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On the parabolic Stefan problem form Ostwald ripening with kinetic undercooling and inhomogeneous driving force

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Abstract

Ostwald ripening is the coarsening phenomenon caused by the diffusion and solidification process which occurs in the last stage of a first-order phase transformation. The force that drives the system towards equilibrium is the gradient of the chemical potential that, according to the Gibbs-Thomson condition, on the interface, is proportional to its mean curvature. A quantitative description of Ostwald ripening has been developed by the LSW theory. We extend the work of Niethammer [14] which deals with kinetic undercooling in the quasi-static case to the parabolic setting with temporally inhomogeneous driving forces on the solid-liquid interfaces. By means of *a priori* estimates, local and global existence results for the parabolic Stefan problem, we derive a first order approximation for the dynamical equations for the heat distribution and particle radii and then prove the convergence to a limiting description using a mean-field equation.

Key words: Ostwald ripening, parabolic Stefan problem, kinetic undercooling, mean-field approximation.
2000 MSC: 35B27, 35R35, 80A22

1. Introduction

1.1. The physical model

Ostwald ripening or coarsening [15] is a diffusion and solidification process occurring in the last stage of a first-order phase transformation. Usually, any first-order phase transformation process results in a two phase mixture with a dispersed (solid) second phase in a background (liquid) phase ([16, 17]). Initially the average size of the dispersed particles is very small. Hence the interfacial energy of the system is very large and the mixture is thus not in thermodynamical equilibrium. The force that drives the system towards equilibrium is the gradient of the chemical potential. According to the Gibbs-Thomson condition, on the interface between the two phases, the value of this driving force is proportional to the mean curvature of the interface. As a result, matter diffuses from regions of high curvature to regions of low curvature. This leads to the growth of large particles at the expense of small ones which eventually shrink to vanish. The outcome of this process, known as the Ostwald ripening is the increase of the average particle size and the reduction of their number so that the mixture becomes coarser over time. A quantitative description of this process was first developed by Lifschitz and Slyozov [11] and independently by Wagner [18] under the assumption that the relative volume fraction of the dispersed phase is very small. The idea of the LSW theory is to make use of the growth velocity of an isolated particle. The interaction between the particles is captured through the average value of the background temperature field. This approach is thus called the *mean field approximation*.

More specifically, the LSW theory produces an equation for $n = n(R, t)$ the *number density* of the particles at time t as a function of radius R. This function is shown to satisfy the following equation:

$$\frac{\partial n(R,t)}{\partial t} + \frac{\partial}{\partial R}\Big(V(R,t)n(R,t)\Big) = 0 \tag{1}$$

where V is the *growth rate* of a particle of radius R:

$$V(R,t) = \frac{1}{R(t)}\left(\frac{1}{\overline{R}(t)} - \frac{1}{R(t)}\right) \tag{2}$$

and $\overline{R}(t)$ is the *average particle radius*:

$$\overline{R}(t) = \frac{\int Rn(R,t)dR}{\int n(R,t)dR}. \tag{3}$$

Note that by definition, $n(R,t)dR$ gives the number of particles at time t with radius in the range $[R, R + dR]$. Hence $\int n(R,t)\,dR$ is the total number of particles present at time t. The system (1) – (3) is analyzed in [11, 18]. It is argued that there exist infinitely many *self-similar solutions*, but only one is believed to describe the typical behavior of the system for large times. This is given by:

$$n_s(R,t) \cong \frac{1}{t^{\frac{4}{3}}}G\left(\frac{R(t)}{\overline{R}(t)}\right) \quad \text{where } G(\cdot) \text{ is some scaling function.} \tag{4}$$

Based on this, the following temporal laws are derived for the average radius and the total number of particles:

$$\overline{R}(t) \cong \left(\overline{R}^3(0) + \frac{4}{9}t\right)^{\frac{1}{3}} \quad \text{and} \quad N(t) \cong \left(\overline{R}^3(0) + \frac{4}{9}t\right)^{-1}. \tag{5}$$

There have been many mathematical works concerning the above description. It is a nontrivial step to connect statements (1) and (5) rigorously to the underlying diffusion and solidification process. The work [13] has given a mathematical justification for (1) and (2) by considering an isotropic approximation which allows the author to restrict attention to the class of spheres with center locations fixed throughout the evolution. In [1, 2, 8] the authors obtained precise expressions for the equations of the centers and the radii by taking also into account the geometry of the distribution thus removing these restrictive hypotheses.

It is the purpose of the present work to contribute further to the overall theory by incorporating *kinetic undercooling* and *temporally inhomogeneous driving forces* in the *parabolic* setting. Our results extend the work of [14] which deals with the quasi-static case.

1.2. Mathematical formulation — free boundary value problem

Now we describe the mathematical set-up for the diffusion and solidification process. In the following, we consider the growth of the solid phase of a substance in an undercooled liquid phase of the same substance. Assuming isotropic growth, one possible model is the following Stefan problem for the temperature field θ and the solid-liquid-interface Γ [7, 10]:

$$\begin{array}{lcll} C\partial_t\theta & = & K\Delta\theta & \text{in } \Omega_l \\ HV & = & -K\nabla\theta\cdot n & \text{on } \Gamma \\ V & = & -M(\theta_M\sigma k + H(\theta - \theta_M)) & \text{on } \Gamma \end{array} \tag{6}$$

where the liquid and solid phases are denoted by Ω_l and $\Omega_s = \mathbb{R}^3\backslash\Omega_l$ and $\Gamma = \partial\Omega_s$ is the solid-liquid interface. Note that these sets are all time dependent. In the above, K is the thermal diffusivity, C is the heat capacity, θ_M is the melting temperature at a flat interface, H is the latent heat, σ is the surface tension, M is a mobility coefficient, k denotes the mean curvature of Γ (which is positive for a ball), n is the outward normal to the solid phase, and V is the normal velocity of the interface. The first interfacial condition on Γ, also known as the *Stefan condition*, ensures local conservation of heat. The second condition, known as the *kinetic undercooling*, couples the geometry of the interface with the evolution of the temperature in the liquid phase Ω_l. The curvature term forces the system to reduce the surface area of the interface Γ. But in the case of undercooled liquid, the second term gives a growing tendency for the solid phase. In other words, these two terms compete against each other. The following equilibrium condition

$$\theta_M \sigma k + H(\theta - \theta_M) = 0 \tag{7}$$

formally derived by setting $V = 0$ or $M = \infty$ is called the *Gibbs-Thomson law* on the interface. It predicts that the melting temperature is reduced for small particles. It is this effect which provides the barrier for nucleation of solid and thus allows for the existence of undercooled liquid phase. Since during Ostwald ripening interfacial velocities are relatively small, the Gibbs-Thomson condition is often used as an approximation of the general growth law. Nevertheless, even for small interfacial velocities, the kinetic term in the boundary condition has a strong regularizing effect on small particles.

The above is one type of *free boundary value problems*. There are many mathematical works that tackle such problems. A local existence result relevant to our present work is [3]. Furthermore, there are many results on the Dirichlet problems in perforated domains. In order to derive the average equations that capture the behavior of the solutions in large spatial scales, it is found out that the *capacity of the holes* is a crucial quantity. Most closely related is the work [4] that considers Dirichlet problems in domains with holes in a similar setting. It proves that if the capacity does not vanish, the type of the limit equation changes. In [5], a simpler Stefan problem with zero boundary was studied in which the solid phase is not allowed to melt completely. This last mentioned work handles the case of finite capacity and hence it does not get a mean-field model in the limit.

1.3. Motivation for the current work

The motivations of the current work are two folds. First we want to extend the work of [14] to the parabolic setting. The cited work deals with kinetic undercooling in the quasi-static case. The work [13] studies both the quasi-static and parabolic case but without the effect of kinetic undercooling. Even though the strategy of attack follows closely to [13, 14], due to the combined presence of the parabolicity and the kinetic undercooling, some additional terms appear in the derivation of energy estimates and the construction of sub- and super-solutions. These terms require extra care in the analysis. Thus we feel that it is worthwhile to investigate more rigorously this case.

In addition, we want to consider the effect of inhomogeneous driving forces both in the spatial and temporal setting. Ideally, we would like to incorporate stochastic perturbations. Possible modification of (6) is the following:

$$\begin{array}{rcll} C\partial_t \theta &=& K\Delta\theta + \xi(x,t) & \text{in } \Omega_l \\ HV &=& -K\nabla\theta\cdot n & \text{on } \Gamma \\ V &=& -M(\theta_M\sigma k + H(\theta-\theta_M)) + \zeta(x,t) & \text{on } \Gamma \end{array} \tag{8}$$

where ξ and ζ are stochastic driving forces. A choice often used is some *white noise* in time and/or space (even though this is far from clear from a modeling point of view). However, a general theory of stochastic perturbation in moving boundary value problems is still not available at present, in particular the incorporation of white noise term into the free boundaries.

In order to understand the estimates involved, in the current paper, we restrict our attention to deterministic driving forces which perturb in time the dynamics of the solid-liquid interface Γ. Specifically, we set $\xi \equiv 0$ and ζ to be some time dependent function which can take on different values on separate parts of Γ. We believe the results obtained here can lead to useful understanding to the ultimate, more general stochastic case.

2. Mean field approximation

To simplify the analysis, it is convenient to non-dimensionalize equation (6). Let

$$y \to \frac{H}{\sigma}y, \quad t \to \frac{\theta_M K H}{\sigma^2}t, \quad v := \frac{\theta_M - \theta}{\theta_M}, \quad \lambda := \frac{C\theta_M}{H}, \quad \text{and} \quad \beta := \frac{K}{MH\sigma}.$$

Then (6), together with some inhomogeneous driving force $g(t)$ acting on the interface Γ can be written as

$$\begin{array}{rcll} \lambda\partial_t v &=& \Delta v & \text{in } \Omega_l \\ V &=& \nabla v\cdot n & \text{on } \Gamma \\ v + g(t) &=& k + \beta V & \text{on } \Gamma \end{array} \tag{9}$$

We will construct an approximation of the solution by making use of the idea that in the vicinity of a particle the solution should look approximately like the one for a single particle. Hence we first consider the single particle problem when the particle is a ball B_R of radius R centered at the origin:

$$\begin{array}{rcll} \lambda\partial_t v & = & \Delta v & \text{in } \mathbb{R}^3\backslash B_R \\ \dot{R} & = & \nabla v\cdot n & \text{on } \partial B_R \\ \beta\dot{R} & = & -\frac{1}{R}+v+g(t) & \text{on } \partial B_R \\ \lim\limits_{r\to\infty} v(r,t) & = & v_\infty(t). & \end{array} \tag{10}$$

Note that the mean-field value $v_\infty(t)$ is imposed as a boundary condition at *infinity*.

In the elliptic (quasi-static) case $\lambda = 0$, the solution of problem (10) at any time $t > 0$ can be explicitly given by

$$v(r,t) = v_\infty(t) + \frac{R(t)(1 - R(t)v_\infty(t) - R(t)g(t))}{r(\beta + R(t))} \tag{11}$$

and

$$\dot{R}(t) = -\frac{1 - R(t)v_\infty(t) - R(t)g(t)}{R(t)(\beta + R(t))}. \tag{12}$$

We first mention that the positivity of β indeed has a profound effect on the dynamics of particles, in particular near the time when the radius is about to vanish. When $R \ll 1$, if $\beta > 0$ (12) becomes:

$$\dot{R} \approx -\frac{1}{R\beta} \quad \text{and hence} \ \ R(t) \approx \left(C - \frac{2t}{\beta}\right)^{\frac{1}{2}}$$

while for $\beta = 0$,

$$\dot{R} \approx -\frac{1}{R^2} \quad \text{and hence} \ \ R(t) \approx (C - 3t)^{\frac{1}{3}}\,.$$

Even though the solution forms (11) and (12) are for the single particle case in the quasi-static situation, we expect them to be still a good approximation with multiple particles if $\lambda \ll 1$ and all the particles are far away from each other. In this case, the overall solution v of (10) is roughly given by the linear combination of the individual solutions:

$$v(y,t) \approx v_\infty(t) + \sum_i \frac{R_i(t)(1 - R_i(t)v_\infty(t) - R_i(t)g_i(t))}{(\beta + R_i(t))|y - y_i|}, \tag{13}$$

where i is the index of the particle with center at y_i and radius R_i.

To complete the picture, we need to specify the quantity $v_\infty(t)$ and its dynamics. Note that it is a spatially constant variable describing the heat distribution far away from the solid-liquid interfaces. This justifies the terminology *mean-field description*. Due to the assumption of small volume fraction (to be prescribed later), the overall background domain Ω is very close to the region Ω_l occupied by the liquid phase. Hence we have

$$v_\infty \approx \frac{1}{|\Omega|}\int_{\Omega_l} v.$$

We now compute

$$\partial_t \int_{\Omega_l} v = \int_{\Omega_l} \partial_t v - \int_{\partial\Omega_l} \dot{R}v = \int_{\Omega_l} \frac{1}{\lambda}\Delta v - \int_{\partial\Omega_l} \dot{R}v = -\int_{\partial\Omega_l} \frac{1}{\lambda}\nabla v\cdot n - \int_{\partial\Omega_l} \dot{R}v = -\frac{1}{\lambda}\int_{\partial\Omega_l} \dot{R} - \int_{\partial\Omega_l} \dot{R}v,$$

so that

$$\partial_t v_\infty \approx -\frac{1}{|\Omega|\lambda}\int_{\partial\Omega_l} \dot{R} - \frac{1}{|\Omega|}\int_{\partial\Omega_l} \dot{R}v.$$

Since λ is small, the second term is negligible. Note that $\partial\Omega_l = \cup_i \partial B(y_i, R_i)$, by (12) we then get

$$\partial_t v_\infty \approx \frac{1}{|\Omega|\lambda}\sum_i \left(\frac{1 - R_i v_\infty - R_i g_i(t)}{R_i(\beta + R_i)}\right) 4\pi R_i^2 \tag{14}$$

The purpose of the current work is to derive rigorously the solution formulae (12), (13) and (14) from the free boundary value problem (9) and give a limiting homogenized description for a large number of particles.

