NORMALIZED JENSEN FUNCTIONAL, SUPERQUADRACITY AND RELATED INEQUALITIES

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Abstract. In this paper we generalize the inequality

$$MJ_n(f,\mathbf{x},\mathbf{q}) \geq J_n(f,\mathbf{x},\mathbf{p}) \geq mJ_n(f,\mathbf{x},\mathbf{q})$$

where

$$J_n(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right),$$

obtained by S.S. Dragomir for convex functions. We provide cases where we can improve the bounds m and M for convex functions, and also, we show that for the class of superquadratic functions nonzero lower bounds of $J_n(f,\mathbf{x},\mathbf{p}) - mJ_n(f,\mathbf{x},\mathbf{q})$ and nonzero upper bounds of $J_n(f,\mathbf{x},\mathbf{p}) - MJ_n(f,\mathbf{x},\mathbf{q})$ can be pointed out. Finally, an inequality related to the Čebyšev functional and superquadracity is also given.

1. Introduction

In this paper we consider the normalized Jensen functional

$$J_n(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right), \tag{1.1}$$

where $\sum_{i=1}^{n} p_i = 1$, $f : I \longrightarrow \mathbb{R}$, and I is an interval in $\mathbb{R}$.

This type of functionals were considered by S. S. Dragomir in [5], where the following theorem was proved:

Theorem 1 ([5, Theorem 1]). *Consider the normalized Jensen functional (1.1) where $f : C \longrightarrow \mathbb{R}$ is a convex function on the convex set C in a real linear space, and $\mathbf{x} = (x_1,...,x_n) \in C^n$, $\mathbf{p} = (p_1,...,p_n)$, $\mathbf{q} = (q_1,...,q_n)$ are nonnegative n-tuples satisfying $\sum_{i=1}^{n} p_i = 1$, $\sum_{i=1}^{n} q_i = 1$, $q_i > 0$, $i = 1,...,n$. Then*

$$MJ_n(f,\mathbf{x},\mathbf{q}) \geq J_n(f,\mathbf{x},\mathbf{p}) \geq mJ_n(f,\mathbf{x},\mathbf{q}), \tag{1.2}$$

provided

$$m := \min_{1\leq i\leq n}\left(\frac{p_i}{q_i}\right), \quad M := \max_{1\leq i\leq n}\left(\frac{p_i}{q_i}\right).$$

In the following section we show when *(1.2)* holds for m^* larger than $\min_{1\leq i\leq n}\left(\frac{p_i}{q_i}\right)$, and M^* smaller than $\max_{1\leq i\leq n}\left(\frac{p_i}{q_i}\right)$. Although $\mathbf{x} = (x_1,...,x_n)$, $x_i \in I$, $i = 1,...,n$

Date: 26 June, 2007.

1991 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Convex Functions, Superquadratic Functions, Jensen Inequality, Jensen Steffensen Inequality, Čebyšev Inequality.

is not necessarily a monotonic n-tuple, we use Jensen-Steffensen's inequality that states that if $f : I \longrightarrow \mathbb{R}$ is convex, where I is an interval in $\mathbb{R}$, then

$$\sum_{i=1}^{n} a_i f(x_i) \geq A_n f(\overline{x}), \tag{1.3}$$

where $\overline{x} := \frac{\sum_{i=1}^{n} a_i x_i}{A_n}$, $\mathbf{x} = (x_1, ..., x_n)$ is any monotone n-tuple in I^n, and $\mathbf{a} = (a_1, ..., a_n)$ is a real n-tuple that satisfies the condition:

$$0 \leq A_i \leq A_n, \quad i = 1, ..., n, \text{ where } A_i = \sum_{j=1}^{i} a_j, \text{ and } A_n > 0 \tag{1.4}$$

(see for instance [6, page 43]).

