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## On the existence of solution for a Cahn-Hilliard/Allen-Cahn equation

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# ON THE EXISTENCE OF SOLUTION FOR A CAHN-HILLIARD/ALLEN-CAHN EQUATION

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ABSTRACT. We consider the existence of the solution for the Cahn-Hilliard/Allen-Cahn equation, which was studied in [11]. This mean field partial differential equation contains qualitatively microscopic information on particle-particle interactions and multiple particle dynamics. For a bounded potential or in one-dimensional case, the existence was proved in [12]. In this paper, we improve the existence for a standard double-well potential in dimension $1 \leq n \leq 4$ with Neumann boundary conditions in view of the free energy.

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**keywords :** Allen-Cahn equation, Cahn-Hilliard equation, mean curvature flow, quartic bistable potential

## 1. INTRODUCTION

We consider a scalar Cahn-Hilliard/Allen-Cahn equation;

$$\begin{cases} u_t = -\delta\Delta(\Delta u - W'(u)) + (\Delta u - W'(u)) & \text{in } \Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $\nu$ is the unit normal on $\partial\Omega$, $\delta > 0$ is a diffusion constant and $W \in C^3(\Omega; \mathbb{R}_+ \cup \{0\})$ is a quartic bistable potential which has zeros at $\pm 1$, that is, there exists a constant $c > 0$ such that

$$|W^{(i)}(s)| \leq c(1 + |s|^{4-i}) \quad \text{for } s \in \mathbb{R} \;\; i = 0, 1, 2. \tag{1.2}$$

As known results, for the Cahn-Hilliard equation the existence of the solution is proved in [5] by the Galerkin approximation, for the degenerate mobility case in [4]. For the equation (1.1) it is proved in [12] for a $C^2$-bounded potential or in the one-dimensional case. In this paper we show the existence of the solution of (1.1). We actually prove the existence for the quartic potential $W$ in space dimension $1 \leq n \leq 4$ by the Galerkin method.

For the equation (1.1) the volume $\int_\Omega u \, dx$ is not necessarily preserved because of the Allen-Cahn term, although the volume is preserved for Cahn-Hilliard equation. Thus we have to pay careful attention to estimate Galerkin ansatz by using some inequalities and interpolations and the energy estimates.

The Cahn-Hilliard/Allen-Cahn equation is introduced in [8] and [13] as a simplified model with multiple microscopic mechanism. The Cahn-Hilliard model can describe surface diffusion including particle/particle interactions, while the Allen-Cahn describes a simplified model of adsorption to and desorption from the surface. It is well known that the Allen-Cahn and Cahn-Hilliard equations can serve as diffuse interface models for limiting sharp interface motion. In [11] they considered the $\varepsilon$-scaled problem;

$$u_t^\varepsilon = -\varepsilon^2\delta\Delta(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}) + (\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}) \quad \text{in } \Omega \times [0,T). \tag{1.3}$$

For the Allen-Cahn equation $u_t^\varepsilon = \Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2}$ or the Cahn-Hilliard equation $u_t^\varepsilon = \Delta(\Delta u^\varepsilon - \frac{W'(u^\varepsilon)}{\varepsilon^2})$, there are several studies about the singular limit as $\varepsilon$ tends to 0. It is well-known that the limit evolution of the Allen-Cahn equation is the mean curvature flow, which is proved in the several methods, formally by Fife in [7], Rubinstein, Sternberg and Keller in [19], from the viscosity solution by Evans and Spruck in [6] and Chen, Giga and Goto in [3], in the sense of Brakke's motion [2] by Ilmanen in [10]. The Cahn-Hilliard equation was constructed to describe mass conservative phase separation. By considering an appropriate singular limit (as $\varepsilon$ tends to 0) it can describe the motion of interphase boundaries separating two phases of differing composition during the later stages of coarsening. In a suitable scale it is proved that the limit evolution is the Mullins-Sekerka model, a result which was formally proved in [18] and rigorously in [1].