3. Rescaling of the problem

In this section, we introduce the spatial rescaling of the Stefan problem (9) so as to derive a limiting homogenized equation.

We consider the case that the solid phase $\Omega_s = \Omega\backslash\Omega_l$ consists of a collection of N disjoint balls, i.e.

$$\Omega_s = \bigcup_{i=1}^{N} B(y_i, R_i) \quad \text{and} \quad \Gamma = \bigcup_{i=1}^{N} \partial B(y_i, R_i). \tag{15}$$

We further assume that the centers of the balls do not move and the spherical shapes are preserved during the evolution (see **Remark 3.1** for a discussion). Strictly speaking, there is no solution satisfying the above assumptions. As in [13, 14], we replace the second condition of (9) by the following integral condition:

$$V_i := V\big|_{\partial B_i} = \frac{1}{|\partial B_i|}\int_{\partial B_i} \nabla v(y,t)\cdot n ds \quad (\text{where } ds \text{ is the area element and } B_i = B(y_i, R_i).) \tag{16}$$

Since $V_i = \dot{R}_i$, $k_i := k|_{\partial B_i} = \dfrac{1}{R_i}$, and $g_i := g\big|_{\partial B_i}$, the third condition of (9) is transformed into

$$v = \beta \dot{R}_i(t) + \frac{1}{R_i(t)} - g_i(t) \quad \text{on } \partial B(y_i, R_i(t)). \tag{17}$$

Note that now v is constant on each of $\partial B(y_i, R_i(t))$.

To model the facts that the volume occupied by the solid phase is very small compared to the vessel's volume (i.e. $\mathrm{Vol}(\cup_i B_i) \ll \mathrm{Vol}(\Omega)$) while the inter-particle distances are very large compared with the particle size, we apply the same spatial rescaling as in [13, 14]. We use δ and δ^a to denote the typical length scales for the inter-particle distance and the particle radii and consider the regime $0 < \delta^a \ll \delta$. Now introduce the following change of variables

$$x = \delta^a y \quad \text{and} \quad u(x,t) = v(y,t); \tag{18}$$

$$R_i^\delta(t) := \tfrac{R_i(t)}{\delta^a} \text{ and } B_i^\delta(t) := B\Big(x_i, \delta^a R_i^\delta(t)\Big) = B\Big(y_i, R_i(t)\Big). \tag{19}$$

Let further

$$N^\delta(t) := \Big\{i:\ 1 \le i \le N \text{ such that } R_i^\delta(t) > 0\Big\} \quad \text{and} \quad t_i^\delta := \sup\Big\{t : R_i^\delta(t) > 0\Big\}$$

be the index of particles at time t and the maximum existence time of B_i^δ. Define also the following domains:

$$\Omega^\delta(t) := \Omega\backslash \bigcup_{i\in N^\delta(t)} \overline{B_i^\delta(t)},\ \ \Omega_T^\delta := \bigcup_{t\in(0,T)} \Big(\Omega^\delta(t)\times\{t\}\Big) \text{ and } \Omega_T := \Omega\times(0,T), \tag{20}$$

where T is some finite fixed time instant.

With the above scaling variables, we are working in the regime that the particles are separated from each other by distances of at least of order $O(\delta)$. Hence $\left|N^\delta(t)\right| = O\left(\delta^{-3}\right)$. A simple such setting is to have the particles located on a regular three dimensional lattice of lattice length δ although this is not absolutely necessary.

Now using the variables x and R_i^δ's, upon choosing $\delta^a = \delta^4$ (see the **Remark 3.1** right afterward), the system of equations (9), adjoined with the Neumann condition on $\partial\Omega_T = \partial\Omega\times(0,T)$ leads to the following initial boundary value problem (IBVP):

$$\begin{aligned}
\lambda\delta u_t &= \delta^8 \Delta u, \quad \text{in } \Omega_T^\delta,\\
u(x,t) + g_i(t) &= \frac{1}{R_i^\delta(t)} + \frac{\beta}{4\pi\delta^4 (R_i^\delta(t))^2}\int_{\partial B_i^\delta(t)} \nabla u\cdot n ds, \quad x\in\partial B_i^\delta(t),\ \ t\in(0,t_i^\delta),\\
\dot{R}_i^\delta(t) &= \frac{1}{4\pi\delta^4 (R_i^\delta(t))^2}\int_{\partial B_i^\delta(t)} \nabla u\cdot n ds, \quad t\in(0,t_i^\delta),\\
R_i^\delta(t) &= 0, \quad t > t_i^\delta,\\
\nabla u\cdot n &= 0, \quad \text{on } \partial\Omega_T,\\
u(x,0) &= u_0(x), \quad \text{in } \Omega^\delta(0),\\
R_i^\delta(0) &= R_{i0}^\delta, \quad \text{for } i\in N^\delta(0).
\end{aligned} \tag{21}$$

The main purpose of this paper is to give a limiting description as δ converges to zero. The following are some remarks about scalings and assumptions used in the problem.

Remark 3.1.

1. *As the size of the solid grains is assumed small compared with the mean distance between them, the direct interactions between the particles are thus negligible and they behave as if they were isolated. Hence, we assume that they stay spherical and their centers do not move in space. On the other hand, models incorporating the non-spherical shape and the particle motion have been considered [1, 2, 8] in which it is shown that these additional features only constitute to higher order effects and hence they will not affect the mean field limit.*
2. *In this model we consider a large number of particles with small volume fraction. We assume that the centers of the spherical particles are separated by the length scale δ, i.e. $\inf_{i\neq j}|x_i - x_j| > c\delta$ for some fixed constant $c > 0$. Furthermore, the size of the particles is of the order δ^a with $a > 1$ leading to the well-separatedness of the particles.*
   The quantity δ^{a-3} gives the order of the capacity *of the balls in Ω. In order to obtain the mean-field model in the limit $\delta \longrightarrow 0$ it is necessary for the capacity to* vanish. *Hence we take $a = 4$. In this case the capacity is of order δ and the volume fraction is of order δ^9.*
   The choice of a scaling $\lambda = \delta^9$ will be clear from the energy type identities derived in **Section 5**.
3. *The initial data u_0 takes the following form:*

$$u_0^\delta = u_{0\infty}^\delta + \sum_i \frac{\left(1 - R_{i0}^\delta u_{0\infty} - R_{i0}^\delta g_{i0}\right)\delta^4 R_{i0}^\delta}{\left(R_{i0}^\delta + \beta\right)|x - x_i|}\eta\left(\frac{|x - x_i|}{\delta}\right) \tag{22}$$

   for some constant $u_{0\infty}^\delta$. In the above, η is a smooth cut-off function such that $\eta(r) \equiv 1$ for $0 \le r \le \frac{1}{8}$ and $\eta(r) \equiv 0$ for $r \ge \frac{1}{4}$. Furthermore, the initial radii R_{i0}^δ's satisfy

$$\sup_i R_{i0}^\delta \le R_0^\delta < \infty \tag{23}$$

4. *The inhomogeneous driving forces satisfy:*

$$\sup_i \sup_{t\ge 0}\left\{ |g_i(t)|\,, \left|R_i^\delta(t)\dot{g}_i(t)\right| \right\} \le M < \infty. \tag{24}$$

   The above are sufficient to derive the a priori estimates. However, in order to have a limit equation in closed *form, we do need to make the assumption that each g_i is a function of the radius R_i. This is stated as follows:*

   there exists a function $G \in C^1(R_+ \times R_+)$ and a function $h \in C^1(R_+)$ such that $g_i(t) = G(t, R_i^\delta(t)) + h(t)$. (25)

   See **Remark 9.3** *for further discussion.*

The rest of the paper is organized as follows: In **Section 4** local existence of a unique solution for the problem (21) is established under the assumption of regular initial data, while *a priori* estimates are presented in **Section 5**. **Section 6** refers to the radii regularity; we first present an appropriate maximum principle and construct super- and sub-solutions for our problem in order to derive a global regularity theorem for the heat distribution and the evolving radii. After this, our approach follows quite closely to that of [13]. More specifically, in **Section 7** we construct a first order approximation for the heat distribution, while the construction of a first order approximation for the radii is analyzed in **Section 8**. Finally the derivation of the limit equations as $\delta \longrightarrow 0$ is presented in **Section 9**.

The overall strategy is briefly explained here. We extend the local in time solution to globally existing solution, i.e. beyond the times when some balls disappear, by establishing *a priori* estimates using integral inequalities and maximum principle. When both λ and β are positive, as in the case of the parabolic problem with kinetic undercooling, when deriving these estimates we need to control the appearing terms involving $R_i^\delta \dot{R}_i^\delta$ uniformly in δ and globally in time, even after some balls have vanished.

We estimate the growth and decay of the radii $R_i^\delta(t)$'s. First we analyze the one single particle case. The important issue is to investigate the solution as $R \longrightarrow 0^+$ when $\delta \ll 1$. The main conclusion is that $|R\dot{R}| < C < \infty$ and $\lim_{R\to 0^+} R\dot{R} = -\frac{1}{\beta}$ (these results state the regularizing effect of kinetic undercooling) and thus $R \in W^{1,p}([0,T])$ for any $1 \leq p < 2$. The previous is established by constructing proper sub- and super-solutions. It is first done for the case $R \ll 1$ and $\dot{R} < 0$. If $R > O(1)$, we show that $|\dot{R}|$ is uniformly bounded. Moreover, we prove that once $R(t)$ reaches below some small value, $\dot{R}$ will become *negative* and will *stay negative* until the extinction time of $R(t)$. We then employ the previous analysis to prove *a priori* bounds for the multiple particle case. The extension of solution beyond vanishing time follows by the energy estimates from **Proposition 5.3** and standard parabolic theory.

In order to derive the limiting equation for the dynamics of the mean field variable and radii as $\delta \longrightarrow 0$, we produce a first order approximation for the heat distribution. In particular, we prove that far away from the particles, the heat distribution u^δ is close to the mean field variable u_∞^δ. Further, we establish the main result of this paper in **Theorem 8.1** which gives the dynamics of the radii as $\delta \longrightarrow 0$: the radii satisfy the following dynamical equation in the weak sense:

$$\dot{R}_i^\delta = -\frac{1 - u_\infty^\delta R_i^\delta - g_i R_i^\delta}{R_i^\delta (R_i^\delta + \beta)} + O(\delta^\gamma), \quad 0 < \gamma < \frac{1}{2}.$$

Finally, we discuss the limit of u^δ and R_i^δ's as $\delta \to 0$. In order to obtain an equation which is closed in the limit, we do need to invoke the assumption (25) on the form of the inhomogeneous forces g_i's. We denote that this assumption is useful for the definition of a white noise model.

4. Local in Time Existence and Uniqueness

We assume that for $T > 0$ the evolution of the radii R_i^δ for any $1 \leq i \leq N$ is given in $(0,T)$, and R_i^δ, g_i are sufficiently smooth. We define first for any $t \in (0,T)$ the vectors $R^\delta := (R_1^\delta, \cdots, R_N^\delta)$, $\dot{R}^\delta := (\dot{R}_1^\delta, \cdots, \dot{R}_N^\delta)$. By the following Theorem, we prove the existence of a unique weak solution for the problem (21) under the assumption of regular initial data. We define as $\|f\| := \left(\int_0^T |f(t)|^2 dt\right)^{1/2}$ the L^2-norm in $(0,T)$ and let $H^1(0,T) := \left\{ f \in L^2(0,T) : \int_0^T \left(\|f\|^2 + \|f_t\|^2\right) dt < +\infty \right\}$ be the usual Sobolev space in $(0,T)$. For t fixed let $H^1\left(\Omega^\delta(t)\right)$ be the Sobolev space in $\Omega^\delta(t)$, while $L^\infty\left(0,T; L^2\left(\Omega^\delta\right)\right) := \left\{ g : \Omega_T^\delta \to \mathbb{R} \text{ such that } \left\| \int_{\Omega^\delta(t)} |g(x,t)|^2 \, dx \right\|_{L^\infty(0,T)} < +\infty \right\}$ and $L^2\left(0,T; H^1\left(\Omega^\delta\right)\right) := \left\{ g : \Omega_T^\delta \to \mathbb{R} \text{ such that } \int_0^T \|g(\cdot,t)\|^2_{H^1(\Omega^\delta(t))} \, dt < +\infty \right\}$.

Theorem 4.1. *Let R_i^δ and g_i be given such that for some $T > 0$ and $0 < c < \infty$, they satisfy:*

$$\sup_{i\in N}\left(\|R_i^\delta\|_{L^\infty(0,T)} + \left\| \frac{1}{R_i^\delta} + \beta \dot{R}_i^\delta \right\|_{L^\infty(0,T)} + \|(R_i^\delta)^{-1}\|_{L^\infty(0,T)} + \|g_i\|_{L^\infty(0,T)} \right) < c. \tag{26}$$

Consider the problem

$$\begin{array}{rcll} \lambda u_t &=& \delta^8 \Delta u, & \text{in } \Omega_T^\delta, \\ u(x,t) &=& \frac{1}{R_i^\delta(t)} + \beta \dot{R}_i^\delta(t) - g_i(t), & x \in \partial B_i^\delta(t), \\ \nabla u \cdot n &=& 0, & \text{on } \partial\Omega_T, \\ u(x,0) &=& u_0(x), & \text{in } \Omega^\delta(0). \end{array} \tag{27}$$

If $u_0 \in H^1\left(\Omega^\delta(0)\right)$, then the above problem admits a unique weak solution $u \in L^\infty\left(0,T; L^2\left(\Omega^\delta\right)\right) \cap L^2\left(0,T; H^1\left(\Omega^\delta\right)\right)$.

