{0,h},\quad n=2,\dots,N,
\end{aligned}\tag{4.44}
$$

after using (4.13). Now, (4.38) along with (4.13), yield

$$
\left\|\omega\otimes\partial_h\left(\tfrac{e^{n-1}+e^{n-2}}{2}\right)\right\|_{0,h}\le\ C\,\left[\,\|\eta^{n-1}\|_{0,h}+\|e^{n-1}\|_{0,h}+\|e^{n-2}\|_{0,h}+k^{-1}\ \|\nu^{n-1}\|_{0,h}\right],\quad n=2,\dots,N.\tag{4.45}
$$

Thus, (4.45) and (4.44) yield

$$
\begin{aligned}
\|\nu^n\|_{0,h}-\|\nu^{n-1}\|_{0,h}\le C\,k\,\Big[&\,\|\nu^n\|_{0,h}+\|\nu^{n-1}\|_{0,h}+k\,\left(\|e^n\|_{0,h}+\|e^{n-1}\|_{0,h}+\|e^{n-2}\|_{0,h}\right)\\
&+k\,\|\eta^{n-1}\|_{0,h}+\|\eta^n-\eta^{n-1}\|_{0,h}\\
&+\left\|T_h(r^{n-\frac12})(e^n+e^{n-1})-T_h(r^{n-\frac32})(e^n+e^{n-1})\right\|_{0,h}\,\Big]
\end{aligned}\tag{4.46}
$$

for $n=2,\dots,N$. Now, we observe that, for $v\in X_h$, we have

$$\Lambda_h(r^{n-\frac12})\Big(T_h(r^{n-\frac12})v-T_h(r^{n-\frac32})v\Big)=\Lambda_h(r^{n-\frac32})T_h(r^{n-\frac32})v-\Lambda_h(r^{n-\frac12})T_h(r^{n-\frac32})v \tag{4.47}$$
$$=V^n\otimes T_h(r^{n-\frac32})v,\quad n=2,\dots,N,$$

where $V^n\in X_h$ is given by

$$V^n_j:=s^2(r^{n-\frac32})\,\tfrac{1+q\,\gamma(r^{n-\frac32},y_j)}{\alpha^2\,q}-s^2(r^{n-\frac12})\,\tfrac{1+q\,\gamma(r^{n-\frac12},y_j)}{\alpha^2\,q},\quad j=1,\dots,J-1. \tag{4.48}$$

Using (4.47), (4.13) and (4.48) we have

$$\|T_h(r^{n-1})v-T_h(r^{n-2})v\|_{0,h}\le C\,|V^n|_{\infty,h}\,\|v\|_{0,h} \tag{4.49}$$
$$\le C\,k\,\|v\|_{0,h},\quad \forall\,v\in X_h,\quad n=2,\dots,N.$$

Combining (4.46) and (4.49) we obtain

$$\|\nu^n\|_{0,h}-\|\nu^{n-1}\|_{0,h}\le C_\star\,k\Big[\,\|\nu^n\|_{0,h}+\|\nu^{n-1}\|_{0,h}+k\,\big(\|e^n\|_{0,h}+\|e^{n-1}\|_{0,h}+\|e^{n-2}\|_{0,h}\big) \tag{4.50}$$
$$+k\,\|\eta^{n-1}\|_{0,h}+\|\eta^n-\eta^{n-1}\|_{0,h}\Big],\quad n=2,\dots,N.$$