In Section 2 we also show that for a class of superquadratic functions defined below, nonzero lower bounds of $J_n(f, \mathbf{x}, \mathbf{p}) - mJ_n(f, \mathbf{x}, \mathbf{q})$ and of $J_n(f, \mathbf{x}, \mathbf{p}) - m^* J_n(f, \mathbf{x}, \mathbf{q})$ and nonzero upper bounds of $J_n(f, \mathbf{x}, \mathbf{p}) - MJ_n(f, \mathbf{x}, \mathbf{q})$ and of $J_n(f, \mathbf{x}, \mathbf{p}) - M^* J_n(f, \mathbf{x}, \mathbf{q})$ are obtained. In addition, we get in the last section an inequality related to the Čebyšev's type functional and superquadracity.

Definition 1 ([2, Definition 1])**.** *A function f defined on an interval $I = [0, a]$ or $[0, \infty)$ is superquadratic, if for each x in I there exists a real number $C(x)$ such that*

$$f(y) - f(x) \geq f(|y - x|) + C(x)(y - x) \tag{1.5}$$

for all $y \in I$.

For example, the functions x^p, $p \geq 2$ and the functions $-x^p$, $0 \leq p \leq 2$ are superquadratic functions as well as the function $f(x) = x^2 \log x$, $x > 0$, $f(0) = 0$.

In Section 2 we use also the following lemmas and theorem for superquadratic functions:

Lemma 1 ([2, Lemma 2.1])**.** *Let f be a superquadratic function with $C(x)$ as in (1.5).*

(i) Then $f(0) \leq 0$

(ii) If $f(0) = f'(0) = 0$, then $C(x) = f'(x)$ wherever f is differentiable at $x > 0$.

(iii) If $f \geq 0$, then f is convex and $f(0) = f'(0) = 0$.

Lemma 2 ([3, Lemma 2.3])**.** *Suppose that f is superquadratic. Let $x_i \geq 0$, $i = 1, ..., n$ and let $\overline{x} := \sum_{i=1}^{n} a_i x_i$, where $a_i \geq 0$, $i = 1, ..., n$ and $\sum_{i=1}^{n} a_i = 1$. Then*

$$\sum_{i=1}^{n} a_i f(x_i) - f(\overline{x}) \geq \sum_{i=1}^{n} a_i f(|x_i - \overline{x}|). \tag{1.6}$$

The following Theorem 2 was proved in [1, Theorem 1] for differentiable positive superquadratic functions f, but because of Lemma 1 *(iii)* it holds also when f is not always differentiable.

Theorem 2. *Let $f : I \longrightarrow \mathbb{R}$, where I is $[0, a]$ or $[0, \infty)$, be nonnegative superquadratic function. Let $\mathbf{x}$ be a monotone nonnegative n-tuple in I^n and $\mathbf{a}$ satisfies (1.4). Let*

$$\overline{x} := \frac{\sum_{i=1}^{n} a_i x_i}{A_n} \tag{1.7}$$

Then

$$\sum_{i=1}^{n} a_i f(x_i) - A_n f(\overline{x}) \geq (n-1) A_n f\left(\frac{\sum_{i=1}^{n} a_i |x_i - \overline{x}|}{(n-1) A_n}\right). \tag{1.8}$$

2. The main results

In this section we use the following notations:

Let $\mathbf{x}_{\uparrow} = \left(x_{(1)}, ..., x_{(n)}\right)$ be the *increasing rearrangement* of $\mathbf{x} = (x_1, ..., x_n)$. Let π be the permutation that transfers $\mathbf{x}$ into $\mathbf{x}_{\uparrow}$ and let $(\overline{p}_1, ..., \overline{p}_n)$ and $(\overline{q}_1, ..., \overline{q}_n)$ be the n-tuples obtained by the same permutation π on $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ respectively. Then for an n-tuple $\mathbf{x} = (x_1, ..., x_n)$, $x_i \in I$, $i = 1, ..., n$ where I is an interval in $\mathbb{R}$ we get the following results:

