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A RIEMANN-HILBERT PROBLEM IN A RIEMANN SURFACE*

Dedicated to Professor Peter D. Lax on occasion of his 85th birthday

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Abstract One of the inspirations behind Peter Lax's interest in dispersive integrable systems, as the small dispersion parameter goes to zero, comes from systems of ODEs discretizing 1-dimensional compressible gas dynamics [17]. For example, an understanding of the asymptotic behavior of the Toda lattice in different regimes has been able to shed light on some of von Neumann's conjectures concerning the validity of the approximation of PDEs by dispersive systems of ODEs.

Back in the 1990s several authors have worked on the long time asymptotics of the Toda lattice [2, 7, 8, 19]. Initially the method used was the method of Lax and Levermore [16], reducing the asymptotic problem to the solution of a minimization problem with constraints (an "equilibrium measure" problem). Later, it was found that the asymptotic method of Deift and Zhou (analysis of the associated Riemann-Hilbert factorization problem in the complex plane) could apply to previously intractable problems and also produce more detailed information.

Recently, together with Gerald Teschl, we have revisited the Toda lattice; instead of solutions in a constant or steplike constant background that were considered in the 1990s we have been able to study solutions in a periodic background.

Two features are worth noting here. First, the associated Riemann-Hilbert factorization problem naturally lies in a hyperelliptic Riemann surface. We thus generalize the Deift-Zhou "nonlinear stationary phase method" to surfaces of nonzero genus. Second, we illustrate the important fact that very often even when applying the powerful Riemann-Hilbert method, a Lax-Levermore problem is still underlying and understanding it is crucial in the analysis and the proofs of the Deift-Zhou method!

Key words Riemann–Hilbert problem; Toda lattice

2000 MR Subject Classification 37K40; 37K45; 35Q15; 37K10

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1 Introduction. A Discretization of a System from Gas Dynamics

Consider the following system of 1-dimensional compressible gas dynamics.

$$\begin{aligned} &u_t + p_x = 0, \\ &V_t - u_x = 0, \\ &p = p(V); \quad p'(V) < 0. \end{aligned} \tag{1.1}$$

Here u, p, V denote velocity, pressure and specific volume (reciprocal of the density) respectively. These equations express the conservation of momentum, conservation of mass and the equation of state.

Consider also the following semi-discretization:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}u_k + \frac{p_{k+1/2} - p_{k-1/2}}{\Delta} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}V_{k+1/2} - \frac{u_{k+1} - u_k}{\Delta} = 0, \end{aligned} \tag{1.2}$$

where $p_{k\pm1/2} = p(V_{k\pm1/2})$.

Let X_k be chosen such that $\frac{\mathrm{d}X_k}{\mathrm{d}t} = u_k$ and at $t = 0$,

$$\frac{X_{k+1} - X_k}{\Delta} = V_{k+1/2}. \tag{1.3}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}t}\frac{X_{k+1}-X_k}{\Delta} = \frac{\mathrm{d}}{\mathrm{d}t}V_{k+1/2}$. So, (1.3) holds for all time.

We end up with

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{p(\frac{X_{k+1}-X_k}{\Delta}) - p(\frac{X_k - X_{k-1}}{\Delta})}{\Delta} = 0. \tag{1.4}$$

This is a discretization of the hyperbolic equation

$$X_{tt} + [p(X_x)]_x = 0. \tag{1.5}$$

As is well known hyperbolic equations suffer a shock at a certain finite time. A natural question (asked by von Neumann in his study of dispersive schemes) is the following: after the appearance of violent oscillations (known numerically to be of frequency $O(1/\Delta)$), is there a weak limit of the solution to the dispersive ODE system? If yes, does the weak limit satisfy the original equations?

It is now known, thanks to the detailed analysis of the 1990s [19], [7], which was based on [16], that, at least in the case of the Toda lattice $p(V) = e^{-V}$, weak limits exist but do not satisfy the original equation.

In this article, we show how to extend the analysis of the Toda lattice when the background is periodic.

2 Long Time Asymptotics of the Periodic Toda Lattice under Short-Range Perturbations and the Riemann-Hilbert method

Consider the doubly infinite Toda lattice in Flaschka's variables

$$\begin{aligned} \dot{b}(n,t) &= 2(a(n,t)^2 - a(n-1,t)^2), \\ \dot{a}(n,t) &= a(n,t)(b(n+1,t) - b(n,t)), \end{aligned} \tag{2.1}$$

$(n, t) \in \mathbb{Z} \times \mathbb{R}$, where the dot denotes differentiation with respect to time.

In the case where one has a constant background (same on both left and right infinity) the long-time asymptotics were first computed by Novokshenov and Habibullin in the 1980s and later made rigorous by the author in 1993 [8], under the additional assumption that no solitons are present. (The full case (with solitons) was only recently presented by Krüger and Teschl).

Here we will consider a quasi-periodic algebro-geometric background solution (a_q, b_q), to be described in the next section, plus a short-range perturbation (a, b) satisfying

$$\sum_{n\in\mathbb{Z}} n^6(|a(n,t) - a_q(n,t)| + |b(n,t) - b_q(n,t)|) < \infty \tag{2.2}$$

for $t = 0$ and hence for all $t \in \mathbb{R}$. The perturbed solution can be computed via the inverse scattering transform. The case where (a_q, b_q) is constant is classical while the more general case we want here was solved only recently in [5].

