A CLASS OF k-LOG-CONVEX FUNCTIONS AND THEIR APPLICATIONS TO SOME SPECIAL FUNCTIONS

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Abstract. Let a and b be two real numbers, f a positive and differentiable function on an interval I. The authors establish the i-log-convex or i-log-concave properties for $i \in \mathbb{N}$ of the function $\frac{[f(bx)]^a}{[f(ax)]^b}$ for $ax \in I$ and $bx \in I$ when the function $u^{k-1}[\ln f(u)]^{(k)}$ for $k \in \mathbb{N}$ is monotonic and apply these properties to to deduce some known and new conclusions related to some special functions such as the gamma function, Riemann's zeta function, complete elliptic integrals, exponential mean and extended mean values.

1. Introduction

By analytical arguments, it was presented in [28] that the function

$$f(x,y) = \frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \tag{1}$$

for all $y \ge 1$ is decreasing in $x \ge 0$.

In [26, 27], the following logarithmically complete monotonicities of $f(x,y)$, as generalizations of the decreasingly monotonic property in [28], are presented:

(1) For given $y > 1$, the function $f(x,y)$ defined by (1) is decreasing and logarithmically concave with respect to $x \in (0,\infty)$, and $\frac{1}{f(x,y)}$ is a logarithmically completely monotonic function of second order in $x \in (0,\infty)$.
(2) For given $0 < y < 1$, the function $f(x,y)$ defined by (1) is increasing and logarithmically convex with respect to $x \in (0,\infty)$, and $f(x,y)$ is a logarithmically completely monotonic function of second order in $x \in (0,\infty)$.
(3) For given $x \in (0,\infty)$, the function $f(x,y)$ defined by (1) is logarithmically concave with respect to $y \in (0,\infty)$, and $\frac{1}{f(x,y)}$ is a logarithmically completely monotonic function of first order in $y \in (0,\infty)$.

In [6, Theorem 2.1], it was proved that the function

$$G_{s,t}(x) = \frac{[\Gamma(1+tx)]^s}{[\Gamma(1+sx)]^t} \tag{2}$$

is decreasing (or increasing respectively) in $x \in [0,\infty)$ if either $s \ge t > 0$ or $0 > s \ge t$ (or both $s > 0$ and $t < 0$ respectively) such that $1 + sx > 0$ and $1 + tx > 0$.

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In [22], the following logarithmically complete monotonicity, logarithmically absolute monotonicity and logarithmically absolute convexity of $G_{s,t}(x)$ defined by (2) are verified: The function $G_{s,t}(x)$ for $x,s,t \in \mathbb{R}$ such that $1+sx>0$ and $1+tx>0$ with $s \neq t$ has the following properties:

(1) $G_{s,t}(x)=\frac{1}{G_{t,s}(x)}$;
(2) For $t>s>0$ and $x \in (0,\infty)$, $G_{s,t}(x)$ is an increasing function and a logarithmically completely monotonic function of second order in x;
(3) For $t>s>0$ and $x \in \left(-\frac{1}{t},0\right)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x;
(4) For $s<t<0$ and $x \in (-\infty,0)$, $G_{s,t}(x)$ is a decreasing function and a logarithmically absolutely monotonic function of second order in x;
(5) For $s<t<0$ and $x \in \left(0,-\frac{1}{s}\right)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x;
(6) For $s<0<t$ and $x \in \left(-\frac{1}{t},0\right)$, $G_{t,s}(x)$ is an increasing function and a logarithmically absolutely convex function in x;
(7) For $s<0<t$ and $x \in \left(0,-\frac{1}{s}\right)$, $G_{t,s}(x)$ is a decreasing function and a logarithmically absolutely convex function in x.

In [11], It was shown that if f is a differentiable and logarithmically convex function in $[0,\infty)$, then the function $\frac{[f(x)]^a}{f(ax)}$ for $a \geq 1$ (or $0<a\leq 1$ respectively) is decreasing (or increasing respectively) in $[0,\infty)$.

Recall [2, p. 190] and [12] that a function $f(x)$ is said to be n-convex on $[a,b]$, with $n \geq 2$, if and only if $f^{(n)}(x)$ exists and $f^{(n)}(x) \geq 0$. If $f^{(n)}(x)>0$, then $f(x)$ is strictly n-convex.

