A GENERAL GENERALIZATION OF JORDAN'S INEQUALITY AND A REFINEMENT OF L. YANG'S INEQUALITY

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Abstract. In this article, for $t \ge 2$, a general generalization of Jordan's inequality $\sum_{k=1}^{n}\mu_k\ \theta^t-x^t\ {}^k \le \frac{\sin x}{x}-\frac{\sin\theta}{\theta} \le \sum_{k=1}^{n}\omega_k\ \theta^t-x^t\ {}^k$ for $n\in\mathbb{N}$ and $\theta\in(0,\pi]$ is established, where the coefficients μ_k and ω_k defined by recursing formulas (11) and (12) are the best possible. As an application, L. Yang's inequality is refined.

1. Introduction

The well known Jordan's inequality (see [2, 6], [4, p. 143], [8, p. 269] and [11, p. 33]) states that

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1 \tag{1}$$

for $0<|x|\le\frac{\pi}{2}$. The equality in (1) is valid if and only if $x=\frac{\pi}{2}$.

Jordan's inequality has important applications in analysis and other branches of mathematics. Therefore, many mathematicians have struggled to refine, generalize and apply it. For more detailed information, please refer to [8, pp. 274–275] and [1, 5, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 31, 32, 33], especially [15], and the references therein.

In [1, 10, 14, 16, 17, 18, 19], among other things, Jordan's inequality had been refined as

$$\frac{1}{\pi^3}x\bigl(\pi^2-4x^2\bigr) \le \sin x-\frac{2}{\pi}x \le \frac{\pi-2}{\pi^3}x\bigl(\pi^2-4x^2\bigr). \tag{2}$$

In [33], a stronger sharp double inequality for $x\in\bigl(0,\frac{\pi}{2}\bigr]$ was obtained:

$$\frac{12-\pi^2}{16\pi^5}\bigl(\pi^2-4x^2\bigr)^2 \le \frac{\sin x}{x}-\frac{2}{\pi}-\frac{1}{\pi^3}\bigl(\pi^2-4x^2\bigr) \le \frac{\pi-3}{\pi^5}\bigl(\pi^2-4x^2\bigr)^2. \tag{3}$$

Recently in [12], the following general refinement of Jordan's inequality was showed:

$$\frac{2}{\pi}+\sum_{k=1}^{n}\alpha_k\bigl(\pi^2-4x^2\bigr)^k \le \frac{\sin x}{x} \le \frac{2}{\pi}+\sum_{k=1}^{n}\beta_k\bigl(\pi^2-4x^2\bigr)^k, \tag{4}$$

where the constants

$$\alpha_k=\frac{(-1)^k}{(4\pi)^k k!}\sum_{i=1}^{k+1}\left(\frac{2}{\pi}\right)^i c_{i-1}^k \sin\left(\frac{k+i}{2}\pi\right) \tag{5}$$

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and

$$\beta_k = \begin{cases} \dfrac{1 - \frac{2}{\pi} - \sum_{i=1}^{n-1} \alpha_i \pi^{2i}}{\pi^{2n}}, & k = n \\ \alpha_k, & 1 \le k < n \end{cases} \tag{6}$$

with

$$c_i^k = \begin{cases} (i + k - 1)c_{i-1}^{k-1} + c_i^{k-1}, & 0 < i \le k \\ 1, & i = 0 \end{cases} \tag{7}$$

in (4) are the best possible.

In [26], as a generalization of Jordan's inequality (1), the following sharp inequality

$$\begin{aligned} \frac{1}{2\tau^2}\left[(1+\lambda)\left(\frac{\sin\theta}{\theta} - \cos\theta\right) - \theta\sin\theta\right]\left(1 - \frac{x^\tau}{\theta^\tau}\right)^2 \\ \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\left(1 - \frac{x^\lambda}{\theta^\lambda}\right) \\ \le \left[1 - \frac{\sin\theta}{\theta} - \frac{1}{\lambda}\left(\frac{\sin\theta}{\theta} - \cos\theta\right)\right]\left(1 - \frac{x^\tau}{\theta^\tau}\right)^2 \end{aligned} \tag{8}$$