To fix our background solution, consider a hyperelliptic Riemann surface of genus g with real moduli $E_0, E_1, \cdots, E_{2g+1}$. Choose a Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and introduce

$$\underline{z}(n,t) = \underline{\hat{A}}_{p_0}(\infty_+) - \underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - n\underline{\hat{A}}_{\infty_-}(\infty_+) + t\underline{U}_0 - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \tag{2.3}$$

where $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors) and $\underline{\Xi}_{p_0}$, $\underline{U}_0$ are some constants defined in the Appendix. Then our background solution is given in terms of Riemann theta functions by

$$\begin{aligned} a_q(n,t)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2}, \\ b_q(n,t) &= \tilde{b} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\right), \end{aligned} \tag{2.4}$$

where $\tilde{a}$, $\tilde{b} \in \mathbb{R}$ are again some constants.

We can of course view this hyperelliptic Riemann surface as formed by cutting and pasting two copies of the complex plane along bands. Having this picture in mind, we denote the standard projection to the complex plane by π.

Assume for simplicity that the Jacobi operator

$$H(t)f(n) = a(n,t)f(n+1) + a(n-1,t)f(n-1) + b(n,t)f(n), \quad f \in \ell^2(\mathbb{Z}), \tag{2.5}$$

corresponding to the perturbed problem (2.1) has no eigenvalues. Then, for long times the perturbed Toda lattice is asymptotically close to the following limiting lattice defined by

$$\begin{aligned} \prod_{j=n}^{\infty}\left(\frac{a_l(j,t)}{a_q(j,t)}\right)^2 &= \frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\frac{\theta(\underline{z}(n-1,t)+\underline{\delta}(n,t))}{\theta(\underline{z}(n,t)+\underline{\delta}(n,t))} \\ &\quad \times \exp\left(\frac{1}{2\pi \mathrm{i}}\int_{C(n/t)} \log(1-|R|^2)\omega_{\infty_+\,\infty_-}\right), \\ \delta_\ell(n,t) &= \frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\zeta_\ell, \end{aligned} \tag{2.6}$$

where R is the reflection coefficient defined when considering scattering with respect to the periodic background (see the second appendix), ζ_ℓ is a canonical basis of holomorphic differentials, $\omega_{\infty_+\,\infty_-}$ is an Abelian differential of the third kind defined in (A.15), and $C(n/t)$ is a contour on the Riemann surface. More specific, $C(n/t)$ is obtained by taking the spectrum of the unperturbed Jacobi operator H_q between $-\infty$ and a special stationary phase point $z_j(n/t)$, for the phase of the underlying Riemann–Hilbert problem defined in the appendix, and lifting it to the Riemann surface (oriented such that the upper sheet lies to its left). The point $z_j(n/t)$ will move from $-\infty$ to $+\infty$ as n/t varies from $-\infty$ to $+\infty$. From the products above, one easily recovers $a_l(n,t)$. More precisely, we have the following.

Theorem 2.1 Let C be any (large) positive number and δ be any (small) positive number. Consider the region $D = \{(n,t) : |\frac{n}{t}| < C\}$. Then one has

$$\prod_{j=n}^{\infty} \frac{a_l(j,t)}{a(j,t)} \to 1 \tag{2.7}$$

uniformly in D, as $t \to \infty$.

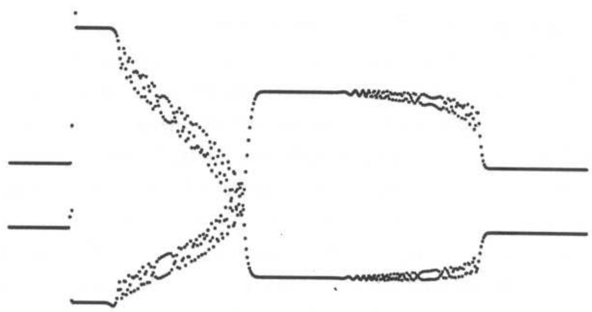

Fig.1 Numerically computed solution of the Toda lattice, with initial condition a period two solution perturbed at one point in the middle.

Remark 2.2 (i) A naive guess would be that the perturbed periodic lattice approaches the unperturbed one in the uniform norm. However, this is wrong: In Figure 1 the two observed lines express the variables $a(n,t)$ of the Toda lattice (see (2.1) below) at a frozen (fairly large) time t. In areas where the lines seem to be continuous this is due to the fact that we have plotted a huge number of particles and also due to the 2-periodicity in space. So one can think of the two lines as the even- and odd-numbered particles of the lattice. We first note the single soliton which separates two regions of apparent periodicity on the left. Also, after the soliton, we observe three different areas with apparently periodic solutions of period two. Finally there are some transitional regions in between which interpolate between the different period two regions. The theorem above gives a rigorous and complete mathematical explanation of this picture.