We first give some remark before the proof. Note that as $\frac{1}{R_i^\delta}$, $\frac{1}{R_i^\delta} + \beta \dot{R}_i^\delta$ are uniformly bounded it follows that $R_i^\delta \in H^1(0,T)$ for any $1 \leq i \leq N$. In [12], B. Niethammer proved the analogous local existence result for the case $\beta = g = 0$. In our case, the proof follows the same steps, under the assumption that the terms R_i^δ, $\frac{1}{R_i^\delta} + \beta \dot{R}_i^\delta$, $\dot{R}_i^\delta$, g_i, appearing at the phase boundary condition are uniformly bounded in $(0,T)$ for any $1 \leq i \leq N$. The first step is to transform the time-dependent space domain of the problem into the initial space domain at $t = 0$ consisting of N spheres of radii $R_i^\delta(0)$, $1 \leq i \leq N$. This may be achieved if $R_i^\delta(t)$ and $\frac{1}{R_i^\delta(t)}$ are uniformly bounded for any i and any

$t \in (0,T)$ (i.e. T is less than the first extinction time). The problem then is transformed for t fixed into an initial and boundary value problem where the boundary value along the phase boundary is defined by $\frac{1}{R_i^\delta(t)} + \beta \dot{R}_i^\delta(t) - g_i(t)$. Under the assumption (26) it follows by standard parabolic theory that if $R_i^\delta \in H^1(0,T)$ for any $1 \le i \le N$ then a unique solution u exists.

Proof. We first transform the domain Ω_T^δ to a fixed domain by means of some diffeomorphism

$$\phi\Big(\cdot, R^\delta\Big) : \Omega^\delta(0) \to \Omega^\delta(t),$$

and define

$$\Phi(x,t) := \phi\Big(x, R^\delta(t)\Big), \quad \tilde{v}(x,t) := u\Big(\Phi(x,t), t\Big),$$

where Φ is smooth in space if R_i^δ, $\frac{1}{R_i^\delta}$ are uniformly bounded.

Differentiating in space we get $\nabla u = D\Phi^{-T} \nabla \tilde{v}$, and

$$|\partial B_i^\delta(t)|^{-1} \int_{\partial B_i^\delta(t)} \nabla u \cdot n = |\partial B_i^\delta(0)|^{-1} \int_{\partial B_i^\delta(0)} D\Phi^{-T} \nabla \tilde{v} \cdot n,$$

while taking the derivative in time the next equalities follow

$$\tilde{v}_t = \nabla u \cdot \partial_t \Phi,$$
$$\partial_t \Phi = \tfrac{\partial \phi}{\partial R_1^\delta} \dot{R}_1^\delta + \cdots + \tfrac{\partial \phi}{\partial R_N^\delta} \dot{R}_N^\delta = (\nabla_{R^\delta} \phi) \cdot (\partial_t R^\delta).$$

The function $\tilde{v}$ solves

$$\begin{aligned}
\lambda \sqrt{\det(D\Phi^T D\Phi)} \partial_t \tilde{v} - \delta^8 \mathrm{div}\Big(\sqrt{\det(D\Phi^T D\Phi)} (D\Phi^T D\Phi)^{-1} \nabla \tilde{v} \Big) &= \lambda \sqrt{\det(D\Phi^T D\Phi)} D\Phi^{-T} \nabla \tilde{v} \cdot \partial_t \Phi, \text{ in } \Omega_T^0, \\
\tilde{v} &= \frac{1}{R_i^\delta} + \beta \dot{R}_i^\delta - g_i, \quad x \in \partial B_i^\delta(0), \\
\nabla \tilde{v} \cdot n &= 0, \text{ on } \partial \Omega_T, \\
\tilde{v}(x,0) &= u_0(x), \text{ in } \Omega^\delta(0),
\end{aligned}$$

where $\Omega_T^0 := \Big(\Omega \setminus \cup_i B_i^\delta(0)\Big) \times (0,T)$.

For any $t \in (0,T)$ fixed we consider the solution w of the problem

$$\begin{aligned}
\mathrm{div}\Big(\sqrt{\det(D\Phi^T D\Phi)} (D\Phi^T D\Phi)^{-1} \nabla w \Big) &= 0, \text{ in } \Omega^\delta(0), \\
w &= \frac{1}{R_i^\delta} + \beta \dot{R}_i^\delta - g_i, \quad x \in \partial B_i^\delta(0), \\
w \cdot n &= 0, \text{ on } \partial \Omega.
\end{aligned}$$

If R_i^δ, $\frac{1}{R_i^\delta}$, $\frac{1}{R_i^\delta} + \beta \dot{R}_i^\delta$, g_i are uniformly bounded for any $1 \le i \le N$ then $w = w(x, R^\delta(t))$, $\nabla_{R^\delta} w$ are smooth. We note that $\partial_t w = \nabla_{R^\delta} w \cdot \partial_t R^\delta$, thus $\partial_t w$ and $\partial_t R^\delta$ have the same regularity if the term $\nabla_{R^\delta} w$ is smooth. Setting $v := \tilde{v} - w$, then v satisfies

$$\begin{aligned}
\lambda \partial_t v - \frac{\delta^8}{\sqrt{\det(D\Phi^T D\Phi)}} \mathrm{div}\Big(\sqrt{\det(D\Phi^T D\Phi)} (D\Phi^T D\Phi)^{-1} \nabla v \Big) &= f_1 \cdot \nabla v + f_2, \\
v = 0, \quad x \in \partial B_i^\delta(0), &
\end{aligned}$$

where $f_1 \in \Big(L^2(L^\infty)\Big)^3$ and f_2 (including the term $\nabla_{R^\delta} w \partial_t R^\delta$) is in $L^2(L^\infty)$, as long as $R_i^\delta \in H^1(0,T)$, $1 \le i \le N$. If $u_0 \in H^1\Big(\Omega^\delta(0)\Big)$, then by standard theory for parabolic problems, [9], it follows that there exists a unique solution v or equivalently as long as w is smooth, there exists unique $\tilde{v} = v + w \in L^\infty\Big(L^2\Big(\Omega^\delta\Big); 0, T\Big) \cap L^2\Big(H^1\Big(\Omega^\delta\Big); 0, T\Big)$. But for R_i^δ, $\frac{1}{R_i^\delta}$ uniformly bounded for any i and any $t \in (0,T)$, the function Φ is smooth, consequently using the definition $\tilde{v}(x,t) := u(\Phi(x,t), t)$, it follows that $u \in L^\infty\Big(0, T; L^2\Big(\Omega^\delta\Big)\Big) \cap L^2\Big(0, T; H^1\Big(\Omega^\delta\Big)\Big)$. □

4.1. *Weak Formulation of Solution*

Let $\xi = \xi(x,t)$ such that ξ equals a constant on ∂B_i^δ for any i, and let $(\cdot,\cdot)_{\Omega^\delta}$ be the inner product in $L^2(\Omega^\delta)$. For simplicity we use the symbol R_i in place of R_i^δ. Multiplying the parabolic equation of (21) by ξ and integrating in Ω^δ, then by means of the boundary condition on $\partial\Omega$ and of the fact that $\xi = c_i(t)$ on ∂B_i^δ for any i we arrive at

$$\begin{aligned} 0 =&(\lambda u_t,\xi)_{\Omega^\delta} - \delta^8(\Delta u,\xi)_{\Omega^\delta} = (\lambda u_t,\xi)_{\Omega^\delta} + \delta^8(\nabla u,\nabla\xi)_{\Omega^\delta} + \delta^8\sum_{i=1}^N\int_{\partial B_i^\delta}\nabla u\cdot n\xi ds \\ =&(\lambda u_t,\xi)_{\Omega^\delta} + \delta^8(\nabla u,\nabla\xi)_{\Omega^\delta} + \delta^8\sum_{i=1}^N \xi_{|\partial B_i^\delta}\int_{\partial B_i^\delta}\nabla u\cdot nds. \end{aligned} \tag{28}$$

We multiply the second equation of (21) by $\xi|_{\partial B_i^\delta}$ and integrate on ∂B_i^δ to get

$$\int_{\partial B_i^\delta}\Big(u(x,t)+g_i(t)-\frac{1}{R_i(t)}\Big)\xi ds - \beta\delta^4\frac{|\partial B_i^\delta|}{|\partial B_i^\delta|}\xi|_{\partial B_i^\delta}\int_{\partial B_i^\delta}\nabla u\cdot nds = 0. \tag{29}$$

Replacing in (28) the term $\xi|_{\partial B_i^\delta}\int_{\partial B_i^\delta}\nabla u\cdot nds$ by (29) leads to

$$\lambda(u_t,\xi)_{\Omega^\delta} + \delta^8(\nabla u,\nabla\xi)_{\Omega^\delta} + \frac{\delta^4}{\beta}\sum_{i=1}^N\int_{\partial B_i^\delta}\xi\Big(u-\frac{1}{R_i^\delta(t)}\Big)ds + \frac{\delta^4}{\beta}\sum_{i=1}^N g_i(t)\int_{\partial B_i^\delta}\xi ds = 0,$$

which gives the following weak formulation for any t smaller than the first extinction time:

$$\lambda(u_t,\xi)_{\Omega^\delta} + \delta^8(\nabla u,\nabla\xi)_{\Omega^\delta} + \frac{\delta^4}{\beta}\sum_{i=1}^N\int_{\partial B_i^\delta} u\,\xi\,ds = \frac{\delta^4}{\beta}\sum_{i=1}^N\Big(\frac{1}{R_i^\delta(t)} - g_i(t)\Big)\int_{\partial B_i^\delta}\xi ds. \tag{30}$$

In order to extend the local in time solution to globally existing solution, in particular beyond the times when some balls disappear, we would need *a priori* estimates. They will be established by means of integral inequalities and maximum principle.

In the following, for simplicity, we omit the super-script δ if it is clear from the context. They will be recovered in the later parts. In addition, we will use M or $M(T,\Omega)$ to denote general constants that might depend on the time interval $[0,T]$ and the domain Ω but not on δ.

5. Preliminary Identities

In this section, we present some preliminary identities in line of energy type estimates. As the domain $\Omega^\delta = \Omega^\delta(t)$ is time dependent, we find it convenient to extend u to the whole domain $\Omega \supset \Omega^\delta$ by means of:

$$u\Big|_{B_i} = u\Big|_{\partial B_i}, \quad \text{for all } i.$$

The extended function is still denoted by u. Furthermore, we use introduce the notation $f_i(t) = R_i(t)\dot R_i(t)$.

Proposition 5.1. *Let u be the solution of* (21). *Then we have*

$$\begin{aligned} &\lambda\int_\Omega u(t) + \frac{\lambda 2\pi\delta^{12}}{3}\sum_{i=1}^N R_i^2(t) + \frac{4\pi\delta^{12}}{3}\sum_{i=1}^N R_i^3(t) + \lambda 4\pi\delta^{12}\beta\sum_{i=1}^N\int_0^t f_i^2(r)\,dr \\ =\;& \lambda\int_\Omega u(0) + \frac{\lambda 2\pi\delta^{12}}{3}\sum_{i=1}^N R_i^2(0) + \frac{4\pi\delta^{12}}{3}\sum_{i=1}^N R_i^3(0) + \frac{\lambda 4\pi\delta^{12}\beta}{3}\left(\sum_{i=1}^N R_i^2(t)f_i(t) - \sum_{i=1}^N R_i^2(0)f_i(0)\right) \\ &- \frac{\lambda 4\pi\delta^{12}}{3}\sum_{i=1}^N R_i^3(t)g_i(t) + \frac{\lambda 4\pi\delta^{12}}{3}\sum_{i=1}^N R_i^3(0)g_i(0) + \lambda 4\pi\delta^{12}\int_0^t\sum_{i=1}^N R_i(r)f_i(r)g_i(r)\,dr. \end{aligned} \tag{31}$$

Proof. We integrate (21) on Ω to get

$$\lambda \int_{\Omega} u_t - \lambda \int_{\Omega \setminus \Omega^\delta} u_t = \delta^8 \int_{\partial \Omega^\delta} \frac{\partial u}{\partial n},$$

Note that the part of $\partial \Omega^\delta$ on solid-liquid interfaces, we use the outward normal to the B_i's. Hence

$$\begin{aligned}
\lambda \frac{d}{dt} \int_{\Omega} u - \lambda \sum_i \frac{4\pi}{3} (\delta^4 R_i)^3 \left(\frac{1}{R_i} - g_i + \beta \dot{R}_i \right)_t &= -\sum_i 4\pi \delta^{12} R_i^2 \dot{R}_i \\
\lambda \frac{d}{dt} \int_{\Omega} u - \frac{\lambda 4\pi \delta^{12}}{3} \sum_i R_i^3 \left(-\frac{\dot{R}_i}{R_i^2} - \dot{g}_i + \beta \ddot{R}_i \right) + 4\pi \delta^{12} \sum_i R_i^2 \dot{R}_i &= 0 \\
\lambda \frac{d}{dt} \int_{\Omega} u + \frac{\lambda 4\pi \delta^{12}}{3} \sum_i \left(R_i \dot{R}_i + R_i^3 \dot{g}_i - \beta R_i^3 \ddot{R}_i \right) + \frac{4\pi \delta^{12}}{3} \sum_i \frac{d}{dt} R_i^3 &= 0 \\
\lambda \frac{d}{dt} \int_{\Omega} u + \frac{\lambda 4\pi \delta^{12}}{3} \sum_i \left(\frac{1}{2} \frac{d}{dt} R_i^2 + R_i^3 \dot{g}_i - \beta R_i^3 \ddot{R}_i \right) + \frac{4\pi \delta^{12}}{3} \sum_i \frac{d}{dt} R_i^3 &= 0
\end{aligned}$$

Upon integrating in time from 0 to t and employing integration by parts, we obtain (31). □

Remark 5.2. *For conceptual understanding and to compare with known results, we simplify the above identity for the case* $g_i(t) \equiv 0$.