was obtained for $0 < x \le \theta \in \left(0, \frac{\pi}{2}\right]$, $\tau \ge 2$ and $\tau \le \lambda \le 2\tau$. The equalities in (8) holds if and only if $x = \theta$. The coefficients of the term $\left(1 - \frac{x^\tau}{\theta^\tau}\right)^2$ are the best possible. If $1 \le \tau \le \frac{5}{3}$ and either $\lambda \ne 0$ or $\lambda \ge 2\tau$ then inequality (8) is reversed. Specially, when $\theta = \frac{\pi}{2}$, inequality (8) becomes

$$\begin{aligned} \frac{4\lambda + 4 - \pi^2}{4\tau^2\pi^{2\tau+1}}\left(\pi^\tau - 2^\tau x^\tau\right)^2 \le \frac{\sin x}{x} - \frac{2}{\pi} - \frac{2}{\lambda\pi^{\lambda+1}}\left(\pi^\lambda - 2^\lambda x^\lambda\right) \\ \le \frac{\lambda\pi - 2\lambda - 2}{\lambda\pi^{2\tau+1}}\left(\pi^\tau - 2^\tau x^\tau\right)^2 \end{aligned} \tag{9}$$

for $0 < x \le \frac{\pi}{2}$, $\tau \ge 2$ and $\tau \le \lambda \le 2\tau$. If $1 \le \tau \le \frac{5}{3}$ and either $\lambda \ne 0$ or $\lambda \ge 2\tau$ then inequality (9) is reversed. If taking $(\tau, \lambda) = (2, 2)$ in (9), then inequality (3) can be deduced.

For recent developments of the refinements, generalizations and applications of Jordan's inequality, please refer to the expository and summary article [15].

The first aim of this paper is to generalize inequalities (4) and (8). One of the main results of this paper is the following Theorem 1.

Theorem 1. *For $0 < x \le \theta < \pi$, $n \in \mathbb{N}$ and $t \ge 2$, inequality*

$$\sum_{k=1}^{n} \mu_k \left(\theta^t - x^t\right)^k \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} \le \sum_{k=1}^{n} \omega_k \left(\theta^t - x^t\right)^k \tag{10}$$

holds with the equalities if and only if $x = \theta$, where the constants

$$\mu_k = \frac{(-1)^k}{k! t^k} \sum_{i=1}^{k+1} a_{i-1}^k \theta^{k-i-kt} \sin\left(\theta + \frac{k+i-1}{2}\pi\right) \tag{11}$$

and

$$\omega_k = \begin{cases} \dfrac{1 - \frac{\sin\theta}{\theta} - \sum_{i=1}^{n-1} \mu_i \theta^{ti}}{\theta^{tn}}, & k = n \\ \mu_k, & 1 \le k < n \end{cases} \tag{12}$$

with

$$a_i^k = \begin{cases} a_i^{k-1} + [i + (k-1)(t-1)] a_{i-1}^{k-1}, & 0 < i \le k \\ 1, & i = 0 \\ 0, & i > k \end{cases} \tag{13}$$

in (10) *are the best possible.*

Remark 1. Taking $t = 2$ in (10) yields inequality (4). Letting $n = 2$ in (10) leads to (8) for $\lambda = \tau = 2$.

The second aim of this paper is to apply Theorem 1 to refine L. Yang's inequality [27] as follows.

Theorem 2. *Let $0 \le \lambda \le 1$, $0 < x \le \theta < \pi$, $t \ge 2$ and $A_i > 0$ with $\sum_{i=1}^n A_i \le \pi$ for $n \in \mathbb{N}$. If $m \in \mathbb{N}$ and $n \ge 2$, then*

$$L_m(n,\lambda) \le H(n,\lambda) \le R_m(n,\lambda), \tag{14}$$

where

$$L_m(n,\lambda) = \binom{n}{2} \lambda^2 \pi^2 \left[\frac{\sin\theta}{\theta} + \sum_{k=1}^m 2^{-kt} \mu_k \left(2^t \theta^t - \lambda^t \pi^t\right)^k \right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right), \tag{15}$$

$$H(n,\lambda) = (n-1) \sum_{k=1}^n \cos^2(\lambda A_k) - 2\cos(\lambda\pi) \sum_{1 \le i < j \le n} \cos(\lambda A_i)\cos(\lambda A_j), \tag{16}$$

$$R_m(n,\lambda) = \binom{n}{2} \lambda^2 \pi^2 \left[\frac{\sin\theta}{\theta} + \sum_{k=1}^m 2^{-kt} \omega_k \left(2^t \theta^t - \lambda^t \pi^t\right)^k \right]^2 \cos^2\left(\frac{\lambda}{2}\pi\right), \tag{17}$$

and μ_k and ω_k are defined by (11).