Definition 1 ([12, 18])**.** For $k \in \mathbb{N}$, a k-times differentiable function $f(x)>0$ is said to be k-log-convex (or k-log-concave, respectively) on an interval I if $[\ln f(x)]^{(k)} \geq 0$ (or $[\ln f(x)]^{(k)} \leq 0$, respectively) on I.

It is clear that a 1-log-convex function (or 1-log-concave function, respectively) is equivalent to a positive and increasing (or decreasing, respectively) function and that a 2-log-convex function is positive and convex. Conversely, a convex function may not be 2-log-convex. See [12, p. 7, Remark. 1.16].

The aim of this paper is to generalize the monotonicity results obtained in [6, 11, 22, 26, 27, 28] and stated above.

Our main results are following two theorems.

Theorem 1. *Let a and b be two real numbers, $f(x)$ a positive function on an interval I, and*

$$g_{a,b}(x)=\frac{[f(bx)]^a}{[f(ax)]^b} \tag{3}$$

defined for $ax \in I$ and $bx \in I$. The function $g_{a,b}(x)$ has the following properties:

(1) *For all real numbers a, b and x such that $ax \in I$ and $bx \in I$, it is valid that $g_{b,a}(x)=\frac{1}{g_{a,b}(x)}$.*
(2) *When $f(x)$ is 2-log-convex on I,*
    (a) *if either $ab>0$ and $bx>ax$ for all x or $ab<0$ and $bx<ax$ for all x, then $g_{a,b}(x)$ is 1-log-convex in x;*
    (b) *if either $ab>0$ and $bx<ax$ for all x or $ab<0$ and $bx>ax$ for all x, then $g_{a,b}(x)$ is 1-log-concave in x.*
(3) *When $f(x)$ is 2-log-concave on I,*

(a) *if either* $ab > 0$ *and* $bx > ax$ *for all* x *or* $ab < 0$ *and* $bx < ax$ *for all* x, *then* $g_{a,b}(x)$ *is* 1*-log-concave in* x*;*

(b) *if either* $ab > 0$ *and* $bx < ax$ *for all* x *or* $ab < 0$ *and* $bx > ax$ *for all* x, *then* $g_{a,b}(x)$ *is* 1*-log-convex in* x*.*

Theorem 2. *For* $i \in \mathbb{N}$, *the function* $g_{a,b}(x)$ *defined by* (3) *has the following properties:*

(1) *For either* $b > a > 0$ *and* $x > 0$ *or* $0 > b > a$ *and* $x > 0$,
  (a) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is increasing on* I, *then* $g_{a,b}(x)$ *is* i*-log-convex;*
  (b) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is decreasing on* I, *then* $g_{a,b}(x)$ *is* i*-log-concave.*

(2) *For* $b > 0 > a$ *and* $x > 0$,
  (a) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is increasing on* I, *then* $g_{a,b}(x)$ *is* i*-log-concave;*
  (b) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is decreasing on* I, *then* $g_{a,b}(x)$ *is* i*-log-convex.*

(3) *For either* $b > a > 0$ *and* $x > 0$ *or* $0 > b > a$ *and* $x < 0$,
  (a) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is increasing on* I, *then* $g_{a,b}(x)$ *is* $(2i-1)$*-log-concave and* $(2i)$*-log-convex;*
  (b) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is decreasing on* I, *then* $g_{a,b}(x)$ *is* $(2i-1)$*-log-convex and* $(2i)$*-log-concave.*

(4) *For* $b > 0 > a$ *and* $x < 0$,
  (a) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is increasing on* I, *then* $g_{a,b}(x)$ *is* $(2i-1)$*-log-convex and* $(2i)$*-log-concave;*
  (b) *if the function* $u^{i-1}[\ln f(u)]^{(i)}$ *for all* $i \in \mathbb{N}$ *is decreasing on* I, *then* $g_{a,b}(x)$ *is* $(2i-1)$*-log-concave and* $(2i)$*-log-convex.*

2. Applications of theorems

Before verifying Theorem 1 and Theorem 2, we would like to apply them to deduce some known and new conclusions related to some special functions such as the gamma function, Riemann's zeta function, complete elliptic integrals, exponential mean and extended mean values.