1. *For the quasi-static problem* $\lambda = 0$ *with* $\beta \geq 0$ *the following volume conservation condition is obtained:*

$$\delta^3 \sum_{i=1}^{N} R_i^3(t) = \delta^3 \sum_{i=1}^{N} R_i^3(0),$$

   as in [13, 14].

2. *For the parabolic case* $\lambda > 0$*:*
   (a) *If* $\beta = 0$, *then setting* $\lambda := \delta^9$ *in* (31), *we obtain the result of [13]:*

$$\int_{\Omega} u(t) + \frac{4}{3}\pi \sum_{i=1}^{N} \delta^3 R_i^3(t) + \frac{2}{3}\pi \sum_{i=1}^{N} \delta^{12} R_i^2(t) = \int_{\Omega} u(0) + \frac{4}{3}\pi \sum_{i=1}^{N} \delta^3 R_i^3(0) + \frac{2}{3}\pi \sum_{i=1}^{N} \delta^{12} R_i^2(0).$$

   (b) *If* $\beta > 0$, *then* (31) *gives*

$$\begin{aligned}
&\lambda \int_{\Omega} u(t) + \frac{\lambda 2\pi \delta^{12}}{3} \sum_{i=1}^{N} R_i^2(t) + \frac{4\pi \delta^{12}}{3} \sum_{i=1}^{N} R_i^3(t) + \lambda 4\pi \delta^{12} \beta \sum_{i=1}^{N} \int_0^t f_i^2(r)\, dr \\
&= \lambda \int_{\Omega} u(0) + \frac{\lambda 2\pi \delta^{12}}{3} \sum_{i=1}^{N} R_i^2(0) + \frac{4\pi \delta^{12}}{3} \sum_{i=1}^{N} R_i^3(0) + \frac{\lambda 4\pi \delta^{12} \beta}{3} \left(\sum_{i=1}^{N} R_i^2(t) f_i(t) - \sum_{i=1}^{N} R_i^2(0) f_i(0) \right).
\end{aligned} \tag{32}$$

Next we derive the identity for $\|u\|_{L^2(\Omega)}$.

Proposition 5.3. *Let u be the solution of* (21)*. Then we have*

$$
\begin{aligned}
&\frac{\lambda}{2}\int_\Omega u^2(t)+\delta^8\int_0^t\int_\Omega |\nabla u|^2(s)\,ds+2\pi\delta^{12}\sum_i R_i^2(t)+\lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i(t)\\
&+4\pi\delta^{12}\beta\int_0^t\sum_i f_i^2(s)\,ds+\lambda 4\pi\delta^{12}\beta\int_0^t\sum_i\frac{f_i^2(s)}{R_i(s)}\,ds\\
=\;&\frac{\lambda}{2}\int_\Omega u^2(0)+\frac{4\pi\delta^{12}}{2}\sum_i R_i^2(0)+\lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i(0)+\lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i(t)f_i(t)\\
&-\lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i(0)f_i(0)+\lambda\frac{2\pi\delta^{12}}{3}\beta^2\sum_i R_if_i^2(t)-\lambda\frac{2\pi\delta^{12}}{3}\beta^2\sum_i R_if_i^2(0)\\
&-\lambda 2\pi\delta^{12}\beta^2\int_0^t\sum_i\frac{f_i^3(s)}{R_i(s)}\,ds+4\pi\delta^{12}\int_0^t\sum_i R_if_ig_i(s)\,ds\\
&-\lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^2(t)g_i(t)+\lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^2(0)g_i(0)+\lambda 4\pi\delta^{12}\sum_i\int_0^t f_i(s)g_i(s)\,ds\\
&-\lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i^2(t)f_i(t)g_i(t)+\lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i^2(0)f_i(0)g_i(0)+\lambda 4\pi\delta^{12}\beta\sum_i\int_0^t f_i^2g_i\,ds\\
&+\lambda\frac{2\pi\delta^{12}}{3}\sum_i R_i(t)^3g_i^2(t)-\lambda\frac{2\pi\delta^{12}}{3}\sum_i R_i(0)^3g_i^2(0)-\lambda 2\pi\delta^{12}\int_0^t\sum_i R_if_ig_i^2(s)\,ds.
\end{aligned}
$$

Proof. Multiplying (21) by u and integrating on Ω^δ, we get

$$
\begin{aligned}
\lambda\int_{\Omega^\delta}u_tu &= \delta^8\int_{\Omega^2}\triangle uu\\
\lambda\int_\Omega u_tu-\lambda\int_{\Omega\setminus\Omega^\delta}u_tu &= -\delta^8\int_{\partial\Omega^\delta}\frac{\partial u}{\partial n}u-\delta^8\int_{\Omega^\delta}|\nabla u|^2
\end{aligned}
$$

Using the boundary conditions in (21), it follows that

$$
\frac{\lambda}{2}\frac{d}{dt}\int_\Omega u^2+\delta^8\int_\Omega|\nabla u|^2-\lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^3\left(\frac{1}{R_i}-g_i+\beta\dot R_i\right)_t\left(\frac{1}{R_i}-g_i+\beta\dot R_i\right)
$$
$$
+\,4\pi\delta^{12}\sum_i R_i^2\dot R_i\left(\frac{1}{R_i}-g_i+\beta\dot R_i\right)=0
$$

$$
\frac{\lambda}{2}\frac{d}{dt}\int_\Omega u^2+\delta^8\int_\Omega|\nabla u|^2-\lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i^3\left(-\frac{\dot R_i}{R_i^2}-\dot g_i+\beta\ddot R_i\right)\left(\frac{1}{R_i}-g_i+\beta\dot R_i\right)
$$
$$
+\,4\pi\delta^{12}\sum_i\left(R_i\dot R_i-R_i^2\dot R_ig_i+\beta R_i^2\dot R_i^2\right)=0
$$

$$
\frac{\lambda}{2}\frac{d}{dt}\int_\Omega u^2+\delta^8\int_\Omega|\nabla u|^2+4\pi\delta^{12}\sum_i\left(\frac{d}{dt}\frac{R_i^2}{2}+\beta R_i^2\dot R_i^2-R_i^2\dot R_ig_i\right)
$$
$$
+\,\lambda\frac{4\pi\delta^{12}}{3}\sum_i\left(R_i\dot R_i+R_i^3\dot g_i-\beta R_i^3\ddot R_i\right)\left(\frac{1}{R_i}-g_i+\beta\dot R_i\right)=0
$$

Expanding the above, and integrating in time from 0 to t together with integration by parts gives the stated identity. □

Remark 5.4. *Again, we give the simplified form of the above in the case $g_i(t) \equiv 0$.*

1. $\lambda = 0, \beta = 0$.
$$\sum_i \delta^3 R_i^2(t) + \frac{1}{2\pi\delta}\int_0^t \int_\Omega |\nabla u|^2\, ds = \sum_i \delta^3 R_i^2(0);$$
as in [14].
2. $\lambda > 0, \beta = 0$.
$$\frac{1}{2}\int_\Omega u^2(t) + \frac{1}{\delta}\int_0^t \int_\Omega |\nabla u|^2\, ds + 2\pi\delta^3 \sum_i R_i^2(t) + \frac{2}{3}\pi\delta^{12}\sum_i R_i(t)$$
$$= \frac{1}{2}\int_\Omega u^2(0) + 2\pi\delta^3 \sum_i R_i^2(0) + \frac{2}{3}\pi\delta^{12}\sum_i R_i(0).$$
(in accordance to [13]) where we have set $\lambda := \delta^9$.
3. $\lambda = 0, \beta > 0$.
$$\delta^3 \sum_i R_i^2(t) + \frac{1}{2\pi\delta}\int_0^t \int_\Omega |\nabla u|^2\, ds + 2\beta\int_0^t \sum_i \delta^3 f_i^2\, ds = \delta^3 \sum_i R_i^2(0),$$
as in [14].
4. $\lambda > 0, \beta > 0$.
$$\begin{aligned} & \frac{\lambda}{2}\int_\Omega u^2(t) + \delta^8 \int_0^t \int_\Omega |\nabla u|^2(s)\, ds + 2\pi\delta^{12}\sum_i R_i^2(t) + \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i(t) \\ & +4\pi\delta^{12}\beta \int_0^t \sum_i f_i^2(s)\, ds + \lambda 4\pi\delta^{12}\beta\int_0^t \sum_i \frac{f_i^2(s)}{R_i(s)}\, ds \\ = \; & \frac{\lambda}{2}\int_\Omega u^2(0) + \frac{4\pi\delta^{12}}{2}\sum_i R_i^2(0) + \lambda\frac{4\pi\delta^{12}}{3}\sum_i R_i(0) + \lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i(t) f_i(t) \\ & -\lambda\frac{4\pi\delta^{12}}{3}\beta\sum_i R_i(0) f_i(0) + \lambda\frac{2\pi\delta^{12}}{3}\beta^2 \sum_i R_i f_i^2(t) - \lambda\frac{2\pi\delta^{12}}{3}\beta^2\sum_i R_i f_i^2(0) \\ & -\lambda 2\pi\delta^{12}\beta^2 \int_0^t \sum_i \frac{f_i^3(s)}{R_i(s)}\, ds. \end{aligned} \tag{33}$$

Note that when both λ and β are positive, as in the case of parabolic problem with kinetic undercooling, when deriving the *a priori* estimates extra care is needed due to the terms involving $f_i = R_i \dot{R}_i$ which appear on the right hand sides of (32) and (33). Their estimates will be derived next using maximum principle. These will be used in combination with the integral identities to derive estimates which are uniform in δ and global in time, even after some balls have vanished.

Solving for $\int_0^t \int_\Omega |\nabla u|^2(s)\, ds$ in the estimate of **Proposition 5.3**, we observe that if all quantities are smooth (to be proved in our analysis) then
$$\int_0^t \int_\Omega |\nabla u|^2(s)\, ds \le c\frac{\lambda}{\delta^8} + c\delta^4 - \frac{\lambda}{2\delta^8}\int_\Omega u^2(t) \le c\frac{\lambda}{\delta^8} + c\delta^4.$$
Since we expect that in the limit $\delta \to 0$ the mean field solution is constant in space, then the right-hand side of the previous inequality must tends to zero, hence we set $\lambda = \delta^9$ as mentioned in **Remark 3.1**. From now on in our proofs we use the value $\lambda = \delta^9$. Recall also the form of the initial condition (22).

6. Regularity of the radii R_i's

6.1. Preliminaries

We first record the following lemma on the maximum principle suitable for our problem. It is the parabolic version of **Lemma 4.2** in [14].

Lemma 6.1. *Let $\{\Omega(t)\}_{t\geq 0}$ be a time dependent Lipschitz domain and $\bigcup_i \{B_i(t)\}_{t\geq 0}$ be a finite collection of disjoint balls such that $\bigcup_i B_i(t) \subset \Omega(t)$ for all $t \geq 0$.*

Let u be a function which is constant on each ∂B_i and satisfy for all $t \geq 0$ the following statements

$$\begin{aligned} u_t - \triangle u &\geq (\leq) \quad 0 \quad \text{in } \Omega(t)\backslash \cup_i B_i(t), \\ u - c_i \int_{\partial B_i(t)} \nabla u \cdot n &\geq (\leq) \quad 0 \quad \text{on } \partial B_i(t), \text{ for all } i, \\ \nabla u \cdot n &\geq (\leq) \quad 0 \quad \text{on } \partial\Omega(t), \end{aligned}$$

where $c_i \geq 0$ for all i. If $u(x,0) \geq (\leq) 0$, then $u \geq (\leq) 0$ in $\Omega(t)\backslash \bigcup_i B_i(t)$ for $t > 0$.

The rigorous proof of the above can be produced following the steps in [14] and hence it is omitted. It can also be intuitively understood. For example, if $u \geq 0$ at $t = 0$, then by strong maximum principle, it cannot reach zero *inside* the domain $\Omega(t)\backslash \bigcup_i B_i(t)$. By means of Hopf lemma, the boundary conditions also prevent the occurrence of zero on $\partial\Omega(t)$ and $\partial B_i(t)$. Hence u will be strictly positive for all $t > 0$.

Equipped with the above result, we are ready to construct sub- and super-solutions which will be used to control the growth and decay of the radii $R_i^\delta(t)$'s. First we present an *a priori* bound using the maximum principle.

Lemma 6.2. *There exist two constants $M_1(T,\Omega)$ and $M_2(T,\Omega)$ such that for any solution u^δ of* (21) *with initial data (22), we have*

$$M_1(T,\Omega) \leq u^\delta(x,t) \leq M_2(T,\Omega) + u^\delta_{\infty,0} + \sum_{i\in N^\delta} \frac{\delta^4}{|x - x_i|}, \tag{34}$$

(In general, M_1 might be negative.) The above leads to that for some constant $M > 0$,

1. *at any particle boundary: for x such that $|x - x_i| = \delta^4 R_i$,*
$$u\Big|_{\partial B_i} \leq M + \frac{1}{R_i}; \tag{35}$$
2. *away from any of particle boundary: for x such that $|x - x_i| \geq \frac{\delta}{4}$ for all i,*
$$|u| \leq M. \tag{36}$$

Proof. The proof of the lower bound in (34) is simply due to the fact that a negative constant with large magnitude $(-M)$ satisfies:

$$(-M) \leq -g_i(t) + \frac{1}{R_i} + \frac{\beta}{4\pi\delta^4 R_i^2} \int_{\partial B_i} \nabla(-M)\cdot n$$

and hence is a sub-solution.