For (1.3), the order of the convergence of $\varepsilon$ is sufficiently large to vanish for the Cahn-Hilliard term and actually it is proved in [11] that the limit evolution is also mean curvature flow but with a different coefficient;

$$V = \mu\sigma\kappa \tag{1.4}$$

where $V$ is the normal velocity and $\kappa$ is the mean curvature of the limit interface, $\sigma$ is a surface tension given by $\sigma = \int_{-1}^{1}\sqrt{W(s)}ds$ and $\mu$ is a mobility constant given by

$$\mu = 2(\int_{\mathbb{R}} \chi q'\, dx)^{-1} \tag{1.5}$$

where $q = \tanh$ is a well-known function which is used in order to describe transition profile of the Allen-Cahn equation and $\chi$ is a solution of the ODE

$$-\delta\chi'' + \chi = q' \text{ in } \mathbb{R} \quad \text{and} \quad \chi(\pm\infty) = 0. \tag{1.6}$$

It is worth mentioning that the mobility is completely different from the one of the Allen-Cahn equation ($V = k$). This implies in particular that the diffusion speeds up the mean curvature flow.

In [12] focusing on the continuous dependence of the diffusion constant $\delta$ they constructed a sequence of solutions which converges to a solution of the Allen-Cahn equation as $\delta$ tends to 0.

Concerning the Allen-Cahn structure we set

$$v := \Delta u - W'(u). \tag{1.7}$$

and we rewrite (1.1) to the following form;

$$
\left\{
\begin{array}{ll}
u_t = -\delta \Delta v + v & \text{in } \Omega \times [0,T), \\
v = \Delta u - W'(u) & \text{in } \Omega \times [0,T), \\
u(x,0) = u_0(x) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega.
\end{array}
\right. \tag{1.8}
$$

For the diffused interface problem, we usually consider the free energy functional given by

$$
E(u) = \int_\Omega \frac{|\nabla u|^2}{2} + W(u)\, dx. \tag{1.9}
$$

In the stationary problem for $\varepsilon$-scaled energy $E_\varepsilon(u) = \int_\Omega \frac{\varepsilon|\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}\, dx$, there are various interesting analysis of the singular limit as $\varepsilon$ tends to zero, in connection with the theory of the minimal surface. As a celebrated result, Modica and Mortola in [16] and Sternberg in [20] proved the $\Gamma$-convergence of the energy to the perimeter of the minimal area surface. In [15] Luckhaus and Modica showed that for the minimization problem of $E_\varepsilon$ under the volume constraint, the Lagrange multiplier in the Euler-Lagrange equation, converges to the mean curvature of the minimal area surface. Moreover in the general critical point, Hutchinson and Tonegawa proved that the chemical potential converges to the mean curvature of the limit interface in varifold sense in [9].

Concerning the time evolution equation, for a pair of solution $(u, v)$ of (1.8) it holds that

$$
\begin{aligned}
\frac{d}{dt}E(u) &= \int_\Omega (-\Delta u + W'(u))u_t\, dx = \int_\Omega -v(-\delta\Delta v + v)\, dx \\
&= -\int_\Omega \delta|\nabla v|^2 + v^2\, dx \le 0,
\end{aligned} \tag{1.10}
$$

which implies that the equation (1.3) is a flow for the free energy functional $E(u)$.

For the mathematical setting we introduce

$$
H^2_{bc}(\Omega) := \left\{ f \in H^2(\Omega) \mid \frac{df}{d\nu} = 0 \ \ on \ \ \partial\Omega \right\} \tag{1.11}
$$

and

$$
H^4_{bc}(\Omega) := \left\{ f \in H^4(\Omega) \mid \frac{df}{d\nu} = \frac{d\Delta f}{d\nu} = 0 \ \ on \ \ \partial\Omega \right\}. \tag{1.12}
$$

We remark that equivalences of norms in these spaces are known, referred to [17], that is, for any $\eta > 0$,

$$
\{\|\Delta u\|^2_{L^2(\Omega)} + \eta\|u\|^2_{L^2(\Omega)}\}^{\frac{1}{2}} \tag{1.13}
$$

are norms on $H^2_{bc}(\Omega)$ which are equivalent to the $H^2(\Omega)$-norm. Similarly,

$$\{\|\Delta^2 u\|^2_{L^2(\Omega)} + \eta\|u\|^2_{L^2(\Omega)}\}^{\frac{1}{2}} \tag{1.14}$$

are norms on $H^4_{bc}(\Omega)$ which are equivalent to the $H^4$-norm.