2. Lemmas

To prove our main results, the following lemmas are necessary.

Lemma 1. *For $x > 0$, let $u_0(x) = \frac{\sin x}{x}$ and $u_k(x) = \frac{u'_{k-1}(x)}{x^r}$ for $k \in \mathbb{N}$ and $r \ge 1$. Then*

$$u_k(x) = \sum_{i=1}^{k+1} \frac{a_{i-1}^k \sin\left(x + \frac{i+k-1}{2}\pi\right)}{x^{kr+i}}, \tag{18}$$

where a_i^k is defined by (13).

Proof. It is apparent that $u_1(x) = x^{-r}\left(\frac{\sin x}{x}\right)' = x^{-1-r}\cos x - x^{-2-r}\sin x$, which tells us that formula (18) is valid for $k = 1$.

Now assume formula (18) holds for some given $k > 1$. Direct computation by using (13) gives

$$u_{k+1} = \sum_{i=1}^{k+1} a_{i-1}^k \left[\frac{1}{x^{kr+i+r}} \cos\left(x + \frac{k+i-1}{2}\pi\right) \right.$$

$$
\begin{aligned}
&- \frac{1}{x^{kr+i+r+1}}\sin\left(x+\frac{k+i-1}{2}\pi\right)\Bigg]\\
&= \frac{a_0^k}{x^{kr+r+1}}\cos\left(x+\frac{k}{2}\pi\right) - \frac{(kr+k+1)a_k^k}{x^{kr+r+k+2}}\sin(x+k\pi)\\
&\quad - \sum_{i=0}^{k-1}\frac{a_i^k(kr+1+i)+a_{i+1}^k}{x^{kr+r+i+2}}\sin\left(x+\frac{k+i}{2}\pi\right)\\
&= \frac{a_0^{k+1}}{x^{kr+r+1}}\sin\left(x+\frac{k+1}{2}\pi\right) + \frac{a_{k+1}^{k+1}}{x^{kr+r+k+2}}\sin[x+(k+1)\pi]\\
&\quad + \sum_{i=0}^{k-1}\frac{a_{i+1}^{k+1}}{x^{kr+r+i+2}}\sin\left(x+\frac{k+i+2}{2}\pi\right)\\
&= \sum_{i=1}^{k+2}\frac{a_{i-1}^{k+1}}{x^{kr+i+r}}\sin\left(x+\frac{k+i}{2}\pi\right).
\end{aligned}
$$

By mathematical induction, Lemma 1 is proved. □

Lemma 2. *For $x>0$ and $k\in\mathbb{N}$, let $v_1(x)=\sum_{i=1}^{k+1}a_{i-1}^k x^{k-i+1}\sin\bigl(x+\frac{k+i-1}{2}\pi\bigr)$ and $v_{j+1}(x)=\frac{1}{x}v_j'(x)$ for $j\in\mathbb{N}$. Then*

$$
v_j(x)=\sum_{i=0}^{k-j+1}b_i^j x^{k-i-j+1}\sin\left(x+\frac{k+i+j-1}{2}\pi\right) \tag{19}
$$

is valid for $j\in\mathbb{N}$, where $b_i^1=a_i^k$, $b_0^j=1$ and

$$
b_i^j=b_i^{j-1}-(k-i-j+3)b_{i-1}^{j-1},\qquad 0<i\le k-j+1,\quad j>1. \tag{20}
$$

Proof. When $j=1$, formula (19) is valid clearly.