2.1. If letting $b = 1$ and $I = [0, \infty)$ in Theorem 1, then [11, Theorem 2.1] can be deduced readily.

2.2. If taking $f(x) = \Gamma(1+x)$ and $I = (-1, \infty)$ in Theorem 1, then [6, Theorem 2.1] can be deduced directly.

2.3. If applying $b = 1$, $f(x) = \Gamma(1 + x)$ and $I = (-1, \infty)$ in Theorem 2, then Theorem 1 in [26, 27] can be deduced straightforwardly.

2.4. If replacing $f(x)$ by $\Gamma(1 + x)$ and I by $(-1, \infty)$ in Theorem 2, then [22, Theorem 1] can be deduced easily.

2.5. By taking suitable function f and appropriate interval I in Theorem 1 or Theorem 2, almost all related inequalities and monotonicity results established in [3, 6, 10, 11, 22, 26, 27, 28] can be recovered simply.

2.6. In [1, p. 807, 23.2.7], the following formula is listed:

$$\Gamma(1+x)\zeta(1+x)=\int_0^\infty \frac{t^x}{e^t-1}\,\mathrm{d}t \tag{4}$$

for $x>0$. In [11], it was pointed out that the function (4) is 2-log-convex. Applying Theorem 1 to the function (4) arrives at

$$\frac{[\Gamma(1+bx)\zeta(1+bx)]^a}{[\Gamma(1+ax)\zeta(1+ax)]^b}\le\frac{[\Gamma(1+by)\zeta(1+by)]^a}{[\Gamma(1+ay)\zeta(1+ay)]^b} \tag{5}$$

for $0<x<y$ and $b>a>0$. Since the function $G_{a,b}(x)$ is increasing in $x>0$ by either [6, Theorem 2.1], or [22, Theorem 1], or Theorem 1 and Theorem 2 of this paper, then inequality

$$\frac{[\Gamma(1+bx)]^a}{[\Gamma(1+ax)]^b}\ge 1 \tag{6}$$

is valid for $x>0$ and $b>a>0$. Substituting (6) into (5) gives

$$\frac{[\zeta(1+bx)]^a}{[\zeta(1+ax)]^b}\le\frac{[\Gamma(1+by)\zeta(1+by)]^a}{[\Gamma(1+ay)\zeta(1+ay)]^b} \tag{7}$$

for $0<x<y$ and $b>a>0$. From this, inequalities (2.7), (2.8) and (2.9) in [11] can be deduced by the similar argument as in [11, pp. 2–3].

2.7. The complete elliptic integral of the first kind is defined in [4, p. 132] by

$$K(k)=\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{8}$$

for $0<k<1$. It can also be defined [11] by

$$R_K(x,y)=\frac{2}{\pi}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{x\sin^2\theta+y\cos^2\theta}} \tag{9}$$

for $x>0$ and $y>0$. It is clear that $K(k)=\frac{\pi}{2}R_K(\kappa^2,1)$ with $\kappa^2=1-k^2$. It was pointed out in [11] that $R_K(x,y)$ is 2-log-convex in both x and y. Consequently, it follows by either Theorem 1 or Theorem 2 of this paper that the functions

$$\frac{[R_K(bx,y)]^a}{[R_K(ax,y)]^b}\quad\text{or}\quad\frac{[R_K(x,by)]^a}{[R_K(x,ay)]^b} \tag{10}$$

for $b>a>0$ are increasing in x or in y respectively.

It is worthwhile to remark that some inequalities of the complete elliptic integrals of all three kinds and their relation inequalities had been researched in [5, 7, 19, 20, 21, 25, 29], see also [14].