Theorem 3. *Let* $\mathbf{p} = (p_1, ..., p_n)$, *where* $0 \leq \sum_{j=1}^{i} \overline{p}_j \leq 1$, $i = 1, ..., n$, $\sum_{i=1}^{n} p_i = 1$, *and* $\mathbf{q} = (q_1, ..., q_n)$, $0 < \sum_{j=1}^{i} \overline{q}_j < 1$, $i = 1, ..., n-1$, $\sum_{i=1}^{n} q_i = 1$, *and* $\mathbf{p} \neq \mathbf{q}$. *Denote*

$$m_i := \frac{\sum_{j=1}^{i} \overline{p}_j}{\sum_{j=1}^{i} \overline{q}_j}, \qquad \overline{m_i} =: \frac{\sum_{j=i}^{n} \overline{p}_j}{\sum_{j=i}^{n} \overline{q}_j}, \qquad i = 1, ..., n \tag{2.1}$$

where $(\overline{p}_1, ..., \overline{p}_n)$ *and* $(\overline{q}_1, ..., \overline{q}_n)$ *are as denoted above, and*

$$m^* := \min_{1 \leq i \leq n} \{m_i, \overline{m_i}\}, \qquad M^* := \max_{1 \leq i \leq n} \{m_i, \overline{m_i}\}. \tag{2.2}$$

If $\mathbf{x} = (x_1, ..., x_n)$ *is any* n*-tuple in* I^n, *where* I *is an interval in* $\mathbb{R}$, *then*

$$M^* J_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq m^* J_n(f, \mathbf{x}, \mathbf{q}), \tag{2.3}$$

where $f : I \longrightarrow \mathbb{R}$ *is a convex function on the interval* I.

Proof. As $\mathbf{p} \neq \mathbf{q}$ it is clear that $m^* < 1$, $m^* \geq \min_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right)$, and $M^* > 1$, $M^* \leq \max_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right)$.

As $\sum_{i=1}^{n} q_i = 1$ and $q_i > 0$ it is obvious that there is an integer k, $2 \leq k \leq n$ such that $x_{(k-1)} \leq \sum_{i=1}^{n} q_i x_i \leq x_{(k)}$.

We apply Jensen-Steffensen's inequality for the increasing $(n+1)$-tuple $\mathbf{y} = (y_1, ..., y_{n+1})$

$$y_i = \begin{cases} x_{(i)}, & i = 1, ..., k-1 \\ \sum_{j=1}^{n} q_j x_j, & i = k \\ x_{(i-1)}, & i = k+1, ..., n+1 \end{cases} \tag{2.4}$$

and to

$$a_i = \begin{cases} \overline{p}_i - m^* \overline{q}_i, & i = 1, ..., k-1 \\ m^*, & i = k \\ \overline{p}_{i-1} - m^* \overline{q}_{i-1}, & i = k+1, ..., n+1 \end{cases} \tag{2.5}$$

where m^* is defined in (2.2).

It is clear that $\mathbf{a}$ satisfies (1.4). Therefore, (1.3) holds for the increasing $(n+1)$-tuple $\mathbf{y}$ and for a convex function f.

Hence

$$\begin{aligned}\sum_{i=1}^{n+1} a_i f(y_i) &= m^* f\left(\sum_{i=1}^{n} q_i x_i\right) + \sum_{i=1}^{n} (p_i - m^* q_i) f(x_i) \\ &\geq f\left(m^* \sum_{i=1}^{n} q_i x_i + \sum_{i=1}^{n} (p_i - m^* q_i) x_i\right) = f\left(\sum_{i=1}^{n} p_i x_i\right).\end{aligned}$$

In other words

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq m^* \left(\sum a_i f(x_i) - f\left(\sum_{i=1}^{n} q_i x_i\right)\right).$$

This completes the proof of the right side inequality in (2.3).