(ii) It is easy to see how the asymptotic formula above describes the picture given by the numerics. Recall that the spectrum $\sigma(H_q)$ of H_q consists of $g+1$ bands whose band edges are the branch points of the underlying hyperelliptic Riemann surface. If $\frac{n}{t}$ is small enough, $z_j(n/t)$ is to the left of all bands implying that $C(n/t)$ is empty and thus $\delta_\ell(n,t) = 0$; so we recover the purely periodic lattice. At some value of $\frac{n}{t}$ a stationary phase point first appears in the first band of $\sigma(H_q)$ and begins to move form the left endpoint of the band towards the right endpoint of the band. (More precisely we have a pair of stationary phase points z_j and z_j^*, one in each sheet of the hyperelliptic curve, with common projection $\pi(z_j)$ on the complex plane.) So $\delta_\ell(n,t)$ is now a non-zero quantity changing with $\frac{n}{t}$ and the asymptotic lattice has a slowly modulated non-zero phase. Also the factor given by the exponential of the integral is non-trivially changing with $\frac{n}{t}$ and contributes to a slowly modulated amplitude. Then, after the stationary phase point leaves the first band there is a range of $\frac{n}{t}$ for which no stationary phase point appears in the spectrum $\sigma(H_q)$, hence the phase shift $\delta_\ell(n,t)$ and the integral remain constant, so the asymptotic lattice is periodic (but with a non-zero phase shift). Eventually a stationary phase point appears in the second band, so a new modulation appears and so on. Finally, when $\frac{n}{t}$ is large enough, so that all bands have been traversed by the stationary phase point(s), the asymptotic lattice is again periodic. Periodicity properties of theta functions easily show that phase shift is actually cancelled by the exponential of the integral and we recover the original periodic lattice with no phase shift at all.

(iii) If eigenvalues are present one can apply appropriate Darboux transformations to add the effect of such eigenvalues. Alternatively one can modify the Riemann-Hilbert by adding small circles around the extra poles coming from the eigenvalues and applying some of the methods in [2]. What we then see asymptotically is travelling solitons in a periodic background. Note that this will change the asymptotics on one side. See [12] for the exact statement and proof.

(iv) It is very easy to also show that in any region $|\frac{n}{t}| > C$, one has

$$\prod_{j=n}^{\infty} \frac{a_l(j,t)}{a(j,t)} \to 1 \tag{2.8}$$

uniformly in t, as $n \to \infty$.

By dividing in (2.6) one recovers the $a(n,t)$. It follows from the theorem above that

$$|a(n,t) - a_l(n,t)| \to 0 \tag{2.9}$$

uniformly in D, as $t \to \infty$. In other words, the perturbed Toda lattice is asymptotically close to the limiting lattice above.

A similar theorem can be proved for the velocities $b(n,t)$.

Theorem 2.3 In the region $D = \{(n,t) : |\frac{n}{t}| < C\}$, of Theorem 2.1 we also have

$$\sum_{j=n}^{\infty} \big(b_l(j,t) - b_q(j,t)\big) \to 0 \tag{2.10}$$

uniformly in D, as $t \to \infty$, where b_l is given by

$$\begin{aligned} &\sum_{j=n}^{\infty} \big(b_l(j,t) - b_q(j,t)\big) \\ &= \frac{1}{2\pi \mathrm{i}} \int_{C(n/t)} \log(1-|R|^2)\Omega_0 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s} \log\left(\frac{\theta(\underline{z}(n,s) + \underline{\delta}(n,t))}{\theta(\underline{z}(n,s))} \right)\Bigg|_{s=t} \end{aligned} \tag{2.11}$$

and Ω_0 is an Abelian differential of the second kind defined in (A.16).

The next question we address here concerns the higher order asymptotics. Namely, what is the rate at which the perturbed lattice approaches the limiting lattice? Even more, what is the exact asymptotic formula? The answer is given by

Theorem 2.4 Let D_j be the sector $D_j = \{(n,t),:\ z_j(n/t) \in [E_{2j}+\varepsilon, E_{2j+1}-\varepsilon]$ for some $\varepsilon > 0$. Then one has

$$\prod_{j=n}^{\infty}\left(\frac{a(j,t)}{a_l(j,t)}\right)^2 = 1 + \sqrt{\frac{\mathrm{i}}{\phi''(z_j(n/t))t}}2\mathrm{Re}\left(\overline{\beta(n,t)}\mathrm{i}\Lambda_0(n,t)\right) + O(t^{-\alpha}) \tag{2.12}$$

and

$$\sum_{j=n+1}^{\infty}\big(b(j,t) - b_l(j,t)\big) = \sqrt{\frac{\mathrm{i}}{\phi''(z_j(n/t))t}}2\mathrm{Re}\left(\overline{\beta(n,t)}\mathrm{i}\Lambda_1(n,t)\right) + O(t^{-\alpha}) \tag{2.13}$$

for any $\alpha < 1$ uniformly in D_j, as $t \to \infty$. Here

$$\phi''(z_j)/\mathrm{i} = \frac{\prod\limits_{k=0,k\neq j}^{g}(z_j - z_k)}{\mathrm{i}R_{2g+2}^{1/2}(z_j)} > 0, \tag{2.14}$$