By induction, suppose that formula (19) holds for some $j>1$. Since $k-j+1>k-(j+1)+1$, it deduced from (20) that $b_{k-j+1}^{j+1}=b_{k-j+1}^j-b_{k-j}^j=0$. Thus,

$$
\begin{aligned}
v_{j+1}(x)&=\frac{1}{x}\left\{\sum_{i=0}^{k-j}b_i^j\left[(k-i-j+1)x^{k-i-j}\sin\left(x+\frac{k+i+j-1}{2}\pi\right)\right.\right.\\
&\qquad\left.\left.+x^{k-i-j+1}\cos\left(x+\frac{k+i+j-1}{2}\pi\right)\right]+b_{k-j+1}^j\cos(x+k\pi)\right\}\\
&=b_0^jx^{k-j}\sin\left(x+\frac{k+j}{2}\pi\right)\\
&\quad+\sum_{i=0}^{k-j-1}\bigl[b_{i+1}^j-(k-i-j+1)b_i^j\bigr]x^{k-i-j+1}\sin\left(x+\frac{k+i+j+1}{2}\pi\right)\\
&=b_0^jx^{k-j}\sin\left(x+\frac{k+j}{2}\pi\right)+\sum_{i=0}^{k-j-1}b_{i+1}^{j+1}x^{k-i-j+1}\sin\left(x+\frac{k+i+j+1}{2}\pi\right)\\
&=\sum_{i=0}^{k-j}b_i^{j+1}x^{k-i-j}\sin\left(x+\frac{k+i+j}{2}\pi\right).
\end{aligned}
$$

By mathematical induction, formula (19) is proved. □

Lemma 3 ([3])**.** *Let f and g be continuous on $[a,b]$ and differentiable in (a,b) such that $g'(x) \neq 0$ in (a,b). If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing) in (a,b), then the functions $\frac{f(x)-f(b)}{g(x)-g(b)}$ and $\frac{f(x)-f(a)}{g(x)-g(a)}$ are also increasing (or decreasing) in (a,b).*

Lemma 4. *Let $0 < x \le \theta < \pi$ and $t \ge 2$, then inequality*

$$\frac{1}{t}\left(\frac{\sin\theta}{\theta^{1+t}} - \frac{\cos\theta}{\theta^t}\right)\left(\theta^t - x^t\right) \le \frac{\sin x}{x} - \frac{\sin\theta}{\theta} \le \left(\frac{1}{\theta^t} - \frac{\sin\theta}{\theta^{1+t}}\right)\left(\theta^t - x^t\right) \tag{21}$$

holds with the equalities if and only if $x = \theta$, where the constants

$$\frac{1}{t}\left(\frac{\sin\theta}{\theta^{1+t}} - \frac{\cos\theta}{\theta^t}\right) \quad \text{and} \quad \left(\frac{1}{\theta^t} - \frac{\sin\theta}{\theta^{1+t}}\right)$$

are the best possible.

Proof. Let $f(x) = \frac{\sin x}{x} - \frac{\sin\theta}{\theta}$, $g(x) = \theta^t - x^t$, $f_1(x) = x\cos x - \sin x$ and $g_1(x) = -tx^{1+t}$. Then

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(0)}{g(x)-g(0)}, \qquad \frac{f'(x)}{g'(x)} = \frac{f_1(x)-f_1(0)}{g_1(x)-g_1(0)}, \qquad \frac{f_1'(x)}{g_1'(x)} = \frac{\sin x}{t(1+t)x^t}.$$

Since $\frac{\sin x}{x^t}$ is decreasing in $(0,\pi]$, then $\frac{f_1'(x)}{g_1'(x)}$ is decreasing, and then, in virtue of Lemma 3, the function $\frac{f'(x)}{g'(x)}$ is decreasing, further $\frac{f(x)}{g(x)}$ is decreasing in $(0,\pi]$, thus,

$$\frac{1}{t}\left(\frac{\sin\theta}{\theta^{1+t}} - \frac{\cos\theta}{\theta^t}\right) = \lim_{x\to\theta^-}\frac{f(x)}{g(x)} \le \frac{f(x)}{g(x)} \le \lim_{x\to 0^+}\frac{f(x)}{g(x)} = \frac{1}{\theta^t}\left(1 - \frac{\sin\theta}{\theta}\right)$$

and the two constants are the best possible. □

3. Proofs of theorems

Proof of Theorem 1. If $n = 1$, inequality (10) becomes (21) in Lemma 4.