The proof of the left side of (2.3) is similar:

We define an increasing $(n+1)$-tuple $\mathbf{z}$

$$z_i = \begin{cases} x_{(i)}, & i = 1, ..., s-1 \\ \sum_{j=1}^{n} p_j x_j, & i = s \\ x_{(i-1)}, & i = s+1, ..., n+1 \end{cases} \tag{2.6}$$

and to

$$b_i = \begin{cases} \overline{q}_i - \frac{\overline{p}_i}{M^*}, & i = 1, ..., s-1 \\ \frac{1}{M^*}, & i = s \\ \overline{q}_{i-1} - \frac{\overline{p}_{i-1}}{M^*}, & i = s+1, ..., n+1 \end{cases}, \tag{2.7}$$

where s satisfies $x_{s-1} \leq \sum_{j=1}^{n} p_j x_j \leq x_s$. As $\mathbf{b}$ satisfies (1.4) and $\sum_{i=1}^{n+1} b_i = 1$, by using Jensen-Steffensen's inequality, we get the left side of (2.3).

This completes the proof. ∎

Remark 1. *If* $\underset{1 \leq i \leq n}{min} \left(\frac{p_i}{q_i}\right) = \frac{\overline{p}_k}{\overline{q}_k}$, $k \neq 1, n$ *and* $\max_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right) = \frac{\overline{p}_s}{\overline{q}_s}$, $s \neq 1, n$ *then it is clear that for* $p_i \geq 0$, *and* $q_i > 0$, *we get that* $m^* > m$ *and* $M^* < M$ *and in these cases (2.3) refines (1.2).*

In Theorem 4 that deals with superquadratic functions we use the same techniques as used in [5] to prove Theorem 1 for convex functions.

Theorem 4. *Under the same conditions and definitions on* $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, m *and* M *as in Theorem 1, if* I *is* $[0, a)$ *or* $[0, \infty)$ *and* $f(x)$ *is a superquadratic function on* I, *then*

$$\begin{aligned}& J_n(f, \mathbf{x}, \mathbf{p}) - m J_n(f, \mathbf{x}, \mathbf{q}) \\ \geq\ & m f\left(\left|\sum_{i=1}^{n} (q_i - p_i) x_i\right|\right) + \sum_{i=1}^{n} (p_i - m q_i) f\left(\left|x_i - \sum_{i=1}^{n} p_j x_j\right|\right)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}& J_n(f, \mathbf{x}, \mathbf{p}) - M J_n(f, \mathbf{x}, \mathbf{q}) \\ \leq\ & -\left(\sum_{i=1}^{n} (M q_i - p_i) f\left(\left|x_i - \sum_{i=1}^{n} q_j x_j\right|\right) + f\left(\left|\sum_{i=1}^{n} (p_i - q_i) x_i\right|\right)\right).\end{aligned} \tag{2.9}$$

Proof. To prove (2.8) we define $\mathbf{y}$ as

$$y_i = \begin{cases} x_i, & i = 1, ..., n \\ \sum_{j=1}^{n} q_j x_j, & i = n+1 \end{cases},$$

and $\mathbf{d}$ as

$$d_i = \begin{cases} p_i - m q_i, & i = 1, ..., n \\ m, & i = n+1 \end{cases}.$$

Then (1.6) for $\mathbf{y}$ and $\mathbf{d}$ is

$$\begin{aligned} \sum_{i=1}^{n} (p_i - m q_i) f(x_i) + m f\left(\sum_{i=1}^{n} q_i x_i\right) &= \sum_{i=1}^{n+1} d_i f(y_i) - f\left(\sum_{i=1}^{n+1} d_i y_i\right) \\ \geq & \sum_{i=1}^{n+1} d_i f\left(\left| y_i - \sum_{j=1}^{n+1} a_j y_j \right|\right) \\ = & \sum_{i=1}^{n} (p_i - m q_i) f\left(\left| x_i - \sum_{j=1}^{n} p_j x_j \right|\right) + m f\left(\left| \sum_{i=1}^{n} (p_i - q_i) x_i \right|\right) \end{aligned}$$

which is (2.8).