The proof of the upper bound in (34) is similar to [13, Lemma 17]. It turns out that the function $\overline{V}$ denoting the right hand side of (34) is automatically a super-solution for large enough $M_2(T,\Omega)$. The main reason is as follows.

- For any $i \in N^\delta(t)$,
$$\begin{aligned} \overline{V}\Big|_{\partial B_i} &= M_2 + u^\delta_{\infty 0} + \frac{1}{R_i} + \sum_{j\neq i} \frac{\delta^4}{\left|x_j - x_i\right|} \geq M_2 + u^\delta_{\infty 0} + \frac{1}{R_i} + O(1)\sum_{j\neq i}\frac{\delta^4}{\delta} \\ &\geq M_2 + u^\delta_{\infty 0} + \frac{1}{R_i} + \frac{O(1)}{\delta^3}\sum_{j\neq i}\delta^3 \geq M_2 + O(1) + \frac{1}{R_i} \end{aligned}$$

  In the above, we have used the fact that $N^\delta(t) = O(\delta^{-3})$ and $\left|x_i - x_j\right| \geq c\delta$ for any $i \neq j$.

- Next we compute the gradient term: again for any $i \in N^\delta(t)$,

$$\begin{aligned}
\frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\overline{V}\cdot n &= \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\left[\sum_{j\in N^\delta(t)}\frac{\delta^4}{|x-x_j|}\right]\cdot n\\
&= \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\left[\frac{\delta^4}{|x-x_i|}\right]\cdot n + \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\left[\sum_{j\neq i}\frac{\delta^4}{|x-x_j|}\right]\cdot n\\
&\geq \frac{\beta}{4\pi\delta^4 R_i^2}\left[-\frac{\delta^4}{\delta^8 R_i^2}\right]4\pi\delta^8 R_i^2 + \frac{O(1)}{\delta^3}\frac{\beta}{4\pi\delta^4 R_i^2}\left[\frac{\delta^4}{\delta^2}\right]4\pi\delta^8 R_i^2 = -\frac{\beta}{R_i^2} + O(\delta^3)
\end{aligned}$$

Hence we always have (with M_2 chosen big enough and δ being small):

$$\overline{V} \geq -g_i + \frac{1}{R_i} + \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla\overline{V}\cdot n.$$

(In principle, we also need to modify the boundary value of $\overline{V}$ so that it satisfies the Neumann boundary condition on $\partial\Omega$. This is similar to [13, Lemma 17]. Let $h = \sum_{i\in N^\delta}\frac{\delta^4}{|x-x_i|}$ and w be the solution of the following equation:

$$\begin{aligned}
\delta w_t &= \triangle w, \quad \text{in } \Omega_T,\\
\nabla w\cdot n &= -\nabla h\cdot n \quad \text{on } \partial\Omega_T,\\
w(0,\cdot) &= w_0 \quad \text{in } \Omega,
\end{aligned}$$

where w_0 solves:

$$\begin{aligned}
-\triangle w_0 &= \int_{\partial\Omega}\nabla h\cdot n\\
\nabla w_0\cdot n &= -\nabla h\cdot n\\
\int_\Omega w_0 &= 0.
\end{aligned}$$

By [13, Lemma 17], w_0 and w satisfy the estimates $\|w_0\|_\infty \leq M\sqrt{\delta}$ and $\|w\|_\infty \leq M$. Due to our kinetic undercooling boundary condition, we will also need to estimate ∇w. This can also be done in a way similar to [13, Lemma 20] so that $\|\nabla w\|_\infty \leq M\delta^\gamma$ for any $\gamma < \frac{1}{2}$. Hence by choosing M_2 large enough, we have:

$$(\overline{V}+w)\Big|_{\partial B_i} \geq g_i - \frac{1}{R_i} + \frac{\beta}{4\pi\delta^4 R_i^2}\int_{\partial B_i}\nabla(\overline{V}+w)\cdot n$$

so that the desired result is still true.) □

Now we proceed to construct sub- and super-solutions so as to control the growth and decay rates of the particle radii.

6.2. Single Particle Scenario

We first consider the case of one single particle which forms the building block for the general multiple particle scenario. In this case, problem (21) is formulated in the following form:

$$\begin{aligned}
\delta u_t &= \triangle u, \quad \text{on } \left\{|x| \geq \delta^4 R(t)\right\},\\
u &= \frac{1}{R} - g(t) + \frac{\beta}{4\pi\delta^4 R^2(t)}\int_{\partial B_{\delta^4 R}}\nabla u\cdot n, \quad \text{on } \left\{|x| = \delta^4 R(t)\right\}, \qquad (37)\\
\dot{R} &= \frac{1}{4\pi\delta^4 R^2}\int_{\partial B_{\delta^4 R}}\nabla u\cdot n.
\end{aligned}$$

The key is to investigate the solution as $R \longrightarrow 0^+$ in the regime $\delta \ll 1$. The main conclusion is that $|R\dot{R}| < C < \infty$ and $\lim_{R\to 0^+} R\dot{R} = -\frac{1}{\beta}$. Hence $R \in W^{1,p}([0,T])$ for any $1 \le p < 2$. This will be established by constructing sub- and super-solutions. It is first done for the case $R \ll 1$ and $\dot{R} < 0$. If $R > O(1)$, we will show that $\left|\dot{R}\right|$ is uniformly bounded. However, once $R(t)$ reaches below some small value, $\dot{R}$ will become *negative* and will *stay negative* until the extinction time of $R(t)$.

6.2.1. Construction of Sub-solution ($\dot{R} \le 0$, $R \ll 1$)

Given $R(t)$, then $U(x,t)$ is a sub-solution if

$$\begin{aligned} &\delta U_t \le \triangle U, \quad \text{on} \quad \left\{|x| \ge \delta^4 R(t)\right\},\\ &U \le \frac{1}{R} - g(t) + \frac{\beta}{4\pi\delta^4 R^2(t)} \int_{\partial B_{\delta^4 R}} \nabla U \cdot n, \quad \text{on} \quad \left\{|x| = \delta^4 R(t)\right\}, \end{aligned}$$

For any constant C, consider the function

$$U_{C,R}(x) = C + \left(\frac{1 - RC - Rg}{R+\beta}\right)\frac{\delta^4 R}{|x|}. \tag{38}$$

By simple computations, $U_{C,R}$ satisfies the following properties:

$$\begin{aligned} U_{C,R}(x) &> 0 \quad \text{for } |x| \ge \delta^4 R,\\ U_{C,R}(x) &\ge C \quad \text{for } |x| \ge \delta^4 R \text{ and } R(C+g) \le 1,\\ U_{C,R}(\delta^4 R) &= \frac{1 + \beta C - Rg}{R+\beta},\\ U_{C,R}(\delta^4 R) &= \frac{1}{R} - g + \frac{\beta}{4\pi\delta^4 R^2}\int_{\partial B_{\delta^4 R}} \nabla U_{C,R} \cdot n,\\ \lim_{R\longrightarrow 0^+} U_{C,R}(\delta^4 R) &= C + \frac{1}{\beta},\\ \lim_{|x|\to\infty} U_{C,R}(x) &= C. \end{aligned}$$

Note that $\left|U_{C,R}\right|$ is uniformly bounded by some constant $M(C,G) < \infty$. Furthermore,

$$\frac{\partial U_{C,R}}{\partial C} = 1 - \frac{\delta^4 R^2}{(R+\beta)|x|} \ge 1 - \frac{R}{R+\beta} = \frac{\beta}{R+\beta} > 0 \quad \text{if} \quad |x| \ge \delta^4 R. \tag{39}$$

so that we can use the constant C to adjust the far-field value in order to ensure that at $t = 0$, $U_{C,R}$ is smaller than the initial data.

Now let $R = R(t)$ be given from the solution of (37) and $C = C(t)$ be some time dependent function (to be specified). Then $\Delta U_{C,R} = 0$ and

$$\begin{aligned} \frac{\partial U_{C(t),R(t)}(x)}{\partial t} &= \frac{\delta^4 \dot{R}}{(R+\beta)^2|x|}\left[(R+\beta)(1 - 2RC - 2Rg) - R + R^2 C + R^2 g\right] + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}\\ &= \frac{\delta^4 \dot{R}}{(R+\beta)^2|x|}\left[\beta - R^2 C - 2R\beta C - 2R^2 g - 2Rg\beta\right] + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}. \end{aligned}$$

Recall the assumptions that $\dot{R} \le 0$ and $R \ll 1$ and also (24) on g. The above can be made negative by choosing $C(t)$ such that $\dot{C}(t)$ is much bigger that $|R\dot{g}|$. Thus $U_{C,R}$ is a sub-solution. So if $C(0)$ is chosen small enough (possible with negative value), we have $u_0 \ge U_{C(0),R(0)}$ and hence $u \ge U_{C,R}$ for $t > 0$. This leads to

$$\dot{R} = \frac{1}{4\pi\delta^4 R^2}\int_{\partial B_{\delta^4 R}} \nabla u \cdot n = \frac{1}{\beta}\left[u - \frac{1}{R} + g\right] \ge \frac{1}{\beta}\left[U_{C,R}(R) - \frac{1}{R} + g\right] = \frac{1}{\beta}\left[\frac{1 + C\beta - Rg}{R+\beta} + g - \frac{1}{R}\right] \gtrsim -\frac{1}{\beta R}. \tag{40}$$

6.2.2. Construction of Super-solution ($\dot{R} < 0, R \ll 1$)

Again let $R(t)$ be taken from the solution of (37), then $V(x,t)$ is a super-solution if

$$\begin{aligned} \delta V_t &\geq \Delta V, \quad \text{on} \quad \left\{|x| \geq \delta^4 R(t)\right\}, && (41)\\ V &\geq \frac{1}{R} - g + \frac{\beta}{4\pi\delta^4 R^2(t)}\int_{\partial B_{\delta^4 R}} \nabla V \cdot n \quad \text{on} \quad \left\{|x| = \delta^4 R(t)\right\}. && (42)\end{aligned}$$

Consider the function

$$V_{C(t),R(t)}(x) = \frac{\delta^4 a(t)}{|x|} + C(t) + \frac{(1 - RC(t) - Rg)\delta^4 R}{(R+\beta)|x|}, \tag{43}$$

where $a(t)$ and $C(t)$ are to be determined. Note that $\Delta V_{C(t),R(t)} = 0$ and

$$\begin{aligned} \frac{\partial V_{C(t),R(t)}}{\partial t} &= \frac{\delta^4 \dot{a}}{|x|} + \frac{\delta^4 \dot{R}}{(R+\beta)^2|x|}[\beta - R^2 C - 2R\beta C - 2R^2 g - 2Rg\beta] + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}\\ &\approx \frac{\delta^4 \dot{a}}{|x|} + \frac{\delta^4 \dot{R}}{\beta |x|} + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g}.\end{aligned}$$

To make (41) hold, we choose $a(t)$ and $C(t)$ such that

$$\dot{a} + \frac{\dot{R}}{\beta} \geq 0 \quad \text{or} \quad a(t) = a_0 - \frac{R(t)}{\beta} > 0, \quad \text{and } \dot{C} \text{ is much bigger than } \dot{g} \text{ (recall again (24)).}$$

As $\dot{R} < 0$, a convenient choice is

$$a(t) = \frac{R(0)}{\beta} - \frac{R(t)}{\beta}.$$

The condition (42) is equivalent to

$$\frac{a(t)}{R(t)} > \beta\delta^4\delta^4 a(t)(-1)\frac{1}{\delta^8 R^2(t)},$$

which is always true as long as $a(t) > 0$. Hence V is a super-solution. So if $C(0)$ is chosen big enough, we have $u_0 \leq V_{C(0),R(0)}$ and also $u \leq V_{C(t),R(t)}$ for $t > 0$.

Now considering the dynamics of $R(t)$, we have

$$\begin{aligned} \dot{R} &= \frac{1}{4\pi\delta^4 R^2}\int_{\partial B_{\delta^4 R}} \nabla u \cdot n = \frac{1}{\beta}\left[u - \frac{1}{R}\right] \leq \frac{1}{\beta}\left[V - \frac{1}{R} + g\right]\\ &= \frac{1}{\beta}\left[\frac{a}{R} + \frac{1 + C\beta}{R+\beta} - \frac{1}{R} + g\right] = \frac{1}{\beta R}\left[a(t) - 1 + \frac{R(1+C\beta) + gR(R+\beta)}{(R+\beta)}\right]\\ &= \frac{1}{\beta R}\left[a_0 - \frac{R(t)}{\beta} - 1 + \frac{R(1+C\beta) + gR(R+\beta)}{(R+\beta)}\right]\\ &= \frac{1}{\beta R}\left[-1 + \frac{R(0)}{\beta} - \frac{R(t)}{\beta} + \frac{R(1+C\beta) + gR(R+\beta)}{(R+\beta)}\right] \lesssim -\frac{1}{\beta R}. && (44)\end{aligned}$$

Combining (40) and (44), we finally have,

$$-\frac{1}{\beta R} \leq \dot{R} \leq -\frac{1 - O(1)}{\beta R}. \tag{45}$$

6.2.3. Construction for Big Radius.

This section considers the case when R is not small. The idea is to modify the previous construction of sub- and super-solutions by a term with small L^∞-norm but large Laplacian value (see also [13, Lemma 18]).