The weak formulation of the problem is the following;

**Definition 1.1.** *We say a function $u \in L^2(0,T;H^2_{bc}(\Omega))$ with $u_t \in L^2(0,T;(H^1(\Omega))^{-1})$ is a weak solution of the equation (1.1) if*

$$\int_\Omega u_t\eta\, dx = \int_\Omega -\delta(\Delta u - W'(u))\Delta\eta + (\Delta u - W'(u))\eta\, dx. \tag{1.15}$$

*for each $\eta \in H^2_{bc}(\Omega)$ and a.e. time $t \in [0,T]$ and*

$$u(x,0) = u_0(x). \tag{1.16}$$

**Remark 1.2.** *For the pairing in LHS of (1.15), we note that $L^2(\Omega) \subset (H^1(\Omega))^{-1} \subset (H^2(\Omega))^{-1}$. Since if $u \in L^2(0,T;H^2_{bc}(\Omega))$ with $u_t \in L^2(0,T;(H^1(\Omega))^{-1})$ then $u \in C([0,T];L^2(\Omega))$, thus the equality (1.16) makes sense.*

Throughout this paper, different positive constants will be denoted by the same letter $c$. We write $c(s)$ when it is helpful to write out the dependence of $c$ on $s$.

Next, we prove existence for the quartic potential $W$ in space dimension $1 \le n \le 4$ by the Galerkin method.

## 2. The existence theorem

We obtain the following existence theorem.

**Theorem 2.1.** *Let $\Omega$ be a bounded domain with a $C^4$-boundary in $\mathbb{R}^n$ for dimension $1 \le n \le 4$. Suppose the initial data $u_0 \in H^1(\Omega)$ then there exists a solution $u$ of the initial boundary problem (1.1) satisfying*

$$u \in C([0,T];H^1(\Omega)) \cap L^2(0,T;H^2_{bc}(\Omega)) \cap L^4((0,T)\times\Omega) \quad \text{for all} \quad T > 0. \tag{2.1}$$

*Additionally, the function $v$ given by (1.7) satisfies $v \in L^2(0,T;H^1(\Omega))$.*

*Moreover in dimension $n = 1,2,3$, if $\partial\Omega$ is $C^\infty$ and the initial data $u_0 \in H^2(\Omega)$, then*

$$u \in C([0,T];H^2_{bc}(\Omega)) \cap L^2(0,T;H^4_{bc}(\Omega)) \quad \text{for all} \quad T > 0. \tag{2.2}$$

**Remark 2.2.** *The same claim also holds under the periodic boundary condition. In the proof for simplicity we consider the explicit and typical potential, given by $W(s) = (1-s^2)^2$ without loss of generality.*

**Remark 2.3.** *For $\varepsilon$-scaled problem, we can also prove the existence of solution if we assume the boundedness of the initial energy.*

*Proof.* **(STEP1)** First we consider the case of the initial value $u_0 \in H^1(\Omega)$. We show the existence of the solution by applying a Galerkin method. Let $\{\lambda_i\}_{i\in\mathbb{N}}$ be eigenvalues and $\{\phi_i\}_{i\in\mathbb{N}}$ be eigenfunctions of Laplacian under the Neumann boundary condition

$$-\lambda_i\phi_i = \Delta\phi_i \quad \text{in } \Omega, \qquad \frac{\partial\phi_i}{\partial\nu} = 0 \quad \text{on } \partial\Omega \quad \text{for } i = 1, 2, \cdots . \tag{2.3}$$

We can assume that the first eigenvalue $\lambda_1 = 0$ and the normalization condition $(\phi_i, \phi_j)_{L^2(\Omega)} = \delta_{ij}$ for $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ without loss of generality. For every $N \in \mathbb{N}$ we consider the following function $u^N$ defined by the Galerkin ansatz

$$u^N(x,t) = \sum_{i=1}^{N} a_i^N(t)\phi_i(x), \tag{2.4}$$

$$\int_\Omega u_t^N\phi_j\,dx + \delta\Delta u^N\Delta\phi_j - \delta W'(u^N)\Delta\phi_j - \Delta u^N\phi_j + W'(u^N)\phi_j\,dx = 0 \tag{2.5}$$

for $j = 1, \cdots, N$, and

$$u^N(x,0) = \sum_{i=1}^{N}(u_0, \phi_i)_{L^2(\Omega)}\,\phi_i(x) \qquad \text{for } j = 1, \cdots, N. \tag{2.6}$$