Assuming that k is enough small (i.e. $3\,k\,\max\{c_\star,C_\star\}\le 1$) and applying a discrete Grönwall argument on (4.50), we conclude that

$$\max_{1\le n\le N}\|\nu^n\|_{0,h}\le\;C\Big[\max_{2\le n\le N}\|\eta^n-\eta^{n-1}\|_{0,h}+k\max_{1\le n\le N-1}\|\eta^n\|_{0,h}$$
$$+k\max_{0\le n\le N}\|e^n\|_{0,h}+\|\nu^1\|_{0,h}\Big].$$

which, along with (4.36) and (4.41), yields

$$\max_{1\le n\le N}\|\nu^n\|_{0,h}\le C\Big[\max_{2\le n\le N}\|\eta^n-\eta^{n-1}\|_{0,h}+k\max_{1\le n\le N}\|\eta^n\|_{0,h}+\|e^1\|_{0,h}\Big] \tag{4.51}$$
$$\le C\Big[\max_{2\le n\le N}\|\eta^n-\eta^{n-1}\|_{0,h}+k\max_{1\le n\le N}\|\eta^n\|_{0,h}\Big].$$

Now, from (4.38) follows that

$$\|\omega\otimes\partial_h(e^n+e^{n-1})\|_{0,h}\le\;C\Big[k^{-1}\,\|\nu^n\|_{0,h}+\|e^n\|_{0,h}+\|e^{n-1}\|_{0,h}+\|\eta^n\|_{0,h}\Big],\quad n=1,\dots,N. \tag{4.52}$$

Then, (4.52) and (4.51) yield that

$$\max_{1\le n\le N}\|\omega\otimes\partial_h(e^n+e^{n-1})\|_{0,h}\le\;C\Big[k^{-1}\max_{2\le n\le N}\|\eta^n-\eta^{n-1}\|_{0,h}+\max_{1\le n\le N}\|\eta^n\|_{0,h}\Big]. \tag{4.53}$$

Finally, (4.37) follows from (4.53) and (4.7), in view of the identity $\partial_h(\omega\otimes v)=\omega\otimes\partial_h v+I_h v$ for $v\in X_h$. □

In view of the results of Lemma 4.5, (4.36) implies that

$$\max_{0\le n\le N}\|U^n-u^n\|_{0,h}=\mathcal{O}(k^2+h^2).$$

Also, the estimates (4.37), (4.32) and (4.33) yield

$$\max_{0\le n\le N}|\omega\otimes e^{n-\frac12}|_{\infty,h}=\mathcal{O}(k^2+h^2+h^3\,k^{-1})$$

which may be viewed as a discrete weighted maximum norm estimate on $e^{n-\frac12}$, which is of optimal order when $h=\mathcal{O}(k)$; however does not yield an optimal order maximum norm estimate, due to vanishing of the coefficient of u_y in the p.d.e. in (1.8) at $y=0$.

5. Numerical implementation

5.1. **The numerical scheme.** Using the definitions $\delta(r) := \frac{\dot{s}(r)}{s(r)}$, $\zeta(r,y) := \frac{(1+q\,\gamma(r,y))\,s^2(r)}{\alpha^2\,q}$, $\xi(r) := \frac{\lambda\,s^2(r)}{\alpha^2\,q^2}$, we note that the p.d.e. in (1.8) may written as

$$-\zeta(r,y)G(r,y) - \partial_y^2 G(r,y) = \mathrm{i}\xi(r)\,u,$$

where we have put $G(r,y) := u_r(r,y) - \mathrm{i}\,\frac{\lambda}{q}\,u(r,y) - \delta(r)y\,u_y(r,y)$. Motivated by this we rewrite the CNFD scheme (4.29)-(4.30) in the following equivalent form using the notation introduced in section 4 and putting $U_j^{n-\frac{1}{2}} := \frac{U_j^{n-1}+U_j^n}{2}$. We seek U_j^n, $0 \le j \le J$, $0 \le n \le N$, approximating u_j^n and given by the equations:
For $n = 0$:

$$U_j^0 = u_0(y_j) \quad \text{for} \quad 1 \le j \le J-1, \quad U_0^0 = U_J^0 = 0. \tag{5.1}$$