To get (2.9), we choose $\mathbf{z}$ and $\mathbf{r}$ as

$$z_i = \begin{cases} x_i, & i = 1, ..., n \\ \sum_{j=1}^{n} p_j x_j, & i = n+1 \end{cases},$$

and

$$r_i = \begin{cases} q_i - \frac{p_i}{M}, & i = 1, ..., n \\ \frac{1}{M}, & i = n+1 \end{cases}$$

where s is any integer $1 \leq s \leq n-1$.

Then, as f is superquadratic and $\sum_{i=1}^{n} r_i = 1$, $r_i \geq 0$, we get that

$$\begin{aligned} & \sum_{i=1}^{n} \left(q_i - \frac{p_i}{M}\right) f(x_i) + \frac{1}{M} f\left(\sum_{i=1}^{n} p_i x_i\right) - f\left(\sum_{i=1}^{n} q_i x_i\right) \\ = & \sum_{i=1}^{n+1} r_i f(z_i) - f\left(\sum_{i=1}^{n+1} r_i z_i\right) \\ \geq & \sum_{i=1}^{n+1} r_i f\left(\left| z_i - \sum_{i=1}^{n+1} r_i z_i \right|\right) \\ = & \sum_{i=1}^{n} \left(q_i - \frac{p_i}{M}\right) f\left(\left| x_i - \sum_{j=1}^{n} q_j x_j \right|\right) + \frac{1}{M} f\left(\left| \sum_{i=1}^{n} (p_i - q_i) x_i \right|\right) \end{aligned}$$

which is equivalent to (2.9). ∎

Remark 2. *If the superquadratic function is also positive and therefore according to Lemma 1 is convex, then (2.8) and (2.9) refine Theorem 1.*

The following result is proved for superquadratic functions using the same technique used in Theorem 3 for convex functions and by using Theorem 2, therefore, the proof is omitted.

Theorem 5. *Let $f(x)$ be a positive superquadratic function on $[0,a]$. Let $\mathbf{x}$, $\mathbf{p}$, $\mathbf{q}$, m^*, M^* be the same as in Theorem 3. Then*

$$\begin{aligned} &J_n(f,\mathbf{x},\mathbf{p}) - m^* J_n(f,\mathbf{x},\mathbf{q}) \\ \geq\ & nf\left(\frac{\sum_{i=1}^n (p_i - m^* q_i)\left|x_i - \sum_{j=1}^n p_j x_j\right| + m^*\left|\sum_{i=1}^n (p_i - q_i) x_i\right|}{n}\right) \geq 0, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &J_n(f,\mathbf{x},\mathbf{p}) - M^* J_n(f,\mathbf{x},\mathbf{q}) \\ \leq\ & -nf\left(\frac{\sum_{i=1}^n \left(q_i - \frac{p_i}{M^*}\right)\left|x_i - \sum_{j=1}^n q_j x_j\right| + \left|\sum_{j=1}^n (q_j - p_j) x_j\right|}{n}\right) \leq 0. \end{aligned} \tag{2.11}$$

In the following we state another generalisation of the Jensen inequality for superquadratic functions, and then we extend Theorems 4 and 5.

Theorem 6. *Assume that $\mathbf{x} = (x_1, ..., x_n)$ with $x_i \geq 0$ for $i \in \{1, ..., n\}$, $\mathbf{p} = (p_1, ..., p_n)$ is a probability sequence and $\mathbf{q} = (q_1, ..., q_k)$ is another probability sequence with $n, k \geq 2$. Then for any superquadratic function $f : [0, \infty) \to \mathbb{R}$ we have the inequality*

$$\begin{aligned} &\sum_{i_1,...,i_k=1}^n p_{i_1}...p_{i_k} f\left(\sum_{j=1}^k q_j x_{i_j}\right) \\ \geq\ & f\left(\sum_{i=1}^n p_i x_i\right) + \sum_{i_1,...,i_k=1}^n p_{i_1}...p_{i_k} f\left(\left|\sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i\right|\right). \end{aligned} \tag{2.12}$$