This yields the following initial value problem of ODE for $a_j^N(t)$ for $j = 1, \cdots, N$

$$\frac{d}{dt}a_j^N(t) + \delta\lambda_j^2 a_j^N(t) + \delta\lambda_j(W'(u^N), \phi_j)_{L^2(\Omega)} + \lambda_j a_j^N + (W'(u^N), \phi_j)_{L^2(\Omega)} = 0. \tag{2.7}$$

and

$$a_j^N(0) = (u_0, \phi_j)_{L^2(\Omega)}. \tag{2.8}$$

By the standard argument of ODE, this initial value problem has a local solution. We want to show that a global solution $\{a_j^N\}_j^N$ exists on $(0,T)$ for any $T > 0$.

By multiplying $\phi_j u_N$ for each $j = 1, \cdots, N$ by both side of (2.7), taking $\sum_{j=1}^N$ and integrating, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |u^N|^2\,dx + \int_\Omega \delta|\Delta u^N|^2 + 12\delta|\nabla u^N|^2|u^N|^2 + |\nabla u^N|^2 + |u^N|^4\,dx \\ &= 4\int_\Omega |u^N|^2\,dx + 4\delta\int_\Omega |\nabla u^N|^2\,dx. \end{aligned} \tag{2.9}$$

For the second term of RHS of (2.9), by interpolation and the equivalence of the norm (1.13), we have

$$
\begin{aligned}
4\delta \int_\Omega |\nabla u^N|^2\, dx &\le c\delta \|u^N\|_{L^2(\Omega)} \|u^N\|_{H^2(\Omega)} \\
&\le c\delta \|u^N\|_{L^2(\Omega)} \{\int_\Omega |\Delta u^N|^2\, dx + \|u^N\|^2_{L^2(\Omega)}\}^{\frac{1}{2}} \\
&\le c\delta \int_\Omega |u^N|^2\, dx + \frac{\delta}{2}\int_\Omega |\Delta u^N|^2\, dx.
\end{aligned} \tag{2.10}
$$

By (2.9) and (2.10), we have

$$
\frac{d}{dt}\int_\Omega |u^N|^2\, dx + \int_\Omega \frac{\delta}{2}|\Delta u^N|^2 + |\nabla u^N|^2 + |u^N|^4\, dx \le c\int_\Omega |u^N|^2\, dx. \tag{2.11}
$$

Thus by Gronwall's inequality for an arbitrary fixed $T > 0$, we have

$$
\int_\Omega |u^N|^2\, dx \le c\, e^{cT} \int_\Omega |u^N(x,0)|^2\, dx. \tag{2.12}
$$

By the definition of $a_j^N(0)$ in (2.8), we have

$$
\begin{aligned}
\int_\Omega |u^N(x,0)|^2\, dx &= \sum_{j=1}^N |a_j(0)|^2 = \sum_{j=1}^N \{\int_\Omega u_0(x)\phi_j(x)\, dx\}^2 \\
&\le \sum_{j=1}^\infty \{\int_\Omega u_0(x)\phi_j(x)\, dx\}^2 = \int_\Omega |u_0|^2\, dx.
\end{aligned} \tag{2.13}
$$

Thus by (2.12), (2.11) and (2.13), $u^N$ is bounded independently of $N$ in $L^\infty(0,T;L^2(\Omega))$, $L^2(0,T;H^2(\Omega))$ and $L^4((0,T)\times\Omega)$.

Since $\|u^N\|_{L^2(\Omega)} = \sum_{i=1}^N (a_i^N(t))^2$, by $L^\infty(0,T;L^2(\Omega))$-bound of $u^N$ we obtain a priori bound of $a_j^N$ for $j = 1, \cdots, N$. Thus the ODE (2.7) and (2.8) have a global solution.