(where $\phi(p,n/t)$ is the phase function defined in (B.17) and $R_{2g+2}^{1/2}(z)$ the square root of the underlying Riemann surface),

$$\begin{aligned}\Lambda_0(n,t) &= \omega_{\infty_-\,\infty_+}(z_j) + \sum_{k,\ell} c_{k\ell}(\underline{\hat{\nu}}(n,t))\int_{\infty_+}^{\infty_-}\omega_{\hat{\nu}_\ell(n,t),0}\zeta_k(z_j),\\ \Lambda_1(n,t) &= \omega_{\infty_-,0}(z_j) - \sum_{k,\ell} c_{k\ell}(\underline{\hat{\nu}}(n,t))\omega_{\hat{\nu}_\ell(n,t),0}(\infty_+)\zeta_k(z_j),\end{aligned} \tag{2.15}$$

with $c_{k\ell}(\underline{\hat{\nu}}(n,t))$ some constants defined by

$$(c_{\ell k}(\underline{\hat{\nu}}))_{1\le\ell,k\le g} = \left(\sum_{j=1}^{g} c_k(j)\frac{\mu_\ell^{j-1}d\pi}{R_{2g+2}^{1/2}(\hat{\mu}_\ell)}\right)^{-1}_{1\le\ell,k\le g} \tag{2.16}$$

where $c_k(j)$ are defined in (A.6), $\omega_{q,0}$ an Abelian differential of the second kind with a second order pole at q,

$$\begin{aligned}\beta &= \sqrt{\nu}\mathrm{e}^{\mathrm{i}(\pi/4-\arg(R(z_j)))+\arg(\Gamma(\mathrm{i}\nu))-2\nu\alpha(z_j))}\left(\frac{\phi''(z_j)}{\mathrm{i}}\right)^{\mathrm{i}\nu}\mathrm{e}^{-t\phi(z_j)}t^{-\mathrm{i}\nu}\\ &\quad\times\frac{\theta(\underline{z}(z_j,n,t)+\underline{\delta}(n,t))}{\theta(\underline{z}(z_j,0,0))}\frac{\theta(\underline{z}(z_j^*,0,0))}{\theta(\underline{z}(z_j^*,n,t)+\underline{\delta}(n,t))}\\ &\quad\times\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log\left(\frac{1-|R|^2}{1-|R(z_j)|^2}\right)\omega_{p\,p^*}\right),\end{aligned} \tag{2.17}$$

where $\Gamma(z)$ is the gamma function,

$$\nu = -\frac{1}{2\pi}\log(1-|R(z_j)|^2) > 0 \tag{2.18}$$

and $\alpha(z_j)$ is a real constant defined by

$$\frac{1}{2}\int_{C(n/t)}\omega_{p\,p^*} = \pm\log(z-z_j) \pm \alpha(z_j) + O(z-z_j), \quad p = (z,\pm), \tag{2.19}$$

where $\omega_{p\,q}$ is the meromorphic differential of the third kind with poles at p, q with residues $1, -1$ respectively.

The last theorem and its proof have the following interpretation: even when a Riemann-Hilbert problem needs to be considered on an algebraic variety, a localized parametrix Riemann-Hilbert problem need only be solved in the complex plane and the local solution can then be glued to the global Riemann-Hilbert solution on the variety.

Similarly one can study the local higher asymptotics in the small (decaying actually) "resonance" regions that we excluded in the last theorem above. There is a Painlevé region where the higher order asymptotics are given in terms of a solution of Painlevé II and a "collisionless shock" region where the higher order asymptotics are given in terms of the elliptic cosine function cn.

The proof of all three theorems is given in [15]. It is a stationary phase type argument. One reduces the given Riemann-Hilbert problem to a localized parametrix Riemann-Hilbert problem. This is done via the solution of a scalar global Riemann-Hilbert problem which is solved explicitly with the help of the Riemann-Roch theorem. The reduction to a localized parametrix Riemann-Hilbert problem is done with the help of a theorem reducing general Riemann-Hilbert problems to singular integral equations. (A generalized Cauchy transform is defined appropriately for each Riemann surface.) The localized parametrix Riemann-Hilbert problem is solved explicitly in terms of parabolic cylinder functions. The argument follows [3] up to a point but also extends the theory of Riemann-Hilbert problems for Riemann surfaces. The right (well-posed) Riemann-Hilbert factorization problems are no more holomorphic but instead have a number of poles equal to the genus of the surface.

3 The Generalized Toda Shock Problem and the Return of Lax-Levermore

Suppose now that we have different backgrounds at the infinite ends of the lattice. This could mean different genus or even same genus but different isospectral class. What happens then? This is the subject of an ongoing investigation wih Gerald Teschl and our theorem will not be stated in full detail here. We will only note a few things.

First, as in the previous case, the n, t-plane is also divided in several regions. There are "modulated" regions and periodic" regions (which actually become "soliton" regions when eigenvalues are allowed).

The main difference is that not all regions are associated to Riemann surfaces of the same genus. They couldn't, since by assumption we have data of different genus at the infinite ends of the lattice. So, as n/t ranges from $-\infty$ to $+\infty$ the moduli of the underlying Riemann surface change, but also there are singular pinching points where there is a change in genus.

On the level of the Riemann-Hilbert analysis, the underlying Riemann surface can be chosen to have genus compatible with either (left or right) asymptotics of the initial data. But now there is an extra issue: instead of the constant band/gap structure of the Riemann surface, there is also an additional n/t dependent band/gap structure coming from a Lax-Levermore type variational problem. The support of the minimizer is a finite union of bands!