Let (R, u) be the solution of (21). In addition, we assume for some fixed constants δ_0, A_1, A_2 and B such that

$$\begin{cases} \delta \le \delta_0; \\ A_1 < R(t) < A_2; \\ \dot{R} \text{ is uniformly bounded by } \frac{B}{\delta}. \end{cases} \tag{46}$$

For super-solution, we consider the following function:

$$\widetilde{V}_{C,R}(x,t) = C + \frac{(1 - RC - Rg)\delta^4 R}{(R+\beta)|x|} - \frac{1}{2}|x - x_i|^2 + \epsilon \tag{47}$$

where $\epsilon \gg \delta$. It holds that

$$\begin{aligned} & \delta \frac{\partial \widetilde{V}_{C(t),R(t)}}{\partial t} - \triangle \widetilde{V}_{C(t),R(t)} \\ = \quad & \delta \left\{ \frac{\delta^4 \dot{R}}{(R+\beta)^2|x|}[\beta - R^2C - 2R\beta C - 2R^2 g - 2Rg\beta] + \left[1 - \frac{\delta^4 R^2}{(R+\beta)|x|}\right]\dot{C} - \frac{\delta^4 R^2}{(R+\beta)|x|}\dot{g} \right\} + 3 \end{aligned} \tag{48}$$

and

$$\widetilde{V}_{C,R} \quad \ge \quad -g\frac{1}{R} + \frac{\beta}{4\pi\delta^4 R^2}\int_{\partial B_{\delta^4 R}} \nabla \widetilde{V} \cdot n. \tag{49}$$

Under the assumption (46), the right hand side of (48) is positive. Hence V is a *super-solution*. As before we obtain that

$$\dot{R} \le \frac{1}{\beta R}\left[-1 - \frac{R}{\beta} + \frac{R(1 + C\beta) + gR(R+\beta)}{R+\beta}\right] < M \tag{50}$$

for some constant M independent of δ.

For sub-solution, similarly, we consider

$$\widetilde{U}_{C,R}(x,t) = C + \frac{(1 - RC - Rg)\delta^4 R}{(R+\beta)|x|} + \frac{1}{2}|x - x_i|^2 - \epsilon. \tag{51}$$

Again by (46), $\widetilde{U}_{C,R}$ will be a *sub-solution*. So we have

$$\dot{R} \ge \frac{1}{\beta R}\left[-1 - \frac{R}{\beta} + \frac{R(1 + C\beta) + gR(R+\beta)}{R+\beta}\right] > -M. \tag{52}$$

Hence we obtain

$$\left|\dot{R}\right| < M. \tag{53}$$

6.3. Multi-particle case

Now we employ the above single particle analysis to prove *a priori* bounds for the multiple particle case. Consider the initial data u_0 given by (22). By **Theorem 4.1**, the solution exists locally in time. The key is to extend the solution globally in time, beyond the vanishing times of some balls.

Let T be some fixed constant. By the uniform estimate (36), on the set $K = \left\{x : |x - x_i| \ge \frac{\delta}{4} \text{ for all } i\right\}$ (i.e. away from each $\partial B_{\delta^4 R_i}$), $|u|_{0\le t\le T}$ is bounded uniformly by some fixed constant. Hence if $\widetilde{C}_i^-(0)$ and $\widetilde{C}_i^+(0)$ are chosen sufficiently small and large respectively, using (47) and (51), we have $\widetilde{U}_{\widetilde{C}_i^-(0),R_i(0)} \le u_0 \le \widetilde{V}_{\widetilde{C}_i^+(0),R_i(0)}$ and hence $\widetilde{U}_{\widetilde{C}_i^-(t),R_i(t)} \le u \le \widetilde{V}_{\widetilde{C}_i^+(t),R_i(t)}$ for as long as $A_1 \le R \le A_2$ and $\left|\dot{R}\right| \le \frac{B}{\delta}$. By (53), it follows that $\left|\dot{R}\right| \le M$. Now given any finite time interval $[0, T]$, choose $A_2 = R_0 + 2MT$. Then the upper bounds are always true for time interval $[0, T]$ (independent of δ).

If some $R_i(t)$ ever reaches some small value A_1, by (50), $\dot{R}_i$ will be negative. Similarly choose C_i^- and C_i^+ to be sufficiently small and large such that $U_{C_i^-(t),R_i(t)}$ and $V_{C_i^+(t),R_i(t)}$ from (38) and (43) satisfy

$$U_{C_i^-(t),R_i(t)} \leq \widetilde{U}_{\widetilde{C}_i^-(t),R_i(t)} \ (\leq u) \quad \text{and} \quad (u \leq) \ \widetilde{V}_{\widetilde{C}_i^+(t),R_i(t)} \leq V_{C_i^+(t),R_i(t)}$$

Now by (44), $\dot{R}$ will *stay negative* and hence $U_{C_i^-(t),R_i(t)}$ and $V_{C_i^+(t),R_i(t)}$ remain to be sub- and super-solutions up to the vanishing moment t_i of R_i. Finally estimates (45) hold.

Let t^* be the first vanishing time of some ball. We have

$$\left|R_i\dot{R}_i\right| \leq M < \infty, \quad \text{and} \quad \sup_i \sup_{t<t^*} R_i(t) \leq M < \infty \tag{54}$$

hence $R \in W^{1,p}([0,t^*])$ for all $1 \leq p < 2$.

With the above, the extension of solution beyond t^* follows as in [13, pp. 158-159, 165]: by the energy estimates of **Proposition 5.3**, we have $\sup_{t<t^*} \|u\|_{L^2(\Omega(t))}$ and $\|\nabla u\|_{L^2(\Omega_{t^*})}$ bounded independently of δ. Hence standard parabolic theory leads to $u(\cdot,t) \longrightarrow u(\cdot,t^*)$ in $L^2(\Omega)$ as $t \longrightarrow t^*$. However, in general $u(\cdot,t^*)$ does not belong to $H^1(\Omega)$ so that we cannot directly invoke the local in time existence result **Theorem 4.1**. On the other hand, the H^1 condition is only needed near each existing particles. Near the location where a ball has just vanished, a regular heat equation is well-posed with L^2 initial data. A localization procedure is used to construct the solution starting from $u(\cdot,t^*)$.

6.4. *Iteration Step*

The purpose of this step is to improve the constant $1-O(1)$ in the right-hand side of (45). This is not necessary for the later part from the point of view of estimates – all is needed is that $R \in W^{1,p}([0,T])$, but we feel it is of independent interest as it gives the limiting asymptotics of $R(t)$ near its extinction time in the strong form.

From the form of the super-solution, we need to progressively reduce a_0 in (43). The expression for the super-solution is simplified as

$$V_0(x,t) = \frac{\delta^4}{\beta|x|}(R(0)-R(t)) + A + Bt.$$

for some A and B large enough (but independent of time and δ).

Let t_1 be such that $R(t_1) = \dfrac{R(0)}{2}$, then

$$V_0(x,t_1) = \frac{\delta^4 R(0)}{2\beta|x|} + A + Bt_1 \geq u(x,t_1) \quad \text{(where } u \text{ is the true solution).}$$

Note that

$$\frac{1}{\beta} + A + Bt_1 + \frac{\delta^4}{\beta|x|}(R(t_1)-R(t)) \geq \frac{\delta^4 R(0)}{2\beta|x|} + A + Bt_1 \quad \text{for all } t \geq t_1 \quad \text{and} \quad |x| \geq \delta^4 R(t).$$

Hence by the similar argument as before, the function

$$V_1(x,t) = \frac{1}{\beta} + A + Bt_1 + \frac{\delta^4}{\beta|x|}(R(t_1)-R(t)) + A + B(t-t_1) = \frac{1}{\beta} + 2A + Bt + \frac{\delta^4}{\beta|x|}(R(t_1)-R(t))$$

is again a super-solution for $t \geq t_1$. Now we have for $t \geq t_1$ that

$$\begin{aligned}\dot{R} &\leq \frac{1}{\beta}\left[V_1 - \frac{1}{R} + g\right] = \frac{1}{\beta}\left[\frac{1}{\beta} + 2A + Bt + \frac{\delta^4\,(R(t_1)-R(t))}{\beta\delta^4 R(t)} - \frac{1}{R} + g\right] \\ &= \frac{1}{\beta R(t)}\left[-1 + \frac{R(t_1)}{\beta} + R(t)\,(2A + Bt + g)\right].\end{aligned}$$

To continue, let t_2 be the time such that $R(t_2) = \frac{R(0)}{4}$. Set

$$V_2(x,t) = \frac{1}{\beta} + \frac{1}{\beta} + 2A + Bt_2 + \frac{\delta^4}{\beta|x|}(R(t_2)-R(t)) + A + B(t-t_2) = \frac{2}{\beta} + 3A + Bt + \frac{\delta^4}{\beta|x|}(R(t_2)-R(t)).$$

It is again a super-solution for $t > t_2$. By induction, let

$$V_n(x,t) = \frac{n}{\beta} + (n+1)A + Bt + \frac{\delta^4}{\beta|x|}(R(t_n) - R(t)) \quad \text{where} \quad R(t_n) = \frac{R(0)}{2^n}.$$

Finally, let

$$V^*(x,t) = \inf_n V_n(x,t), \tag{55}$$

which stands as a super-solution for all $t > 0$, and therefore we obtain

$$\dot{R} \le \frac{1}{\beta R}\left[-1 + \frac{R(t_n)}{\beta} + R(t)\left(\frac{n}{\beta} + (n+1)A + Bt + g(t)\right)\right] \quad \text{for} \quad t_n \le t \le t_{n+1}.$$

The above shows that

$$R\dot{R} \le -\frac{1}{\beta} \quad \text{as} \quad R \longrightarrow 0^+.$$

Combining all the previous analysis of sub- and super-solutions together with the energy estimates from **Section 5**, we have the following regularity theorem for solution of (21).

Theorem 6.3. *Let the initial data u_0^δ, R_{i0}^δ and the inhomogeneous driving forces g_i satisfy the conditions (22), (23) and (24), then for any time $T > 0$ and δ small enough:*

1. *there is a solution u of (21) in $L^\infty\left(0,T;L^2(\Omega)\right)\bigcap L^2\left(0,T;H^1(\Omega)\right)$ satisfying:*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{L^2(\Omega)} + \frac{1}{\delta}\int_0^T \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt \le M < \infty \tag{56}$$

2. *the radii R_i's satisfy $\sup_i \sup_{t\ge 0} R_i(t) < \infty$ and $\sup_i \|R_i\|_{W^{1,p}([0,\min(t_i,T)])} \le M < \infty$ for any $1 \le p < 2$ (where t_i is the vanishing moment for the i-th particle). Furthermore, we have that*

$$\left|R_i\dot{R}_i\right| \le M < \infty \quad \textit{and} \quad \lim_{t\to t_i} R_i\dot{R}_i = -\frac{1}{\beta}, \tag{57}$$

so that $R_i(t) \approx A\,(t_i - t)^{\frac{1}{2}}$ as $t \to t_i$.

With the above regularity result for the heat distribution and evolving radii, our approach now follows quite closely to that of [13]. The steps include: (i) construction of a first order approximation for the heat distribution (**Section 7**); (ii) construction of a first order approximation for the radii (**Section 8**); and (iii) derivation of the limit equations as $\delta \longrightarrow 0$ (**Section 9**). We will still outline the main steps to keep the paper self-contained and to emphasize the essential features, in particular the derivation of the limit equations. On the other hand, there are some differences in the procedure which we will point out in appropriate places.

7. First Order Approximation

The goal here is to produce a good approximation for the heat distribution which are then used to derive the limiting equation for the dynamics of the mean field variable and radii as $\delta \longrightarrow 0$. This is facilitated by the following expression:

$$\zeta^\delta = u^\delta_\infty(t) + \sum_i \left(\frac{1 - R_i(t)u^\delta_\infty(t) - R_i(t)g_i(t)}{R_i(t) + \beta}\right)\frac{\delta^4 R_i(t)}{|x - x_i|}. \tag{58}$$

Using the above, we will construct sub- and super-solutions to control the difference between the actual solution u^δ and the approximation ζ^δ.

For this, we define:

$$u^\delta_\pm = \zeta + w + z \pm M\delta^\gamma \tag{59}$$

where the corrections w^δ and z^δ satisfy:

$$\begin{aligned} \delta w_t &= \triangle w - \delta\partial_t u^\delta_\infty(t) \quad \text{in } \Omega \\ \nabla w\cdot n &= -\nabla\zeta\cdot n \quad \text{on } \partial\Omega \\ w(0,\cdot) &= w_0(\cdot) \end{aligned} \tag{60}$$

and

$$\begin{aligned} \delta z_t &= \triangle z - \delta\sum_i \left(\frac{\Big(1-R_i(t)u^\delta_\infty(t)-R_ig_i(t)\Big)R_i(t)}{R_i(t)+\beta}\right)_t \frac{\delta^4}{|x-x_i|} \quad \text{in } \Omega^\delta_t \\ z &= \frac{\beta}{4\pi\delta^4 R_i^2(t)}\int_{\partial B_i^\delta}\nabla z\cdot n \quad \text{on } \partial B_i^\delta \\ \nabla z\cdot n &= 0 \quad \text{on } \partial\Omega \\ z(0,\cdot) &= z_0(\cdot) \end{aligned} \tag{61}$$

which deal with various inhomogeneous terms of the equation. Their initial data are chosen as $z_0\equiv 0$ and $w_0 = u_0-\zeta_0$ so that all the boundary conditions are satisfied at $t=0$. The M_γ is initially chosen so that $u^\delta_- \le u_0 \le u^\delta_+$.