In order to estimate of $\|\nabla u^N\|_{L^\infty(0,T;L^2(\Omega))}$ we consider the energy $E(u^N)$. We set $b_j^N(t)$ and $v^N(x,t)$ such as

$$
b_j^N = -\lambda_j a_j^N(t) - (W'(u^N), \phi_j)_{L^2(\Omega)} \tag{2.14}
$$

and

$$
v^N(x,t) = \sum_{j=1}^N b_j^N(t)\phi_j(x). \tag{2.15}
$$

By the definition of $v^N$ and $b_j^N$, we have

$$-\int_\Omega \delta|\nabla v^N|^2 + |v^N|^2 \, dx = \frac{d}{dt}\int_\Omega \frac{|\nabla u^N|^2}{2} + W(u^N) \, dx. \tag{2.16}$$

By integrating with respect to $t \in (0, T]$, we have

$$\int_0^t \int_\Omega \delta|\nabla v^N|^2 + |v^N|^2 \, dx \, dt + E(u^N(t)) = E(u^N(0)) \tag{2.17}$$

Since the initial data $u_0$ is $H^1$-function, the initial energy $E(u_0)$ is well-defined in dimension $1 \le n \le 4$ by the Sobolev inequality and we denote $E_0 := E(u_0)$. We claim that there exists a constant $c_{E_0} = c_{E_0}(E_0, \|u_0\|_{H^1(\Omega)}) > 0$ such that

$$E(u^N(0)) \le c_{E_0}. \tag{2.18}$$

Indeed, by (2.6) we have

$$\begin{aligned} \int_\Omega |\nabla u^N(0)|^2 \, dx &= \sum_{i=1}^N (u_0, \phi_i)^2_{L^2(\Omega)} \int_\Omega |\nabla \phi_i|^2 \, dx = \sum_{i=1}^N (u_0, \phi_i)^2_{L^2(\Omega)} \lambda_i \\ &\le \sum_{i=1}^\infty (u_0, \phi_i)^2_{L^2(\Omega)} \lambda_i = \int_\Omega |\nabla u_0|^2 \, dx \le E_0. \end{aligned} \tag{2.19}$$

For the second term of the energy,

$$\int_\Omega W(u^N(0)) \, dx \le c(1 + \int_\Omega |u^N(0)|^4 \, dx). \tag{2.20}$$

By (2.13) and (2.19), we notice that

$$\|u^N(0)\|_{H^1(\Omega)} \le \|u_0\|_{H^1(\Omega)}. \tag{2.21}$$

In dimension $n = 1$, by (2.21) and the Sobolev inequality, we have

$$\|u^N(0)\|_{C(\overline{\Omega})} \le c\|u^N(0)\|_{H^1(\Omega)} \le c\|u_0\|_{H^1(\Omega)}. \tag{2.22}$$

In dimension $n = 2, 3, 4$, similarly by (2.21) and the Sobolev inequality, we have

$$\|u^N(0)\|_{L^4(\Omega)} \le c\|u^N(0)\|_{H^1(\Omega)} \le c\|u_0\|_{H^1(\Omega)}. \tag{2.23}$$

By (2.20), (2.22) and (2.23), the potential term of the energy is bounded. Thus the claim (2.18) holds and by (2.17) and (2.18), $\|\nabla u^N\|_{L^\infty(0,T;L^2(\Omega))}$ is bounded. Thus $u^N \in L^\infty(0, T; H^1(\Omega))$ and there exists a constant $c_{H^1} > 0$ independent of N satisfying

$$\|u^N\|_{L^\infty(0,T;H^1(\Omega))} \le c_{H^1}. \tag{2.24}$$

Moreover, we choose $t = T$ in (2.17), by (2.18) we obtain

$$\|v^N\|_{L^2(0,T;H^1(\Omega))} \le c(\delta). \tag{2.25}$$

Let $\Pi_N$ be a projection of $L^2(\Omega)$ onto $span\{\phi_1, \cdots, \phi_N\}$. For all $\zeta \in L^2(0,T;H^1(\Omega))$ by (2.7), (2.14) and (2.15) we have