As already noted in [4] and numerous publications in the application of the Riemann-

Hilbert theory to orthogonal polynomials and random matrices (see [1] for a clear detaied exposition), when one needs to use the full power of the Deift-Zhou method an underlying Lax-Levermore problem has to be understood. In fact, a minimizer (or in some cases a maxi-minimizer [10, 11, 13]) has to be constructed![1] In our case, the Lax-Levermore minimizer lives on a curve in a hyperelliptic surface. Conjugation by the log-transform of the minimizer enables us to transform our given Riemann-Hilbert problem to one that is explicitly solvable.

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[1] While the original Lax-Levermore problem was defined on the real line, the maxi-min extension is in general defined in the complex plane or even in an infinite-sheeted Riemann surface.

A Algebro-geometric Quasi-periodic Finite-gap Solutions

We present some facts on our background solution (a_q, b_q) which we want to choose from the class of algebro-geometric quasi-periodic finite-gap solutions, that is the class of stationary solutions of the Toda hierarchy. In particular, this class contains all periodic solutions. We will use the same notation as in [18], where we also refer to for proofs.

To set the stage let $\mathbb{M}$ be the Riemann surface associated with the following function

$$R_{2g+2}^{1/2}(z), \qquad R_{2g+2}(z) = \prod_{j=0}^{2g+1}(z - E_j), \qquad E_0 < E_1 < \cdots < E_{2g+1}, \tag{A.1}$$

$g \in \mathbb{N}$. $\mathbb{M}$ is a compact, hyperelliptic Riemann surface of genus g. We will choose $R_{2g+2}^{1/2}(z)$ as the fixed branch

$$R_{2g+2}^{1/2}(z) = -\prod_{j=0}^{2g+1}\sqrt{z - E_j}, \tag{A.2}$$

where $\sqrt{.}$ is the standard root with branch cut along $(-\infty, 0)$.

A point on $\mathbb{M}$ is denoted by $p = (z, \pm R_{2g+2}^{1/2}(z)) = (z, \pm)$, $z \in \mathbb{C}$, or $p = (\infty, \pm) = \infty_\pm$, and the projection onto $\mathbb{C} \cup \{\infty\}$ by $\pi(p) = z$. The points $\{(E_j, 0), 0 \le j \le 2g+1\} \subseteq \mathbb{M}$ are called branch points and the sets

$$\Pi_\pm = \{(z, \pm R_{2g+2}^{1/2}(z)) \mid z \in \mathbb{C} \setminus \bigcup_{j=0}^{g}[E_{2j}, E_{2j+1}]\} \subset \mathbb{M} \tag{A.3}$$

are called upper, lower sheet, respectively.

Let $\{a_j, b_j\}_{j=1}^g$ be loops on the surface $\mathbb{M}$ representing the canonical generators of the fundamental group $\pi_1(\mathbb{M})$. We require a_j to surround the points E_{2j-1}, E_{2j} (thereby changing sheets twice) and b_j to surround E_0, E_{2j-1} counterclockwise on the upper sheet, with pairwise intersection indices given by

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{i,j}, \qquad 1 \le i, j \le g. \tag{A.4}$$

The corresponding canonical basis $\{\zeta_j\}_{j=1}^g$ for the space of holomorphic differentials can be constructed by

$$\underline{\zeta} = \sum_{j=1}^{g} \underline{c}(j) \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}}, \tag{A.5}$$

where the constants $\underline{c}(.)$ are given by

$$c_j(k) = C_{jk}^{-1}, \qquad C_{jk} = \int_{a_k} \frac{\pi^{j-1} d\pi}{R_{2g+2}^{1/2}} = 2\int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{R_{2g+2}^{1/2}(z)} \in \mathbb{R}. \tag{A.6}$$

The differentials fulfill

$$\int_{a_j} \zeta_k = \delta_{j,k}, \quad \int_{b_j} \zeta_k = \tau_{j,k}, \quad \tau_{j,k} = \tau_{k,j}, \quad 1 \le j, k \le g. \tag{A.7}$$

Now pick g numbers (the Dirichlet eigenvalues)

$$(\hat{\mu}_j)_{j=1}^g = (\mu_j, \sigma_j)_{j=1}^g \tag{A.8}$$

whose projections lie in the spectral gaps, that is, $\mu_j \in [E_{2j-1}, E_{2j}]$. Associated with these numbers is the divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ which is one at the points $\hat{\mu}_j$ and zero else. Using this divisor we introduce

$$\begin{aligned} \underline{z}(p,n,t) &= \underline{\hat{A}}_{p_0}(p) - \underline{\hat{\alpha}}_{p_0}(\mathcal{D}_{\underline{\hat{\mu}}}) - n\underline{\hat{A}}_{\infty_-}(\infty_+) + t\underline{U}_0 - \underline{\hat{\Xi}}_{p_0} \in \mathbb{C}^g, \\ \underline{z}(n,t) &= \underline{z}(\infty_+, n, t), \end{aligned} \tag{A.9}$$

where $\underline{\Xi}_{p_0}$ is the vector of Riemann constants

$$\hat{\Xi}_{p_0,j} = \frac{j + \sum\limits_{k=1}^{g} \tau_{j,k}}{2}, \qquad p_0 = (E_0, 0), \tag{A.10}$$

$\underline{U}_0$ are the b-periods of the Abelian differential Ω_0 defined below, and $\underline{A}_{p_0}$ ($\underline{\alpha}_{p_0}$) is Abel's map (for divisors). The hat indicates that we regard it as a (single-valued) map from $\hat{\mathbb{M}}$ (the fundamental polygon associated with $\mathbb{M}$ by cutting along the a and b cycles) to $\mathbb{C}^g$. We recall that the function $\theta(\underline{z}(p,n,t))$ has precisely g zeros $\hat{\mu}_j(n,t)$ (with $\hat{\mu}_j(0,0) = \hat{\mu}_j$), where $\theta(\underline{z})$ is the Riemann theta function of $\mathbb{M}$.