Proof. By the definition of superquadratic functions, we have

$$\begin{aligned} &f\left(\sum_{j=1}^k q_j x_{i_j}\right) \\ \geq\ & f\left(\sum_{i=1}^n p_i x_i\right) + C\left(\sum_{i=1}^n p_i x_i\right)\left(\sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i\right) \\ & + f\left(\left|\sum_{j=1}^k q_j x_{i_j} - \sum_{i=1}^n p_i x_i\right|\right) \end{aligned} \tag{2.13}$$

for any $x_{i_j} \geq 0, i_j \in \{1, ..., n\}$.

Now, if we multiply (2.13) with $p_{i_1}...p_{i_k} \geq 0$, sum over $i_1, ..., i_k$ from 1 to n and take into account that $\sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k} = 1$ we deduce

$$\begin{aligned} &\sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) \\ \geq\ & f\left(\sum_{i=1}^{n} p_i x_i\right) + C\left(\sum_{i=1}^{n} p_i x_i\right) \sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k}\left(\sum_{j=1}^{k} q_j x_{i_j} - \sum_{i=1}^{n} p_i x_i\right) \\ &+ \sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k} f\left(\left|\sum_{j=1}^{k} q_j x_{i_j} - \sum_{i=1}^{n} p_i x_i\right|\right). \end{aligned} \tag{2.14}$$

However

$$\begin{aligned} I &:= \sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k}\left(\sum_{j=1}^{k} q_j x_{i_j} - \sum_{i=1}^{n} p_i x_i\right) \\ &= \sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k}\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - \sum_{i=1}^{n} p_i x_i \end{aligned}$$

and since

$$\begin{aligned} &\sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k}\left(\sum_{j=1}^{k} q_j x_{i_j}\right) \\ =\ & q_1 \sum_{i_1=1}^{n} p_{i_1} x_{i_1} \sum_{i_2,...,i_k=1}^{n} p_{i_2}...p_{i_k} + ... + q_k \sum_{i_k=1}^{n} p_{i_k} x_{i_k} \sum_{i_1,...,i_{k-1}=1}^{n} p_{i_1}...p_{i_{k-1}} \\ =\ & q_1 \sum_{i=1}^{n} p_i x_i + ... + q_k \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i x_i \end{aligned}$$

hence $I = 0$ and by (2.14) we get the desired result (2.12). ∎

Theorem 7. *Assume that* $\mathbf{x} = (x_1, ..., x_n)$ *with* $x_i \in I$, $i = 1, ..., n$, I *is an interval in* $\mathbb{R}$, $\mathbf{p} = (p_1, ..., p_n)$, $\mathbf{r} = (r_1, ..., r_n)$, $r_i > 0$, $i = 1, ..., n$ *are probability sequences, and* $\mathbf{q} = (q_1, ..., q_k)$, *another probability sequence with* $n, k \geq 2$. *Then, for any convex function* f *on* I *we have the inequality*

$$\begin{aligned} & M\left(\sum_{i_1,...,i_k=1}^{n} r_{i_1}...r_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} r_i x_i\right)\right) \\ \geq\ & \left(\sum_{i_1,...,i_k=1}^{n} p_{i_1}...p_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right)\right) \\ \geq\ & m\left(\sum_{i_1,...,i_k=1}^{n} r_{i_1}...r_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} r_i x_i\right)\right) \end{aligned} \tag{2.15}$$

where $m := \underset{1\leq i_1,...,i_k\leq n}{min}\left(\frac{p_{i_1}...p_{i_k}}{r_{i_1}...r_{i_k}}\right)$, $M := \underset{1\leq i_1,...,i_k\leq n}{max}\left(\frac{p_{i_1}...p_{i_k}}{r_{i_1}...r_{i_k}}\right)$.