(2.26)

$$\begin{aligned}
|\int_0^T \int_\Omega u_t^N \zeta \, dxdt| &= |\int_0^T \int_\Omega \partial_t u^N \Pi_N \zeta \, dxdt| \\
&= |\int_0^T \int_\Omega -\delta \nabla v^N \nabla \Pi_N \zeta + \int_0^T \int_\Omega v^N \Pi_N \zeta \, dxdt| \\
&\le c\|\nabla v^N\|_{L^2(\Omega\times(0,T))} \|\nabla \zeta\|_{L^2(\Omega\times(0,T))} + \|v^N\|_{L^2(\Omega\times(0,T))} \|\zeta\|_{L^2(\Omega\times(0,T))} \\
&\le c\|v^N\|_{L^2(0,T;H^1(\Omega))} \|\zeta\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

Thus by (2.26) we have

$$\|\partial_t u^N\|_{L^2(0,T;(H^1(\Omega))^{-1})} \le c. \tag{2.27}$$

Together with $L^4((0,T)\times\Omega)$ and $L^2(0,T;H^2(\Omega))$ boundedness of $u^N$, (2.24), (2.25) and (2.26) by the compactness results in [14], there exist $(u,v)$ and a subsequence, which we denote $\{u^N\}$ and $\{v^N\}$ again, such that

$$u^N \to u \quad \text{weak} - * \ \text{ in } \ L^\infty(0,T;H^1(\Omega)), \tag{2.28}$$

$$u^N \to u \quad \text{weakly in } \ L^2(0,T;H^2(\Omega)) \ \ and \ \ L^4((0,T)\times\Omega), \tag{2.29}$$

$$u^N \to u \quad \text{strongly in } \ C([0,T];L^2(\Omega)), \tag{2.30}$$

$$u_t^N \to u_t \quad \text{weakly in } \ L^2(0,T;\{H^1(\Omega)\}^{-1}), \tag{2.31}$$

$$u^N \to u \quad \text{strongly in } \ L^2(0,T;L^2(\Omega)) \ \text{ and a.e. in } \ \Omega\times(0,T) \tag{2.32}$$

and

$$v^N \to v \quad \text{weakly in } \ L^2(0,T;H^1(\Omega)) \tag{2.33}$$

as $N$ tends to $\infty$. Consequently, we can pass to the limit in (2.4), (2.5) and (2.6) and we obtain

(2.34)

$$\int_\Omega u_t\phi_j + \delta\Delta u \Delta\phi_j - \delta W'(u)\Delta\phi_j - \Delta u \phi_j + W'(u)\phi_j \, dx = 0 \qquad \text{for} \quad j = 1, \cdots, N.$$

Let $\eta$ be an arbitrary function in $L^2(0,T;H^2_{bc}(\Omega))$ and $\{\eta_k\}$ be its approximation given by the form

$$\eta_k(x,t) = \sum_{i=1}^{k} d_i^k(t)\phi_i(x) \tag{2.35}$$

where $\{d_i\}_{i=1}^k$ are smooth functions. As the function of the form (2.35) are dense in $L^2(0,T;H^2_{bc}(\Omega)$ we can take such a sequence. By (2.34) the following equality holds

$$\int_0^T\int_\Omega u_t\eta_k + \delta\Delta u\Delta\eta_k - \delta W'(u)\Delta\eta_k - \Delta u\eta_k + W'(u)\eta_k \, dxdt = 0. \tag{2.36}$$

Taking limit with respect to $k$, for $\eta$ we have

$$\int_0^T\int_\Omega u_t\eta + \delta\Delta u\Delta\eta - \delta W'(u)\Delta\eta - \Delta u\eta + W'(u)\eta \, dxdt = 0. \tag{2.37}$$

Hence in particular, for each $\eta \in H^2_{bc}(\Omega)$ and *a.e.* $0 \le t \le T$

$$\int_\Omega u_t\eta + \delta\Delta u\Delta\eta - \delta W'(u)\Delta\eta - \Delta u\eta + W'(u)\eta \, dx = 0. \tag{2.38}$$

For the convergence of the initial value $u^N(0)$, by the strong convergence of $u^N$ in $C([0,T];L^2(\Omega))$, $u^N(0)$ converges to $u_0$ in $L^2(\Omega)$. Thus we have that $u(0) = u_0$. Then the first claim of the theorem holds.