Then our background solution is given by

$$\begin{aligned} a_q(n,t)^2 &= \tilde{a}^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2}, \\ b_q(n,t) &= \tilde{b} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log\Big(\frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\Big). \end{aligned} \tag{A.11}$$

The constants $\tilde{a}$, $\tilde{b}$ depend only on the Riemann surface (see [18, Section 9.2]).

Introduce the time dependent Baker-Akhiezer function

$$\psi_q(p,n,t) = C(n,0,t)\frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))}\exp\Big(n\int_{E_0}^{p}\omega_{\infty_+\,\infty_-} + t\int_{E_0}^{p}\Omega_0\Big), \tag{A.12}$$

where $C(n,0,t)$ is real-valued,

$$C(n,0,t)^2 = \frac{\theta(\underline{z}(0,0))\theta(\underline{z}(-1,0))}{\theta(\underline{z}(n,t))\theta(\underline{z}(n-1,t))}, \tag{A.13}$$

and the sign has to be chosen in accordance with $a_q(n,t)$. Here

$$\theta(\underline{z}) = \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi\mathrm{i}\left(\langle\underline{m},\underline{z}\rangle + \frac{\langle\underline{m},\tau\,\underline{m}\rangle}{2}\right), \qquad \underline{z} \in \mathbb{C}^g, \tag{A.14}$$

is the Riemann theta function associated with $\mathbb{M}$,

$$\omega_{\infty_+\,\infty_-} = \frac{\prod\limits_{j=1}^{g}(\pi - \lambda_j)}{R_{2g+2}^{1/2}}d\pi \tag{A.15}$$

is the Abelian differential of the third kind with poles at ∞_+ and ∞_- and

$$\Omega_0 = \frac{\prod\limits_{j=0}^{g}(\pi - \tilde{\lambda}_j)}{R_{2g+2}^{1/2}}d\pi, \qquad \sum_{j=0}^{g}\tilde{\lambda}_j = \frac{1}{2}\sum_{j=0}^{2g+1}E_j, \tag{A.16}$$

is the Abelian differential of the second kind with second order poles at ∞_+ respectively ∞_- (see [18, Sects. 13.1, 13.2]). All Abelian differentials are normalized to have vanishing a_j periods.

The Baker-Akhiezer function is a meromorphic function on $\mathbb{M} \setminus \{\infty_\pm\}$ with an essential singularity at $\infty_\pm$. The two branches are denoted by

$$\psi_{q,\pm}(z,n,t) = \psi_q(p,n,t), \qquad p = (z,\pm) \tag{A.17}$$

and it satisfies

$$\begin{aligned} H_q(t)\psi_q(p,n,t) &= \pi(p)\psi_q(p,n,t), \\ \frac{\mathrm{d}}{\mathrm{d}t}\psi_q(p,n,t) &= P_{q,2}(t)\psi_q(p,n,t), \end{aligned} \tag{A.18}$$

where

$$H_q(t)f(n) = a_q(n,t)f(n+1) + a_q(n-1,t)f(n-1) + b_q(n,t)f(n), \tag{A.19}$$

$$P_{q,2}(t)f(n) = a_q(n,t)f(n+1) - a_q(n-1,t)f(n-1) \tag{A.20}$$

are the operators from the Lax pair for the Toda lattice.

It is well known that the spectrum of $H_q(t)$ is time independent and consists of $g+1$ bands

$$\sigma(H_q) = \bigcup_{j=0}^{g} [E_{2j}, E_{2j+1}]. \tag{A.21}$$

For further information and proofs we refer to [18, Chap. 9 and Sect. 13.2].

B The Inverse Scattering Transform and the Riemann–Hilbert Problem

In this section our notation and results are taken from [5] and [6]. Let $\psi_{q,\pm}(z,n,t)$ be the branches of the Baker-Akhiezer function defined in the previous section. Let $\psi_\pm(z,n,t)$ be the Jost functions for the perturbed problem

$$a(n,t)\psi_\pm(z,n+1,t) + a(n-1,t)\psi_\pm(z,n-1,t) + b(n,t)\psi_\pm(z,n,t) = z\psi_\pm(z,n,t) \tag{B.1}$$

defined by the asymptotic normalization

$$\lim_{n\to\pm\infty} w(z)^{\mp n}(\psi_\pm(z,n,t) - \psi_{q,\pm}(z,n,t)) = 0, \tag{B.2}$$

where $w(z)$ is the quasimomentum map

$$w(z) = \exp(\int_{E_0}^{p} \omega_{\infty_+\,\infty_-}), \quad p = (z,+). \tag{B.3}$$