Proof. The proof is similar to the proof of Theorem 1:

We will prove the right side of the inequality. The left side of the inequality is similar.

As

$$\begin{aligned} & m\sum_{i=1}^{n} r_i x_i + \sum_{i_1,i_2,\dots,i_k=1}^{n} \left(p_{i_1}...p_{i_k} - m r_{i_1}...r_{i_k}\right) \sum_{j=1}^{k} q_j x_{i_j} \\ = & \sum_{i_1...i_k}^{n} p_{i_1}...p_{i_k} \sum_{j=1}^{k} q_j x_{ij} = \sum_{i=1}^{n} p_i x_i, \end{aligned}$$

$0 \le m \le 1$, $0 \le p_{i_1}...p_{i_k} - m r_{i_1}...r_{i_k} \le 1$ and $m + \sum_{i_1...i_k=1}^{n} \left(p_{i_1}...p_{i_k} - m r_{i_1}...r_k\right) = 1$ we get as a result of the convexity of f that

$$\begin{aligned} & mf\left(\sum_{i=1}^{n} r_i x_i\right) + \sum_{i_1...i_k=1}^{n} \left(p_{i_1}...p_{i_k} - m r_{i_1}...r_{i_k}\right) f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) \\ \ge & f\left(m\sum_{i=1}^{n} r_i x_i + \sum_{i_1...i_k=1}^{n} \left(p_{i_1}...p_{i_k} - m r_{i_1}...r_{i_k}\right) f\left(\sum_{j=1}^{k} q_j x_{i_j}\right)\right) \\ = & f\left(\sum_{i=1}^{n} p_i x_i\right). \end{aligned}$$

This completes the proof of the right inequality of (2.15). ∎

Below we state the analogue to Theorem 7 for superquadratic functions. The proof is similar to the proof of Theorem 4 and hence it is omitted.

Theorem 8. *Under the same conditions on* $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, m *and* M *as in Theorem 7, if* I *is* $[0,a)$ *or* $[0,\infty)$ *and* $f(x)$ *is a superquadratic function on* I*, then:*

$$\begin{aligned} & \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}...p_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\ & -m\left(\sum r_{i_1}...r_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} r_i x_i\right)\right) \\ \ge & mf\left(\left|\sum_{i=1}^{n} (r_i - p_i) x_i\right|\right) \\ & + \sum_{i_1,\dots,i_k=1}^{n} \left(p_{i_1}p_{i_2}...p_{i_k} - m r_{i_1}...r_{i_k}\right) f\left(\left|\sum_{j=1}^{k} q_j x_{i_j} - \sum_{s=1}^{n} p_s x_s\right|\right) \end{aligned}$$

and

$$\begin{aligned}
&\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\dots p_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} p_i x_i\right) \\
&-M\left(\sum r_{i_1}\dots r_{i_k} f\left(\sum_{j=1}^{k} q_j x_{i_j}\right) - f\left(\sum_{i=1}^{n} r_i x_i\right)\right) \\
\leq\; & -f\left(\left|\sum_{i=1}^{n} (r_i - p_i)\, x_i\right|\right) \\
&-\sum_{i_1,\dots,i_k=1}^{n} (p_{i_1} p_{i_2}\dots p_{i_k} - M r_{i_1}\dots r_{i_k})\, f\left(\left|\sum_{j=1}^{k} q_j x_{i_j} - \sum_{j=1}^{n} r_s x_s\right|\right)
\end{aligned}$$

If f is also positive, then this inequality refines (2.15).

3. Other inequalities

The definition of superquadratic functions and their properties draw our attention to the possibility of using the *Čebyšev functional* and its properties to get new type of reverse Jensen Inequality.