The estimates for w^δ are summarized by the following lemma.

Lemma 7.1. *If we choose the mean-field variable $u^\delta_\infty(t)$ according to*

$$\partial_t u^\delta_\infty(t) = 4\pi\delta^3\sum_i\Big(1-R_i(t)u^\delta_\infty(t)-R_i(t)g_i(t)\Big)\frac{R_i(t)}{R_i(t)+\beta},\quad u^\delta_\infty(0)=u^\delta_{\infty0}, \tag{62}$$

then for any $0<\gamma<\frac{1}{2}$, there exists a M_γ such that:

$$\|w\|_{L^\infty(\Omega_T)} \text{ and } \|\nabla w\|_{L^\infty(\Omega_T)} \le M_\gamma\delta^\gamma. \tag{63}$$

The proof is omitted as it is exactly the same as [13, Lemma 20] using careful energy type estimates from parabolic regularity theory. But for completeness we will indicate the origin of (62). This equation is to ensure that $\int_\Omega w=0$ so that the behavior of u^δ far away from the interfaces is indeed captured by the mean-field variable u^δ_∞. In addition, technically speaking, the estimate for ∇w is proved first which together with the zero mean condition then gives the estimate for w.

With the above in mind, we integrate (60) and obtain:

$$0=\delta\frac{d}{dt}\int_\Omega w = \int_{\partial\Omega}\triangle w - \delta\partial_t u^\delta_\infty = \int_{\partial\Omega}\nabla w\cdot n - \delta\partial_t u^\delta_\infty.$$

Hence

$$\delta\partial_t u^\delta_\infty = \int_{\partial\Omega}\frac{\partial w}{\partial n} = -\int_{\partial\Omega}\frac{\partial\zeta}{\partial n} = -\int_{\partial\Omega}\sum_i\left(\frac{1-R_i(t)u^\delta_\infty(t)-R_i(t)g_i(t)}{R_i(t)+\beta}\right)\delta^4R_i(t)\nabla\frac{1}{|x-x_i|}\cdot n.$$

As $\int_{\partial\Omega}\nabla\frac{1}{|x|}\cdot n = -4\pi$, the above gives (62).

The estimates for z^δ are stated in the following lemma.

Lemma 7.2. *In the following M_T denotes some generic finite constant independent of δ.*

1. *Let t_i^δ be the vanishing time of B_i^δ. Then,*

$$|z(t)|_{\partial B_i^\delta} \le M_T\left|\log(t_i^\delta-t)\right| \quad \text{for } t<t_i^\delta. \tag{64}$$

2. *Let* $A^\delta = \Omega \backslash \cup_i B(x_i, \frac{\delta}{4})$.

$$\sup_{t\in[0,T]} \frac{1}{\delta^2}\int_\Omega (z(t))^2 + \frac{1}{\delta^3}\int_0^T\int_\Omega |\nabla z|^2 + \frac{1}{\delta}\int_0^T\int_{A^\delta} \left|D^2 z\right|^2 \leq M_T. \tag{65}$$

By Sobolev embedding theorem, the above gives

$$\|z\|_{L^2(L^\infty(A^\delta))} \leq M_T\sqrt{\delta}. \tag{66}$$

Proof. The proof is similar to [13, Lemma 21], again using energy type estimate for parabolic equation, but in the current case with the effect of kinetic undercooling in the parabolic setting, some additional terms appear in the derivation of some energy identity. This leads to the need of estimates of the type (64).

Using h to denote the inhomogeneous term in (61) (without the δ factor), we have:

$$\delta z_t = \triangle z - \delta h.$$

Multiplying the above equation by z and extending z from Ω^δ to Ω by $z\big|_{B_{\delta^4 R_i}} = z\big|_{\partial B_{\delta^4 R_i}}$ lead to

$$\begin{aligned}
\delta\int_{\Omega^\delta(t)} z_t z &= \int_{\Omega^\delta(t)} \triangle z z - \delta\int_{\Omega^\delta(t)} hz\\
\delta\int_\Omega z_t z - \delta\int_{\Omega\backslash\Omega^\delta(t)} z_t z &= \int_{\partial\Omega^\delta(t)} z\frac{\partial z}{\partial n} - \int_{\Omega^\delta}|\nabla z|^2 - \delta\int_\Omega hz\\
\delta\int_\Omega z_t z - \delta\sum_i\left(\frac{4\pi\delta^{12}R_i^3}{3}\right)\dot{z}_i z_i &= -\sum_i 4\pi\delta^8 R_i^2 z_i (z_n)_i - \int_{\Omega^\delta}|\nabla z|^2 - \delta\int_\Omega hz
\end{aligned}$$

As $z = \beta\delta^4 \frac{\partial z}{\partial n}\Big|_{\partial B_i^\delta}$, the above becomes:

$$\delta\int_\Omega z_t z + \sum_i \frac{4\pi\delta^8 R_i^2 z_i^2}{\beta\delta^4} + \int_{\Omega^\delta}|\nabla z|^2 = \delta\sum_i\left(\frac{4\pi\delta^{12}R_i^3}{3}\right)\dot{z}_i z_i - \delta\int_\Omega hz \tag{67}$$

or

$$\delta\frac{d}{dt}\int_\Omega \frac{1}{2}z^2 + \frac{4\pi\delta^4}{\beta}\sum_i R_i^2(t)z_i^2(t) + \int_\Omega|\nabla z|^2 = \frac{4\pi\delta^{13}}{3}\sum_i R_i^3(t)\left(\frac{z_i^2}{2}\right)_t - \delta\int_\Omega hz \tag{68}$$

Integrating in time then gives

$$\begin{aligned}
&\delta\int_\Omega \frac{1}{2}z^2(t) + \frac{4\pi\delta^4}{\beta}\int_0^t\sum_i R_i^2(s)z_i^2(s)\,ds + \int_0^t\int_\Omega|\nabla z|^2 + \delta\int_0^t\int_\Omega hz\\
&= \frac{4\pi\delta^{13}}{3}\sum_i R_i^3(t)\left(\frac{z_i^2}{2}\right)(t) - \frac{4\pi\delta^{13}}{3}\int_0^t\sum_i 3R_i^2(s)\dot{R}_i(s)\left(\frac{z_i^2}{2}\right)(s)\,ds + \delta\int_\Omega\frac{1}{2}z^2(0) - \frac{4\pi\delta^{13}}{3}\sum_i R_i^3(0)\left(\frac{z_i^2}{2}\right)(0).
\end{aligned} \tag{69}$$

From the above, we see that the $z_i(t)$'s appear in the right hand side which force us to consider their estimate.

As $\sup_{t\in[0,T]}\sup_i R_i(t),\ |R_i(t)g_i(t)| < \infty$ we simplify equation (61) as:

$$\delta z_t = \triangle z - \delta\sum_i \frac{\delta^4\left(A_i(t) + B_i(t)\dot{R}_i(t)\right)}{|x - x_i|} \tag{70}$$

for some uniformly bounded smooth function A_i and B_i. The desired sub- and super-solutions are given by

$$z_{\text{sub}}(t) = M_1 + \sum_i \frac{\delta^4 a_i(t)}{|x - x_i|} \quad\text{and}\quad z_{\text{super}}(t) = -M_1 - \sum_i \frac{\delta^4 a_i(t)}{|x - x_i|}$$

where $\dot{a}_i(t) = M_2 + M_3 \left|\dot{R}_i\right|$. M_1, M_2 and M_3 are large enough constants. (This is similar to the construction of the super-solution V in (43).) Then (64) follows from:

$$\begin{aligned} |z_i(t)| &\leq M_1 T + \int_0^t \frac{\dot{a}(s)}{R_i(s)}\,ds \leq M_1 T + \int_0^t \frac{M_2 + M_3\left|\dot{R}(s)\right|}{R_i(s)}\,ds = M_1 T + \int_0^t \frac{M_2 R_i + M_3\left|R_i(s)\dot{R}_i(s)\right|}{R_i^2(s)}\,ds \\ &\leq M_1 T + M\int_0^t \frac{1}{R_i^2(s)}\,ds \leq M_1 T + M\int_0^t \frac{1}{(t_i^\delta - s)}\,ds \leq M_1 T + M\left|\log(t_i^\delta - t)\right|. \end{aligned}$$

By **Theorem 6.3**(2), we see that the right hand side of (69) is bounded by a finite constant. Then the same computations of [13, Lemma 21, pp 172-173] can be applied. They first give

$$\int_{\Omega^\delta} z^2 + \frac{1}{\delta}\int_0^t \int_\Omega |\nabla z|^2 \leq M_T \delta^2$$

and then the higher order regularity:

$$\sup_{t\in[0,T]} \frac{1}{\delta^2}\int_\Omega (z(t))^2 + \frac{1}{\delta^3}\int_0^T \int_\Omega |\nabla z|^2 + \frac{1}{\delta}\int_0^T \int_{A^\delta} \left|D^2 z\right|^2 \leq M_T$$

concluding the proof of (66).

(Note here that we do not need to any give special consideration for new initial data right after some balls have vanished such as in [13, p 167]. This is because the summands in ζ^δ (58) corresponding to the vanishing R_i's automatically become zero.) □

Estimates (63) and (66) together with (58) and (59) give the following corollary which says that far away from the particles, the heat distribution u^δ is close to the mean field variable u_∞^δ.

Corollary 7.3. *For any* $0 < \gamma < \frac{1}{2}$*, there is a constant* M_γ *such that*

$$\left\|u^\delta - u_\infty^\delta(t)\right\|_{L^2([0,T],L^\infty(A^\delta))} \leq M_\gamma \delta^\gamma \tag{71}$$

8. Approximation of the Dynamics of $R_i(t)$'s

The following is the main theorem of this paper which gives the dynamics of the radii as $\delta \longrightarrow 0$.

Theorem 8.1. *Let* u_∞^δ *be given as in (62). Then for any* $\varphi \in W^{1,1}([0,T])$*, it holds that*

$$\left|\int_0^T \varphi\left[R_i(R_i+\beta)\dot{R}_i - (u_\infty^\delta R_i + g_i R_i - 1)\right]dt\right| \leq C_\gamma \delta^\gamma \|\varphi\|_{W^{1,1}}. \tag{72}$$

The above means that in the weak sense, the radii satisfy the following dynamical equation:

$$\dot{R}_i = -\frac{1 - u_\infty^\delta R_i - g_i R_i}{R_i(R_i+\beta)}. \tag{73}$$

The proof is the same as [13, Theorem 2.b]. As this is the key result, we present the steps here to illustrate the main idea and estimates.

Proof. Define:

$$\psi_i(x,t) = \frac{\delta^4 R_i(t)}{|x - x_i|}\eta\left(\frac{|x - x_i|}{\delta}\right)$$

where η is a smooth function such that $\eta(s) \equiv 1$ for $0 \leq s \leq \frac{1}{8}$ and $\eta(s) \equiv 0$ for $s \geq \frac{1}{4}$. This function satisfies:

$$\psi_i\Big|_{\partial B_i^\delta} = 1, \quad \frac{1}{4\pi\delta^4}\int_{\partial B_i^\delta} \nabla\psi_i \cdot n = -R_i$$

and the identity,

$$\int_{\Omega^\delta} \psi_i \triangle u = -\int_{\partial B_i^\delta} \psi_i \frac{\partial u}{\partial n} + \int_{\partial B_i^\delta} u \frac{\partial \psi_i}{\partial n} + \int_{\Omega^\delta} u \triangle \psi_i.$$

By the dynamics of $R_i(t)$, we have $\frac{d}{dt}\left(\frac{1}{3}R_i^3(t)\right) = \frac{1}{4\pi\delta^4}\int_{\partial B_i^\delta} \nabla u \cdot n$ from which we compute

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{3}R_i^3\right) &= \frac{1}{4\pi\delta^4}\int_{\partial B_i^\delta} \psi_i \nabla u \cdot n = \frac{1}{4\pi\delta^4}\int_{\partial B_i^\delta} u\frac{\partial \psi_i}{\partial n} - \frac{1}{4\pi\delta^4}\int_{\Omega^\delta} \psi_i \triangle u + \frac{1}{4\pi\delta^4}\int_{\Omega^\delta} u \triangle \psi_i \\
&= \frac{u_i}{4\pi\delta^4}\int_{\partial B_i} \frac{\partial \psi_i}{\partial n} + \frac{1}{4\pi\delta^4}\int_{\Omega^\delta} (u - u_\infty^\delta(t))\triangle\psi_i - \frac{\delta}{4\pi\delta^4}\int_{\Omega^\delta} \psi_i u_t + \frac{u_\infty^\delta(t)}{4\pi\delta^4}\int_{\Omega^\delta} \triangle\psi_i \quad (\text{as } \delta u_t = \triangle u) \\
&= -R_i u_i - \frac{u_\infty^\delta(t)}{4\pi\delta^4}\int_{\partial B_i} \frac{\partial \psi_i}{\partial n} + \frac{1}{4\pi\delta^4}\int_{\Omega^\delta} (u - u_\infty^\delta(t))\triangle\psi_i - \frac{\delta}{4\pi\delta^4}\int_{\Omega^\delta} \psi_i u_t \\
&= -R_i\left(\frac{1}{R_i} - g_i + \beta\dot{R}_i\right) + u_\infty^\delta(t)R_i(t) + \frac{1}{4\pi\delta^4}\int_{\Omega^\delta} (u - u_\infty^\delta(t))\triangle\psi_i - \frac{\delta}{4\pi\delta^4}\int_{\Omega^\delta} \psi_i u_t \quad \left(\text{as } u_i = \frac{1}{R_i} - g_i + \beta\dot{R}_i\right).
\end{aligned}$$