The asymptotics of the two projections of the Jost function are

$$\begin{aligned} \psi_\pm(z,n,t) &= \psi_{q,\pm}(z,0,t)\frac{z^{\mp n}\Big(\prod_{j=0}^{n-1} a_q(j,t)\Big)^{\pm 1}}{A_\pm(n,t)} \\ &\quad \times\Big(1 + \Big(B_\pm(n,t) \pm \sum_{j=1}^{n} b_q(j - \genfrac{}{}{0pt}{}{0}{1}, t)\Big)\frac{1}{z} + O(\frac{1}{z^2})\Big), \end{aligned} \tag{B.4}$$

as $z \to \infty$, where

$$A_+(n,t) = \prod_{j=n}^{\infty} \frac{a(j,t)}{a_q(j,t)}, \quad B_+(n,t) = \sum_{j=n+1}^{\infty} (b_q(j,t) - b(j,t)),$$

$$A_-(n,t) = \prod_{j=-\infty}^{n-1} \frac{a(j,t)}{a_q(j,t)}, \quad B_-(n,t) = \sum_{j=-\infty}^{n-1} (b_q(j,t) - b(j,t)). \tag{B.5}$$

One has the scattering relations

$$T(z)\psi_\mp(z,n,t) = \overline{\psi_\pm(z,n,t)} + R_\pm(z)\psi_\pm(z,n,t), \qquad z \in \sigma(H_q), \tag{B.6}$$

where $T(z)$, $R_\pm(z)$ are the transmission respectively reflection coefficients. Here $\psi_\pm(z,n,t)$ is defined such that $\psi_\pm(z,n,t) = \lim_{\varepsilon\downarrow 0} \psi_\pm(z+\mathrm{i}\varepsilon,n,t)$, $z \in \sigma(H_q)$. If we take the limit from the other side we have $\overline{\psi_\pm(z,n,t)} = \lim_{\varepsilon\downarrow 0} \psi_\pm(z-\mathrm{i}\varepsilon,n,t)$.

The transmission $T(z)$ and reflection $R_\pm(z)$ coefficients satisfy

$$T(z)\overline{R_+(z)} + \overline{T(z)}R_-(z) = 0, \qquad |T(z)|^2 + |R_\pm(z)|^2 = 1. \tag{B.7}$$

In particular one reflection coefficient, say $R(z) = R_+(z)$, suffices.

We will define a Riemann–Hilbert problem on the Riemann surface $\mathbb{M}$ as follows:

$$m(p,n,t) = \begin{cases} \begin{pmatrix} T(z)\psi_-(z,n,t) & \psi_+(z,n,t) \end{pmatrix}, & p=(z,+) \\ \begin{pmatrix} \psi_+(z,n,t) & T(z)\psi_-(z,n,t) \end{pmatrix}, & p=(z,-). \end{cases} \tag{B.8}$$

Note that $m(p,n,t)$ inherits the poles at $\hat{\mu}_j(0,0)$ and the essential singularity at $\infty_\pm$ from the Baker–Akhiezer function.

We are interested in the jump condition of $m(p,n,t)$ on Σ, the boundary of $\Pi_\pm$ (oriented counterclockwise when viewed from top sheet Π_+). It consists of two copies $\Sigma_\pm$ of $\sigma(H_q)$ which correspond to non-tangential limits from $p=(z,+)$ with $\pm\mathrm{Im}(z) > 0$, respectively to non-tangential limits from $p=(z,-)$ with $\mp\mathrm{Im}(z) > 0$.

To formulate our jump condition we use the following convention: When representing functions on Σ, the lower subscript denotes the non-tangential limit from Π_+ or Π_-, respectively,

$$m_\pm(p_0) = \lim_{\Pi_\pm \ni p \to p_0} m(p), \qquad p_0 \in \Sigma. \tag{B.9}$$

Using the notation above implicitly assumes that these limits exist in the sense that $m(p)$ extends to a continuous function on the boundary away from the band edges.

Moreover, we will also use symmetries with respect to the the sheet exchange map

$$p^* = \begin{cases} (z,\mp) & \text{for } p=(z,\pm), \\ \infty_\mp & \text{for } p=\infty_\pm, \end{cases} \tag{B.10}$$

and complex conjugation

$$\overline{p} = \begin{cases} (\overline{z},\pm) & \text{for } p=(z,\pm) \notin \Sigma, \\ (z,\mp) & \text{for } p=(z,\pm) \in \Sigma, \\ \infty_\pm & \text{for } p=\infty_\pm. \end{cases} \tag{B.11}$$

In particular, we have $\overline{p} = p^*$ for $p \in \Sigma$.

Note that we have $\tilde{m}_\pm(p) = m_\mp(p^*)$ for $\tilde{m}(p) = m(p^*)$ (since $*$ reverses the orientation of Σ) and $\tilde{m}_\pm(p) = \overline{m_\pm(p^*)}$ for $\tilde{m}(p) = \overline{m(\overline{p})}$.