For $n = 1, \dots, N$:

$$-\zeta(r^{n-\frac{1}{2}}, y_j)\,G_j^n - \widetilde{\Delta}_h G_j^n = \mathrm{i}\,\xi(r^{n-\frac{1}{2}})\,U_j^{n-\frac{1}{2}} \quad \text{for} \quad 1 \le j \le J-1, \tag{5.2}$$

where

$$\widetilde{\Delta}_h G_j^n := \begin{cases} \frac{G_2^n - 2G_1^n}{h^2} & j = 1, \\ \frac{G_{j+1}^n - 2G_j^n + G_{j-1}^n}{h^2} & j = 2, \cdots, J-2, \\ \frac{-2G_{J-1}^n + G_{J-2}^n}{h^2} & j = J-1, \end{cases}$$

with

$$G_j^n := \frac{U_j^n - U_j^{n-1}}{k} - \mathrm{i}\,\frac{\lambda}{q}\,U_j^{n-\frac{1}{2}} - \delta(r^{n-\frac{1}{2}})\,y_j\,\frac{U_{j+1}^{n-\frac{1}{2}} - U_{j-1}^{n-\frac{1}{2}}}{2h} \quad \text{for} \quad j = 1, \dots, J-1.$$

The scheme requires solving a pentadiagonal linear system of algebraic equations at each time step.

5.2. **Numerical experiments.** We implemented the finite difference scheme CNFD in the form (5.1)-(5.2) in a double precision FORTRAN 77 code using it in various numerical examples to test its accuracy and stability. We used the function $u(r,y) = \exp(2r)(y-1)\sin(2\pi y)$ as exact solution of (1.8) (with an appropriate nonhomogeneous term in the right-hand side of the p.d.e.), putting $\gamma(r,y) = 1+y$, $\alpha = 2$, $p = q + \frac{1}{2}$, and selecting $q = (0.252252311, -1.35135138\,e-2)$, [10]. We experimented with several bottom profiles. For example, in the case of the downsloping bottom given by $s(r) = \exp(r)$, $0 \le r \le 1$, we obtained at $r = 1$ the errors in the discrete L^2 and L^∞ norms shown in Table 1 together with the associated experimental convergence rates. (We took $h = k = \frac{1}{J}$ for the values of J shown). The convergence rates of the table are practically equal to 2 and consistent with the predictions of the theory. We also tried bottom profiles $s(r)$ given for $0 \le r \le 1$ by $r+2$, $-r+2$, $-\exp(-r)$, $\cos(2\pi r)+2$, i.e. upsloping and oscillating profiles as well, and found experimentally that the scheme was stable for $k = h$ and that the L^2- and L^∞-convergence rates were practically equal to 2 again. Hence it seems that although CNFD is designed to approximate the ibvp (1.8) in the downsloping bottom case, it is resilient enough in upsloping and non-monotonic bottom problems as well.

Although the ibvp (1.8) is L^2-conservative, in the sense that its solution preserves the quantity

$$\sqrt{s(r)}\|u(r,\cdot)\|_{L^2(I)}$$

J	L^2-error	L^2-rate	L^∞-error	L^∞-rate
40	0.2510 e-1		0.2493 e-1	
80	0.6424 e-2	1.966	0.6365 e-2	1.969
160	0.1627 e-2	1.981	0.1609 e-2	1.983
320	0.4097 e-3	1.990	0.4048 e-3	1.991
640	0.1028 e-3	1.995	0.1015 e-3	1.995
1280	0.2574 e-4	1.997	0.2542 e-4	1.998

Table 1. Discrete L^2- and L^∞-errors and rates at $r = 1$, $s(r) = \exp(r)$, $k = h = \frac{1}{J}$.

for real γ and q, CNFD does not share a corresponding discrete property. For example, when we integrated numerically (1.8) with $\alpha = 10$, $q = \frac{1}{4}$, $p = q + \frac{1}{2}$, $\gamma(r, y) = 1 + y$, $u_0(y) = y^2(y-1)$, $0 \leq y \leq 1$, we found that in the cases $s(r) = \exp(r)$ and $s(r) = r + 2$, $0 \leq r \leq 1$, the quantity $\sqrt{s(r^n)}\|U^n\|_{0,h}$ was preserved to 4 significant digits.