**(STEP 2)** Next we consider the case of the initial value $u_0 \in H^2(\Omega)$. Adding to the previous calculation, we consider the bound of $\sup_{t\in(0,T)} \|\Delta u^N\|_{L^2(\Omega)}$. By multiplying $\phi_j\Delta^2u^N$ for $j = 1,\cdots,N$ by both side of (2.7), taking $\sum_{j=1}^N$ and integrating, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\Delta u^N|^2 dx + \int_\Omega \delta|\Delta^2 u^N|^2 + (D(\Delta u^N))^2 \, dx \\ &= -\int_\Omega \delta\Delta W'(u^N)\Delta^2 u^N + \Delta W'(u^N)\Delta u^N \, dx. \end{aligned} \tag{2.39}$$

For the RHS of (2.39) by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega |\Delta u^N|^2 dx + \int_\Omega \frac{\delta}{2}|\Delta^2 u^N|^2 + (D(\Delta u^N))^2 \, dx \\ &\le c\int_\Omega \delta|\Delta W'(u^N)|^2 \, dx + c\int_\Omega |\Delta u^N|^2 \, dx. \end{aligned} \tag{2.40}$$

For the first term of (2.40) we claim that

$$\int_\Omega \delta|\Delta W'(u^N)|^2\,dx \le \frac{\delta}{4}\int_\Omega |\Delta^2 u^N|^2\,dx + c\int_\Omega |\Delta u^N|^2\,dx + c. \tag{2.41}$$

Indeed, since $\Delta W'(u^N) = W'''(u^N)|Du^N|^2 + W''(u^N)\Delta u^N$, we have

$$\begin{aligned}
&\int_\Omega |\Delta W'(u^N)|^2\,dx \\
&\le c\int_\Omega |u^N|^2|Du^N|^4\,dx + c\int_\Omega (1+|u^N|^4)|\Delta u^N|^2\,dx \\
&\le c\|u^N\|^2_{L^\infty(\Omega)}\|Du^N\|^4_{L^4(\Omega)} + c\|u^N\|^4_{L^\infty(\Omega)}\int_\Omega |\Delta u^N|^2\,dx + c\int_\Omega |\Delta u^N|^2\,dx.
\end{aligned} \tag{2.42}$$

For the term $\|Du^N\|^4_{L^4(\Omega)}$, by the Sobolev inequality, interpolation and (1.14), we have

$$\begin{aligned}
\|Du^N\|_{L^4(\Omega)} \le& c\|u^N\|_{H^{1+\frac{n}{4}}(\Omega)} \\
&\le c\|u^N\|^{1-\frac{n}{12}}_{H^1(\Omega)}\|u^N\|^{\frac{n}{12}}_{H^4(\Omega)} \\
&\le c(c_{H^1})(\|\Delta^2 u^N\|^2_{L^2(\Omega)}+1)^{\frac{n}{24}}.
\end{aligned} \tag{2.43}$$

For the term $\int_\Omega |\Delta u^N|^2\,dx$, by interpolation and (1.14) we have

$$\|\Delta u^N\|_{L^2(\Omega)} \le \|u^N\|_{H^2(\Omega)} \le c\|u^N\|^{\frac{2}{3}}_{H^1(\Omega)}\|u^N\|^{\frac{1}{3}}_{H^4(\Omega)} \le c(c_{H^1})(1+\|\Delta^2 u^N\|^2_{L^2(\Omega)})^{\frac{1}{6}}. \tag{2.44}$$

Thus by (2.42), (2.43) and (2.44) we have

$$\begin{aligned}
\int_\Omega |\Delta W'(u^N)|^2\,dx \le& c\|u^N\|^2_{L^\infty(\Omega)}(c+\|\Delta^2 u^N\|^2_{L^2(\Omega)})^{\frac{n}{6}} \\
&+ c\|u^N\|^4_{L^\infty(\Omega)}(c+\|\Delta^2 u^N\|^2_{L^2(\Omega)})^{\frac{1}{3}} + c\int_\Omega |\Delta u^N|^2\,dx.
\end{aligned} \tag{2.45}$$