Hence

$$R_i(R_i+\beta)\dot{R}_i - (u_\infty^\delta R_i + g_i R_i - 1) = \frac{1}{4\pi\delta^4}\int_{\Omega^\delta} (u - u_\infty^\delta(t))\triangle\psi_i - \frac{\delta}{4\pi\delta^4}\int_{\Omega^\delta} \psi_i u_t. \tag{74}$$

Now let φ be a test function on $[0,T]$. Then we have

$$\int_0^T \varphi\left[R_i(R_i+\beta)\dot{R}_i - (u_\infty^\delta R_i + g_i R_i - 1)\right] dt = \int_0^T \varphi\left[\int_{\Omega^\delta} \frac{(u - u_\infty^\delta(t))\triangle\psi_i}{4\pi\delta^4}\right] dt - \delta\int_0^T \varphi\left[\int_{\Omega^\delta} \frac{\psi_i u_t}{4\pi\delta^4}\right] dt. \tag{75}$$

The first term of the right hand side of (75) is estimated as,

$$\int_0^T \varphi\left[\int_{\Omega^\delta} \frac{(u - u_\infty^\delta(t))\triangle\psi_i}{4\pi\delta^4}\right] dt \le \|\varphi\|_{L^\infty(0,T])} \left\|u - u_\infty^\delta(t)\right\|_{L^\infty(\mathrm{supp}(\triangle\psi_i))} \times \frac{1}{4\pi\delta^4}\int_{\mathrm{supp}(\triangle\psi_i)} |\triangle\psi_i| \le C_\gamma \delta^\gamma \|\varphi\|_{L^\infty([0,T])}.$$

For the second term, we compute,

$$\begin{aligned}
\int_0^T \varphi \int_{\Omega^\delta} \frac{\psi_i u_t}{4\pi\delta^4} dt &= \int_0^T \frac{\varphi}{4\pi\delta^4}\left[\int_{\Omega^\delta} ((u\psi_i)_t - u\psi_{i_t})\right] dt \\
&= \int_0^T \frac{\varphi}{4\pi\delta^4}\left[\int_{\Omega^\delta} (u\psi_i)_t - \int_{\Omega^\delta} u\frac{\delta^4\dot{R}_i}{|x-x_i|}\eta\left(\frac{|x-x_i|}{\delta}\right)\right] dt.
\end{aligned}$$

Note that $\int_{\Omega^\delta} (u\psi_i)_t = \left(\int_{\Omega^\delta} u\psi_i\right)_t + (u\psi_i)\Big|_{\partial B_i^\delta} (\delta^4\dot{R}_i)(4\pi\delta^8 R_i^2)$. Hence we obtain

$$\begin{aligned}
&\int_0^T \varphi \int_{\Omega^\delta} \frac{\psi_i u_t}{4\pi\delta^4} dt \\
= &-\int_0^T \varphi_t \int_{\Omega^\delta} \frac{u\psi_i}{4\pi\delta^4} dt - \varphi(0)\int_{\Omega^\delta} \frac{u(\cdot,0)\psi_i}{4\pi\delta^4} + \int_0^T \frac{\varphi\delta^8}{4\pi}\left(\frac{1}{R_i} - g_i + \beta\dot{R}_i\right)\dot{R}_i R_i^2 dt - \int_0^T \frac{\varphi\dot{R}}{4\pi}\int_{\Omega^\delta} \frac{u\eta}{|x-x_i|} dt \\
= &-\int_0^T \varphi_t \int_{\Omega^\delta} \frac{u\psi_i}{4\pi\delta^4} dt - \varphi(0)\int_{\Omega^\delta} \frac{u(\cdot,0)\psi_i}{4\pi\delta^4} + \int_0^T \frac{\varphi\delta^8}{4\pi}\left(R_i\dot{R}_i - g_i R_i^2\dot{R}_i + \beta\dot{R}_i^2 R_i^2\right) dt - \int_0^T \frac{\varphi\dot{R}}{4\pi}\int_{\Omega^\delta} \frac{u\eta}{|x-x_i|} dt.
\end{aligned}$$

By the fact that:

$$\|u\|_{L^\infty(L^2(\Omega^\delta))}, \quad \left\|\frac{1}{|x|}\right\|_{L^2(\Omega^\delta)}, \quad \left\|\frac{\psi_i}{4\pi\delta^4}\right\|_{L^\infty(L^2(\Omega^\delta))}, \quad \left\|R_i\dot{R}_i\right\|_{L^\infty([0,T])}, \quad \left\|\dot{R}_i\right\|_{L^1([0,T])} \le M$$

we finally have the conclusion:

$$\left|\int_0^T \varphi\left[R_i(R_i+\beta)\dot{R}_i - (u_\infty^\delta R_i + g_i R_i - 1)\right] dt\right| \le M_\gamma \delta^\gamma \|\varphi\|_{W^{1,1}([0,T])}. \tag{76}$$

9. Limit problem as $\delta \to 0$

This section discuss the limit of u^δ and R_i^δ's as $\delta \longrightarrow 0$. With the estimates derived so far, in principle, all the results of [13, 14] carry over. However, in order to obtain an equation which is *closed* in the limit, we do need to invoke the assumption (25) on the form of the inhomogeneous forces g_i's. This will also motivate the incorporation of white noise in the future work so that the machinery of stochastic analysis is applicable.

Since the estimates are the same as in [13, 14], we will omit the proof of the existence of a limit which is a consequence of general compactness result. Instead, we will concentrate on the derivation of the limit equations. For this, we introduce the empirical measure $\nu^\delta \in L^1\left(0, T; C^0(0, K_T])\right)^*$ of the radii:

$$\left\langle \nu^\delta, \varphi \right\rangle = \int_0^T \frac{1}{N_a^\delta(t)} \sum_{i \in N_a^\delta(t)} \varphi(t, R_i^\delta(t))\, dt \quad \text{for } \varphi \in L^1([0, T]; C^0[0, K_T]) \tag{77}$$

where $K_T = \sup_{i,\delta} \left\| R_i^\delta \right\|_{L^\infty(0,T)}$. Then we have the following convergence result:

Lemma 9.1. *Given any $T < \infty$, there exist a $\nu \in L^1\left(0, T; C^0([0, K_T]\right)^*$ and $u_\infty \in W^{1,p}(0, T)$ ($1 \le p < \infty$) such that for a subsequence of $\delta \longrightarrow 0$, the following hold:*

$$\nu^\delta \rightharpoonup \nu \quad \text{in the weak}* \text{ topology of } L^1\left(0, T; C^0[0, K_T]\right)^* \tag{78}$$

$$u_\infty^\delta \longrightarrow u_\infty \quad \text{uniformly in } (0, T) \tag{79}$$

$$u^\delta \longrightarrow u_\infty \quad \text{in } L^2(0, T; H^1(\Omega)). \tag{80}$$

Furthermore, there exists a family of probability measures $\{\nu_{(t)}\}_{t\ge 0} \subset C^0[0, K_T]^*$ and a non-negative function $\alpha \in L^\infty(0, T)$ such that

$$\langle \nu, \varphi \rangle = \int_0^T \int \varphi(t, R)\, d\nu_{(t)}(R)\, \alpha(t)\, dt \quad \text{for } \varphi \in L^1\left(0, T; C^0[0, K_T]\right)^*. \tag{81}$$

In the above, $\alpha(t) = \lim_{\delta\to 0} \frac{|N_a^\delta|}{N}$ represents the percentage of active particles in the system.

The proof of the above is some application of convergence of measures and L^p spaces. The specific concept used is that of Young measures. For details, see [13, Lemmas 7, 8] and [14, Lemma 5.1].

In order to have a closed equation in the limit, we state here again the assumption about the functional form for the $g_i(t)$'s:

there exists a function $G \in C^1(R_+ \times R_+)$ and a function $h \in C^1(R_+)$ such that $g_i(t) = G(t, R_i(t)) + h(t)$. (25)

We will make some remarks about this assumption after presenting the main theorem which is stated as:

Theorem 9.2. *The mean field variable u_∞ and the distribution ν satisfy:*

$$\partial_t u_\infty(t) = 4\pi \int_O^\infty \Big(1 - R u_\infty(t) - R G(t, R) - R h(t)\Big) \frac{R}{R + \beta}\, d\nu_{(t)}(R)\, \alpha(t) \tag{82}$$

and

$$\int_0^T \int \Big\{\partial_t \varphi(t, R) + V(t, R) \partial_R \varphi(t, R)\Big\}\, d\nu_{(t)}(R) \alpha(t)\, dt + \int \varphi(0, R) d\nu_0(R) = 0 \tag{83}$$

for all $\varphi \in C_0^\infty([0, T] \times R_+)$, where

$$V(t, R) = -\frac{1 - R u_\infty(t) - R G(t, R) - R h(t)}{R(R + \beta)} \tag{84}$$

and ν_0 is the limit of the empirical measure of the initial radii R_{i0}^δ.

Proof. For (82), let $\eta \in C_0^1(0,T)$. Then

$$\int_0^T \eta(t)\left(u_\infty^\delta\right)_t\,dt = \int_0^T \eta(t)\left[4\pi\delta^3\sum_i \left((1 - R_i^\delta u_\infty^\delta - R_i^\delta g_i\right)\frac{R_i^\delta}{R_i^\delta + \beta}\right]dt.$$

For the left hand side of the above, we have

$$\int_0^T \eta(t)\left(u_\infty^\delta\right)_t\,dt = -\int_0^T \eta_t(t)u_\infty^\delta\,dt \longrightarrow -\int_0^T \eta_t(t)u_\infty\,dt = \int_0^T \eta(t)\left(u_\infty\right)_t\,dt.$$

For the right hand side, we express it in terms of the empirical measure ν^δ:

$$\int_0^T \eta(t)\left[4\pi\delta^3\sum_i \left((1 - R_i^\delta u_\infty^\delta - R_i^\delta g_i\right)\frac{R_i^\delta}{R_i^\delta + \beta}\right]dt = \left\langle \nu^\delta,\ \Phi^\delta\right\rangle$$

where $\Phi^\delta(t,R) = 4\pi\eta(t)\left[1 - Ru_\infty^\delta(t) - R(G(t,R) + h(t))\right]\dfrac{R}{R+\beta}$. By the strong convergence of u_∞^δ to u_∞ and the form of g_i's, we have that

$$\left\langle \nu^\delta,\ \Phi^\delta\right\rangle \longrightarrow \int_0^T \eta(t)\int 4\pi\Big(1 - Ru_\infty - RG(t,R) - Rh(t)\Big)\frac{R}{R+\beta}\,d\nu_{(t)}(R)\alpha(t)\,dt$$

which gives (82).

For (83), consider for any $\phi \in C_0^\infty([0,T], R_+)$:

$$\int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\frac{d}{dt}\phi(t, R_i^\delta(t))\right]dt + \frac{1}{|N|}\sum_{i\in N}\phi(0, R_{i0}^\delta)\,dt = 0.$$

The convergence of the second term is trivial. For the first term, we compute:

$$\int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\frac{d}{dt}\phi(t, R_i^\delta(t))\right]dt = \int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_t(t, R_i^\delta(t))\right]dt + \int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_R(t, R_i^\delta(t))\dot{R}_i^\delta\right]dt.$$

The first term on the right becomes:

$$\int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_t(t, R_i^\delta(t))\right]dt = \left\langle \nu^\delta,\ \eta\partial_t\phi\right\rangle \longrightarrow \langle \nu,\ \eta\partial_t\phi\rangle\,.$$

For the second term, we compute:

$$\begin{aligned}
&\int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_R(t, R_i^\delta(t))\dot{R}_i^\delta\right]dt \\
= \;&\int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_R(t, R_i^\delta(t))\left(\dot{R}_i^\delta - V(t, R_i^\delta)\right)\right]dt + \int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_R(t, R_i^\delta(t))V(t, R_i^\delta)\right]dt.
\end{aligned}$$

As ϕ has compact support, only the values of the radii which are bounded away from zero matter in the computation. Hence a trivial modification of the proof of **Theorem 8.1**, in particular the steps (74) and (75) give

$$\int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_R(t, R_i^\delta(t))\left(\dot{R}_i^\delta - V(t, R_i^\delta)\right)\right]dt \longrightarrow 0.$$

Finally we have the converge:

$$\int_0^T \eta(t)\left[\frac{1}{|N|}\sum_{i\in N}\phi_R(t, R_i^\delta(t))V(t, R_i^\delta)\right]dt \longrightarrow \langle \nu,\ \eta\phi_R\rangle$$

which all together gives (83) completing the proof of the **Theorem**.

Remark 9.3. *Here we explain the need to impose the functional form (25) for the inhomogeneous forces. From the derivation of the limit equations, we are forced to deal with summations in the form of*

$$\int_0^T \varphi(t) \sum_i F\left(t,\ R_i(t),\ \{R_j(s)\}_{j,\,0\le s\le t},\ g_i(t)\right) dt \quad \textit{for some nonlinear function } F.$$

The dependence on $\{R_j(s)\}_{j,\,0\le s\le t}$ *is due to the mean-field* $u_\infty^\delta(t)$ *variable. In principle the above can all be expressed in terms of some Young measures. But it is not clear if there is any meaningful equation we can obtain to describe these Young measures. The limit equations will thus not be closed – the usual problem when dealing with weak convergence in nonlinear equations. Imposing some probabilistic independence among the* g_i *does not help immediately due to the non-local dependence in time. A reasonable alternative is to consider* white noise *for the* g_i*'s so that techniques from* $\mathit{It}\widehat{o}$*'s calculus can be used to take advantage of the stochastic cancellation in time. Such an approach is used in many works deriving continuum equations from particle systems with mean-field or long range interactions. This will be investigated in some future works.*

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