We also performed a simulation of a realistic underwater acoustics problem using the CNFD scheme. We integrated the ibvp (1.8) using again $q = (0.252252311, -1.35135138\,e-2)$, $p = q + \frac{1}{2}$, and considered a straight downsloping bottom given by $s(r) = 200\left(1 + \frac{r}{4000}\right)$ (distances in meters) and making an angle of $2.86°$ with respect to the horizontal surface. We used as initial condition at $r = 0$ the normal mode starter given by formula (45) of [12] with $M = 6$, simulating the initial field produced by a time-harmonic point source of frequency f = 25 Hz located at a depth of $z = 100$ m. (We shall frequently refer in the sequel to quantities expressed in the r, z variables. Of course the scheme is implemented in the r, y variables and the results are transformed from or into the r, z domain as required.) We assume that the medium is homogeneous and lossless with a sound speed $c = c_0 = 1500\,\mathrm{m/sec}$, so that $\beta = 0$. We integrated the problem up to $R = 3300\,\mathrm{m}$ using $k = 0.83475\,\mathrm{m}$, and 4000 mesh intervals of equal length in y. As is customary in underwater acoustics we present the numerical results in terms of a one-dimensional transmission loss (TL) plot in the r, z variables. (The TL function was computed by the formula TL$= -20\,\log_{10}\left(\frac{|v(r,z_{\mathrm{rec}})|}{\sqrt{r}}\right)$ where z_{rec} is a receiver depth.) The graph of Figure 2 shows as an example, the TL curves at $z_{\mathrm{rec}} = 30\,\mathrm{m}$ obtained by integrating CNFD (solid line) and by the finite difference scheme (dotted line, 'DSZ scheme') proposed in [12] as a discretization of a problem of the form (1.8) but with the bottom boundary condition (cf. Remark 3.3)

$$u_{yy}(r,1) = \mathrm{i}\,\frac{2}{\alpha}\,s(r)\,\dot{s}(r)\,u_y(r,1), \quad 0 \leq r \leq R,$$

replacing $u_y(r,1) = 0$. The results of the two schemes are in very good agreement. (In [12], the results of the DSZ scheme were compared to those of a standard 'staircase' wide-angle PE code and were found to be in very good agreement.)

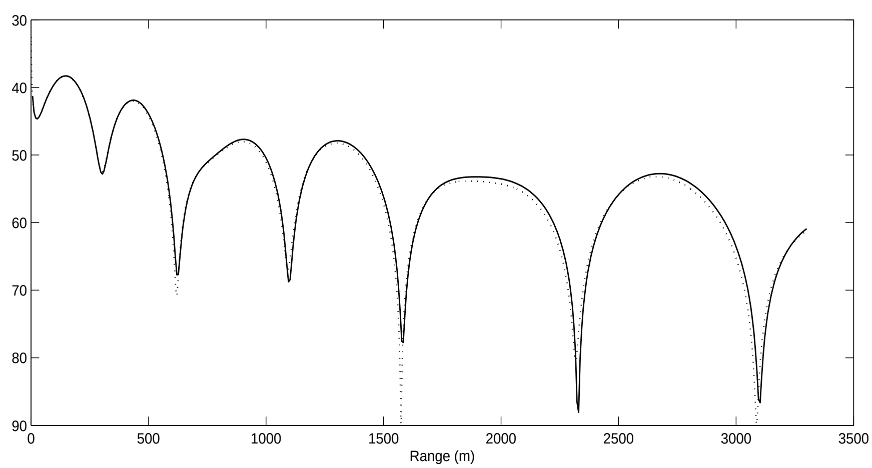

FIGURE 2. TL at $z_{\mathrm{rec}} = 30$ m, downsloping bottom. CNFD (solid line), DSZ (dotted line).