For $n \ge 2$, let $t = r+1$,

$$\varphi(x) = \frac{\sin x}{x} - \frac{\sin\theta}{\theta} - \sum_{k=1}^{n-1}\mu_k\left(\theta^{r+1} - x^{r+1}\right)^k, \quad \psi(x) = \left(\theta^{r+1} - x^{r+1}\right)^n,$$

$$\varphi_1(x) = \frac{\varphi(x)}{x^r}, \quad \varphi_{i+1}(x) = \frac{\varphi_i'(x)}{x^r}, \quad \psi_1(x) = \frac{\psi'(x)}{x^r}, \quad \psi_{i+1}(x) = \frac{\psi_i'(x)}{x^r},$$

where $2 \le i \le n$. Then for $1 \le k \le n-2$,

$$\varphi_k(x) = u_k(x) - [-(r+1)]^k k!\mu_k - \sum_{i=1}^{n-k-1}\frac{(i+k)!}{i!}\mu_{i+k}\left(\theta^{1+r} - x^{1+r}\right)^i,$$

$$\varphi_{n-1}(x) = u_{n-1}(x) - (n-1)![-(r+1)]^{n-1}\mu_{n-1} \quad \text{and} \quad \varphi_n(x) = u_n(x),$$

where $u_k(x)$ for $1 \le k \le n$ is defined by (18).

In view of (18), it is deduced that $[-(1+r)]^k k!\mu_k = u_k(\theta)$ for $1 \le k \le n-1$, hence $\varphi_i(\theta) = 0$ for $1 \le i \le n-1$. A simple calculation gives $\psi_i(x) = [-(1+r)]^i \prod_{\ell=0}^{i-1}(n-\ell)(\theta^{r+1} - x^{r+1})^{n-i}$ for $1 \le i \le n$, consequently $\psi_i(\theta) = 0$ for $1 \le i \le n-1$. As a result, for $1 \le i \le n-1$,

$$\frac{\varphi(x)}{\psi(x)} = \frac{\varphi(x)-\varphi(\theta)}{\psi(x)-\psi(\theta)}, \qquad \frac{\varphi'(x)}{\psi'(x)} = \frac{\varphi_1(x)-\varphi_1(\theta)}{\psi_1(x)-\psi_1(\theta)},$$

$$\frac{\varphi_i'(x)}{\psi_i'(x)} = \frac{\varphi_{i+1}(x)-\varphi_{i+1}(\theta)}{\psi_{i+1}(x)-\psi_{i+1}(\theta)}, \qquad \frac{\varphi_{n-1}'(x)}{\psi_{n-1}'(x)} = \frac{\varphi_n(x)}{\psi_n(x)} = \frac{u_n(x)}{n![-(r+1)]^n}.$$

Let $h_1(x)=x^{nr+n+1}$ and $h_{i+1}(x)=\frac{1}{x}h_i'(x)$ for $1\le i\le n$ and $n\in\mathbb{N}$. Then it is easy to see that $h_{i+1}(x)=\prod_{\ell=1}^{i}(nr+n-2\ell+3)x^{nr+n-2i+1}$ for $1\le i\le n$. Utilization of Lemma 1 and Lemma 2 leads to

$$\frac{\varphi_{n-1}'(x)}{\psi_{n-1}'(x)}=\frac{\sum_{i=1}^{n+1}a_{i-1}^{n}x^{n-i+1}\sin\bigl(x+\frac{n+i-1}{2}\pi\bigr)}{n![-(1+r)]^{n}x^{rn+n+1}}=\frac{v_1(x)}{n![-(1+r)]^{n}h_1(x)}$$

and, since $v_i(0)=h_i(0)=0$ for $1\le i\le n+1$,

$$\frac{v_1(x)}{h_1(x)}=\frac{v_1(x)-v_1(0)}{h_1(x)-h_1(0)},\qquad \frac{v_j'(x)}{h_j'(x)}=\frac{v_{j+1}(x)-v_{j+1}(0)}{h_{j+1}(x)-h_{j+1}(0)},$$
$$\frac{v_n'(x)}{h_n'(x)}=\frac{v_{n+1}(x)-v_{n+1}(0)}{h_{n+1}(x)-h_{n+1}(0)}=\frac{(-1)^{n}\sin x}{\prod_{\ell=1}^{i}(nr+n-2\ell+3)x^{nr-n+1}}$$