With this notation, using (B.6) and (B.7), we obtain

$$m_+(p,n,t) = m_-(p,n,t) \begin{pmatrix} |T(p)|^2 & -\overline{R(p)} \\ R(p) & 1 \end{pmatrix}, \tag{B.12}$$

where we have extended our definition of T to Σ such that it is equal to $T(z)$ on Σ_+ and equal to $\overline{T(z)}$ on Σ_-. Similarly for $R(z)$. In particular, the condition on Σ_+ is just the complex conjugate of the one on Σ_- since we have $R(p^*) = \overline{R(p)}$ and $m_\pm(p^*,n,t) = \overline{m_\pm(p,n,t)}$ for $p \in \Sigma$.

To remove the essential singularity at $\infty_\pm$ and to get a meromorphic Riemann–Hilbert problem we set

$$m^2(p,n,t) = m(p,n,t) \begin{pmatrix} \psi_q(p^*,n,t)^{-1} & 0 \\ 0 & \psi_q(p,n,t)^{-1} \end{pmatrix}. \tag{B.13}$$

Its divisor satisfies

$$(m_1^2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)^*}, \qquad (m_2^2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)}, \tag{B.14}$$

and the jump conditions become

$$m_+^2(p,n,t) = m_-^2(p,n,t) J^2(p,n,t)$$

$$J^2(p,n,t) = \begin{pmatrix} 1 - |R(p)|^2 & -\overline{R(p)\Theta(p,n,t)}\mathrm{e}^{-t\phi(p)} \\ R(p)\Theta(p,n,t)\mathrm{e}^{t\phi(p)} & 1 \end{pmatrix}, \tag{B.15}$$

where

$$\Theta(p,n,t) = \frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))} \frac{\theta(\underline{z}(p^*,0,0))}{\theta(\underline{z}(p^*,n,t))} \tag{B.16}$$

and

$$\phi(p, \frac{n}{t}) = 2\int_{E_0}^p \Omega_0 + 2\frac{n}{t}\int_{E_0}^p \omega_{\infty_+\,\infty_-} \in \mathrm{i}\mathbb{R} \tag{B.17}$$

for $p \in \Sigma$. Note

$$\frac{\psi_q(p,n,t)}{\psi_q(p^*,n,t)} = \Theta(p,n,t)\mathrm{e}^{t\phi(p)}.$$

Observe that

$$m^2(p) = \overline{m^2(\overline{p})}$$

and

$$m^2(p^*) = m^2(\overline{p}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which follow directly from the definition (B.13). They are related to the symmetries

$$J^2(p) = \overline{J^2(\overline{p})} \quad \text{and} \quad J^2(p) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} J^2(p^*)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we come to the normalization condition at ∞_+. To this end note

$$m(p,n,t) = \left(A_+(n,t)(1 - B_+(n-1,t)\frac{1}{z}) \quad \frac{1}{A_+(n,t)}(1 + B_+(n,t)\frac{1}{z}) \right) + O(\frac{1}{z^2}), \tag{B.18}$$

for $p = (z,+) \to \infty_+$, with $A_\pm(n,t)$ and $B_\pm(n,t)$ are defined in (B.5). The formula near ∞_- follows by flipping the columns. Here we have used

$$T(z) = A_-(n,t)A_+(n,t)\left(1 - \frac{B_+(n,t) + b_q(n,t) - b(n,t) + B_-(n,t)}{z} + O(\frac{1}{z^2})\right). \tag{B.19}$$

Using the properties of $\psi(p,n,t)$ and $\psi_q(p,n,t)$ one checks that its divisor satisfies

$$(m_1) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)^*}, \qquad (m_2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)}. \tag{B.20}$$

Next we show how to normalize the problem at infinity. The use of the above symmetries is necessary and it makes essential use of the second sheet of the Riemann surface.

Theorem B.1 The function

$$m^3(p) = \frac{1}{A_+(n,t)} m^2(p,n,t) \tag{B.21}$$

with $m^2(p,n,t)$ defined in (B.13) is meromorphic away from Σ and satisfies:

$$\begin{aligned} &m^3_+(p) = m^3_-(p)J^3(p), \quad p \in \Sigma, \\ &(m^3_1) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)^*}, \quad (m^3_2) \geq -\mathcal{D}_{\underline{\hat{\mu}}(n,t)}, \end{aligned} \tag{B.22}$$

$$\begin{aligned} m^3(p^*) &= m^3(p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ m^3(\infty_+) &= \begin{pmatrix} 1 & * \end{pmatrix}, \end{aligned} \tag{B.23}$$

where the jump is given by

$$J^3(p,n,t) = \begin{pmatrix} 1 - |R(p)|^2 & -\overline{R(p)\Theta(p,n,t)}\mathrm{e}^{-t\phi(p)} \\ R(p)\Theta(p,n,t)\mathrm{e}^{t\phi(p)} & 1 \end{pmatrix}. \tag{B.24}$$

Setting $R(z) \equiv 0$ we clearly recover the purely periodic solution, as we should. Moreover, note

$$m^3(p) = \begin{pmatrix} \frac{1}{A_+(n,t)^2} & 1 \end{pmatrix} + \begin{pmatrix} \frac{B_+(n,t)}{A_+(n,t)^2} & -B_+(n-1,t) \end{pmatrix}\frac{1}{z} + O(\frac{1}{z^2}). \tag{B.25}$$

for $p = (z,-)$ near ∞_-.

Existence of a solution of the normalized Riemann-Hilbert problem follows by construction; for uniqueness see [15].