REFERENCES

[1] G. D. Akrivis, Finite difference methods for the wide-angle 'parabolic' equation, SIAM J. Numer. Anal. 36, (1999), 317-329.

[2] G. D. Akrivis and V. A. Dougalis, Finite difference discretization with variable mesh of the Schrödinger equation in a variable domain, Bull. Greek Math. Soc. 31, (1990), 19-28.

[3] G. D. Akrivis, V. A. Dougalis and G. E. Zouraris, Error estimates for finite difference methods for a wide-angle 'parabolic' equation, SIAM J. Numer. Anal. 33, (1996), 2488–2509.

[4] G. D. Akrivis, V. A. Dougalis and G. E. Zouraris, Finite difference schemes for the 'Parabolic' Equation in a variable depth environment with a rigid bottom boundary condition, SIAM J. Numer. Anal. 39, (2001), 539-565.

[5] D. C. Antonopoulou, V. A. Dougalis, F. Sturm and G. E. Zouraris, "Conservative initial-boundary value problems for the wide-angle PE in waveguides with variable bottoms," in Proceedings of the 9th European Conference on Underwater Acoustics (9th EQUA), M. E. Zakharia, D. Cassereau and F. Luppé (Editors), vol. 1, 2008, 375-380.

[6] D. C. Antonopoulou, V. A. Dougalis and G. E. Zouraris, Galerkin Methods for Parabolic and Schrödinger Equations with dynamical boundary conditions and applications to underwater acoustics, SIAM J. Numer. Anal. 47, (2009), 2752-2781.

[7] A. Bamberger, B. Engquist, L. Halpern L. and P. Joly, Parabolic wave equation approximations in heterogeneous media, SIAM J. Appl. Math. 48, (1988), 99-128.

[8] P. Chevret, Ph. Blanc-Benon and D. Juvé, A numerical model for sound propagation through a turbulent atmosphere near the ground, J. Acoust. Soc. Am. 100, (1996), 3587-3599.

[9] J. F. Claerbout, Fundamentals of Geophysical Data Processing, with Applications to Petroleum Prospecting, McGraw-Hill, New-York, 1976.

[10] M. D. Collins, Higher-order Padé approximations for accurate and stable elastic parabolic equations with application to interface wave propagation, J. Acoust. Soc. Am. 89, (1991), 1050-1057.

[11] Dougalis V. A., Sturm F., Zouraris G. E.: *Boundary Conditions for the Wide-Angle PE at a Sloping Bottom*. Proceedings of the 8th European Conference on Underwater Acoustics (8th EQUA), S. M. Jesus and O. C. Rodriguez, eds. **1**, 51–56 (2006)

[12] V. A. Dougalis, F. Sturm and G. E. Zouraris, On an initial-boundary value problem for a wide-angle parabolic equation in a waveguide with a variable bottom, Math. Meth. in Appl. Sciences 32, (2009), 1519-1540.

[13] L. C. Evans, Partial Differential Equations, American Mathematical Society, 1998.

[14] O. A. Godin, Reciprocity and energy conservation within the parabolic equation, Wave Motion 29, (1999), 175-194.

[15] R. R. Green, A high-angle one-way wave equation for seismic wave propagation along rough and sloping interfaces, J. Acoust. Soc. Am. 77, (1985), 1991-1998.

[16] J. E. Lagnese, General boundary value problems for differential equations of Sobolev type, SIAM J. Math. Anal. 3, (1972), 105-119.

[17] D. Lee and S. T. McDaniel, Ocean acoustic propagation by finite difference methods, Comput. Math. Appl. 14, (1987), 305-423.

[18] F. D. Tappert, "The parabolic approximation method, Wave Propagation and Underwater Acoustics", in Lecture Notes in Phys. 70, J. B. Keller and J. S. Papadakis (Editors), Springer-Verlag, Berlin, 1977, 224-287.