For $\|u^N\|_{L^\infty(\Omega)}$, in dimension $n=1$ we have

$$\|u^N\|_{L^\infty(\Omega)} \le c\|u^N\|_{H^1(\Omega)} \le c\|u^N\|_{L^\infty(0,T;H^1(\Omega))} \le c. \tag{2.46}$$

In dimension $n=2$, since $H^{1+\varepsilon}(\Omega)\subset L^\infty(\Omega)$ for all $\varepsilon>0$, by taking $\varepsilon=\frac{1}{6}$ and interpolation, we have

$$\|u^N\|_{L^\infty(\Omega)} \le c\|u^N\|_{H^{1+\varepsilon}(\Omega)} \le c\|u^N\|^{1-\varepsilon}_{H^1(\Omega)}\|u^N\|^{\varepsilon}_{H^4(\Omega)} \le c(1+\|\Delta^2 u^N\|^2_{L^2(\Omega)})^{\frac{1}{12}}. \tag{2.47}$$

In dimension $n = 3$, since $\partial\Omega$ is smooth we can use Agmon's inequality and by (2.44) we have

$$\|u^N\|_{L^\infty(\Omega)} \le c\|u^N\|_{H^1(\Omega)}^{\frac{1}{2}} \|u^N\|_{H^2(\Omega)}^{\frac{1}{2}} \le c(c + \|\Delta^2 u^N\|_{L^2(\Omega)}^2)^{\frac{1}{12}}. \tag{2.48}$$

Thus together with (2.42), (2.43), (2.44), (2.45), (2.46) and (2.48) by the Cauchy-Schwarz inequality, we have

$$\int_\Omega |\Delta W'(u^N)|^2 \, dx \le \frac{\delta}{4} \int_\Omega |\Delta^2 u^N|^2 \, dx + c \int_\Omega |\Delta u^N|^2 \, dx + c. \tag{2.49}$$

Thus by (2.40) and (2.49), we have

$$\frac{d}{dt} \int_\Omega |\Delta u^N|^2 dx + \int_\Omega \frac{\delta}{4} |\Delta^2 u^N|^2 + (D(\Delta u^N))^2 \, dx \le c \int_\Omega |\Delta u^N|^2 \, dx + c. \tag{2.50}$$

By applying Gronwall's inequality again, we have

$$\sup_{t \in (0,T)} \int_\Omega |\Delta u^N|^2 \, dx \le c \int_\Omega |\Delta u^N(0)|^2 \, dx + c. \tag{2.51}$$

By the definition of $u^N$ in (2.4) and $a_j^N(0)$ in (2.8), we have

$$\int_\Omega |\Delta u^N(0)|^2 \, dx = \sum_{j=1}^N (\lambda_j a_j(0))^2 \le \sum_{j=1}^\infty (\lambda_j a_j(0))^2 = \int_\Omega |\Delta u_0|^2 \, dx. \tag{2.52}$$

By (2.50), (2.51) and (2.52), we have $u^N \in L^\infty(0,T; H^2_{bc}(\Omega)) \cap L^2(0,T; H^4_{bc}(\Omega))$. Thus we can take a subsequence satisfying

$$u^N \to u \quad \text{weak} - * \ \text{ in } \ L^\infty(0,T; H^2_{bc}(\Omega)) \tag{2.53}$$

and

$$u^N \to u \quad \text{weakly in } \ L^2(0,T; H^4(\Omega)) \tag{2.54}$$

adding to the previous convergence from (2.28) to (2.32) as $N$ tends to $\infty$. Therefore, all the claim of the theorem holds.

□

**Remark 2.4.** *If we consider the convergence of* (1.1) *to the Allen-Cahn equation with respect to the diffused constant $\delta$, we can not let $\delta \to 0$ in this proof since the estimates of the top term in* (2.11) *and* (2.40) *depend on $\delta$. But we can drop these terms by their positivity and it follows that if $u_0 \in H^1(\Omega)$ then $u \in C([0,T]; H^1(\Omega))$ and if $u_0 \in H^2(\Omega)$ then $u \in C([0,T]; H^2_{bc}(\Omega))$ independent of $\delta$.*

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