For a function $C : [0,\infty) \to \mathbb{R}$ we consider the Čebyšev type functional

$$T(C,\mathbf{x},\mathbf{p}) := \sum_{i=1}^{n} p_i x_i C(x_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i C(x_i).$$

It is well known that, if C is monotonic nondecreasing function on $[0,\infty)$ then the sequences $\mathbf{x}$ and $C(\mathbf{x}) := (C(x_1),\dots,C(x_n))$ are synchronous and for any probability sequence $\mathbf{p}$ we have the *Čebyšev inequality*

$$T(C,\mathbf{x},\mathbf{p}) \geq 0.$$

If certain bounds for the values of the function $C(x_i)$ are known, namely

$$-\infty < m \leq C(x_i) \leq M < \infty \qquad \text{for any } i \in \{1,\dots,n\} \tag{3.1}$$

then the following inequality due to Cerone & Dragomir [4] holds:

$$|T(C,\mathbf{x},\mathbf{p})| \leq \frac{1}{2}(M-m)\sum_{i=1}^{n} p_i \left| x_i - \sum_{j=1}^{n} p_j x_j \right|. \tag{3.2}$$

The constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

We can state now the following reverse of the Jensen inequality for superquadratic functions:

Theorem 9. *Assume that $\mathbf{x} = (x_1,\dots,x_n)$ with $x_i \geq 0$ for $i \in \{1,\dots,n\}$, and $\mathbf{p} = (p_1,\dots,p_n)$ is a probability sequence with $n \geq 2$. Then for any superquadratic function $f : [0,\infty) \to \mathbb{R}$ with $C(x_i)$ satisfying (2.16), where $C(x)$ is as in Definition*

1 we have the inequality,

$$\frac{1}{2}(M-m)\sum_{i=1}^{n}p_j\left|x_j-\sum_{j=1}^{n}p_ix_i\right|-\sum_{j=1}^{n}p_jf\left(\left|\sum_{i=1}^{n}p_ix_i-x_j\right|\right) \tag{3.3}$$
$$\geq \sum_{j=1}^{n}p_jf(x_j)-f\left(\sum_{i=1}^{n}p_ix_i\right)\left(\geq\sum_{j=1}^{n}p_jf\left(\left|\sum_{i=1}^{n}p_ix_i-x_j\right|\right)\right).$$

Proof. Utilising the definition of the superquadratic functions we have

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\geq f(x_j)+C(x_j)\left(\sum_{i=1}^{n}p_ix_i-x_j\right)+f\left(\left|\sum_{i=1}^{n}p_ix_i-x_j\right|\right) \tag{3.4}$$

for any $j\in\{1,...,n\}$.

If we multiply (3.4) by $p_j\geq 0, j\in\{1,...,n\}$, sum over j from 1 to n and take into account that $\sum_{j=1}^{n}p_j=1$ we get

$$f\left(\sum_{i=1}^{n}p_ix_i\right) \tag{3.5}$$
$$\geq\sum_{j=1}^{n}p_jf(x_j)+\sum_{j=1}^{n}p_jC(x_j)\left(\sum_{i=1}^{n}p_ix_i-x_j\right)+\sum_{j=1}^{n}p_jf\left(\left|\sum_{i=1}^{n}p_ix_i-x_j\right|\right).$$

Since

$$\sum_{j=1}^{n}p_jC(x_j)\left(\sum_{i=1}^{n}p_ix_i-x_j\right)=-T(C,\mathbf{x},\mathbf{p})$$

hence by (3.2) and (3.5) we deduce the desired result (3.3). ∎

Remark 3. *We observe that, as a "by-product" from (3.3) we get the following inequality*

$$\frac{1}{2}T(C,\mathbf{x},\mathbf{p})\geq\sum_{j=1}^{n}p_jf\left(\left|\sum_{i=1}^{n}p_ix_i-x_j\right|\right)$$

while from (3.3) we get

$$\frac{1}{2}(M-m)\sum_{i=1}^{n}p_j\left|x_j-\sum_{j=1}^{n}p_ix_i\right|\geq\sum_{j=1}^{n}p_jf\left(\left|\sum_{i=1}^{n}p_ix_i-x_j\right|\right).$$

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