for $1\le j\le n-1$. Since $\frac{\sin x}{x}$ and $x^{-n(r-1)}$ is decreasing on $(0,\pi)$, then the function $\frac{\sin x}{x^{nr-n+1}}$ is decreasing and $\frac{(-1)^{n}v_n'(x)}{h_n'(x)}$ is deceasing. Accordingly, from Lemma 3, it follows that the functions $\frac{(-1)^{n}v_i'(x)}{h_i'(x)}$ and $\frac{(-1)^{n}v_{i-1}'(x)}{h_{i-1}'(x)}$ for $2\le i\le n$ are decreasing. Thus, the functions $\frac{(-1)^{n}v_1'(x)}{h_1'(x)}$ and $\frac{(-1)^{n}v_1(x)}{h_1(x)}$ are decreasing, and then $\frac{\varphi_{n-1}'(x)}{\psi_{n-1}'(x)}$ is decreasing in $(0,\pi)$. Utilizing Lemma 3 again reveals that the functions $\frac{\varphi_j'(x)}{\psi_j'(x)}$ and $\frac{\varphi_{j-1}'(x)}{\psi_{j-1}'(x)}$ for $2\le j\le n-1$ are decreasing, which implies the deceasingly monotonicity of $\frac{\varphi(x)}{\psi(x)}$ in $\bigl(0,\pi\bigr)$. By L'Hôspital's rule, it is easy to deduce that $\lim_{x\to\theta-}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to\theta-}\frac{\varphi'(x)}{\psi'(x)}=\lim_{x\to\theta-}\frac{\varphi_i'(x)}{\psi_i'(x)}=\frac{u_n(\theta)}{n![-(1+r)]^{n}}=\mu_n$ for $1\le i\le n-1$ and $\lim_{x\to0+}\frac{\varphi(x)}{\psi(x)}=\omega_n$, which implies $\mu_n\le\frac{\varphi(x)}{\psi(x)}\le\omega_n$ and the constants μ_k and ω_k are the best possible.

By the mathematical induction, inequality (10) is proved. $\square$

Proof of Theorem 2. It was proved in [29] and [30, (2.13)] that

$$\begin{aligned}\sin^2(\lambda\pi)\le\cos^2(\lambda A_i)+\cos^2(\lambda A_j)-2\cos(\lambda A_i)\cos(\lambda A_j)\cos(\lambda\pi)\\ \triangleq H_{ij}\le4\sin^2\biggl(\frac{\lambda}{2}\pi\biggr).\end{aligned}\tag{22}$$

Summing up (22) for $1\le i<j\le n$ yields

$$\binom{n}{2}\sin^2(\lambda\pi)\le\sum_{1\le i<j\le n}H_{ij}=H(n,\lambda)\le4\binom{n}{2}\sin^2\biggl(\frac{\lambda}{2}\pi\biggr).\tag{23}$$

By virtue of inequality (10) in Theorem 1,

$$4\sin^2\biggl(\frac{\lambda}{2}\pi\biggr)\le\lambda^2\pi^2\Biggl[\frac{\sin\theta}{\theta}+\sum_{k=1}^{m}2^{-kt}\omega_k\bigl(2^t\theta^t-\lambda^t\pi^t\bigr)^k\Biggr]^2,\tag{24}$$

$$\begin{aligned}\sin^2(\lambda\pi)&=4\cos^2\biggl(\frac{\lambda}{2}\pi\biggr)\sin^2\biggl(\frac{\lambda}{2}\pi\biggr)\\&\ge\lambda^2\pi^2\Biggl[\frac{\sin\theta}{\theta}+\sum_{k=1}^{m}2^{-kt}\mu_k\bigl(2^t\theta^t-\lambda^t\pi^t\bigr)^k\Biggr]^2\cos^2\biggl(\frac{\lambda}{2}\pi\biggr).\end{aligned}\tag{25}$$

Substituting (24) and (25) into (23) leads to (14). The proof of Theorem 2 is complete. □

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