2.8. For two positive numbers s and t, the identric or exponential mean $I(s,t)$ is defined [8, 9, 13] by

$$I(s,t)=\frac{1}{e}\left(\frac{t^t}{s^s}\right)^{1/(t-s)} \tag{11}$$

It was pointed out in [23] that, by standard argument, it is easy to verify that the reciprocal of the identric mean $I_{s,t}(x)=I(x+s,x+t)$ is a logarithmically completely monotonic function in $x>-\min\{s,t\}$ for $s,t\in\mathbb{R}$ with $s\ne t$. By Theorem 1, it is concluded that the function

$$\frac{[I_{s,t}(bx)]^{-a}}{[I_{s,t}(ax)]^{-b}}=\frac{[I_{s,t}(ax)]^b}{[I_{s,t}(bx)]^a} \tag{12}$$

for $b > a > 0$ is increasing in $x > -\min\left\{\frac{\min\{s,t\}}{a}, \frac{\min\{s,t\}}{b}\right\}$.

2.9. The so-called extended mean values $E(r,s;x,y)$ for $x,y>0$ and $r,s\in\mathbb{R}$ can be defined [8, 9, 13, 17] by

$$E(r,s;x,y)=\left[\frac{r}{s}\cdot\frac{y^s-x^s}{y^r-x^r}\right]^{1/(s-r)}, \qquad rs(r-s)(x-y)\neq 0; \tag{13}$$

$$E(r,0;x,y)=\left[\frac{1}{r}\cdot\frac{y^r-x^r}{\ln y-\ln x}\right]^{1/r}, \qquad r(x-y)\neq 0; \tag{14}$$

$$E(r,r;x,y)=\frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r-y^r)}, \qquad r(x-y)\neq 0; \tag{15}$$

$$E(0,0;x,y)=\sqrt{xy}, \qquad x\neq y; \tag{16}$$

$$E(r,s;x,x)=x, \qquad x=y.$$

In [15, 16], the logarithmic convexities of $E(r,s;x,y)$ are obtained: For all fixed $x,y>0$ and $s\in[0,\infty)$ (or $r\in[0,\infty)$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically concave in r (or in s, respectively) on $[0,\infty)$; For all fixed $x,y>0$ and $s\in(-\infty,0]$ (or $r\in(-\infty,0]$, respectively), the extended mean values $E(r,s;x,y)$ are logarithmically convex in r (or in s, respectively) on $(-\infty,0]$. In view of Theorem 1, the following theorem is showed easily.

Theorem 3. *If x and y are positive numbers, $s\in(-\infty,\infty)$ and $b>a\ge 0$, then the function*

$$\frac{[E(br,s;x,y)]^a}{[E(ar,s;x,y)]^b} \tag{17}$$

is decreasing in $r\in(-\infty,\infty)$.

2.10. In [24, Theorem 1], among other things, the following conclusions are proved:

(1) For $i\in\mathbb{N}$, $\alpha\ge 0$ and $\beta\ge 0$, the function

$$x^\alpha\left|\psi^{(i)}(x)\right|=x^\alpha\left|[\ln\Gamma(x)]^{(i+1)}\right|=(-1)^{i+1}x^\alpha[\ln\Gamma(x)]^{(i+1)}$$

in $(0,\infty)$ is strictly decreasing if and only if $0\le\alpha\le i$;

(2) For $\beta\ge\frac{1}{2}$, the function

$$x^\alpha\left|\psi^{(i)}(x+\beta)\right|=x^\alpha\left|[\ln\Gamma(x+\beta)]^{(i+1)}\right|=(-1)^{i+1}x^\alpha[\ln\Gamma(x+\beta)]^{(i+1)}$$

is strictly increasing in $[0,\infty)$ if and only if $\alpha\ge i$.

These imply that the functions

$$x^{2i-1}[\ln\Gamma(x)]^{(2i)} \qquad\text{and}\qquad x^{2i}[\ln\Gamma(x+\beta)]^{(2i+1)}$$

for $i\in\mathbb{N}$ are decreasing in $x>0$ and that the functions

$$x^{2i}[\ln\Gamma(x)]^{(2i+1)} \qquad\text{and}\qquad x^{2i-1}[\ln\Gamma(x+\beta)]^{(2i)}$$

for $i\in\mathbb{N}$ are increasing in $x>0$, where $\beta\ge\frac{1}{2}$. By virtue of Theorem 2, the following two theorems are established.

Theorem 4. *For $x>0$ and $b>a>0$, the function $\frac{[\Gamma(bx)]^a}{[\Gamma(ax)]^b}$ is $(2i+1)$-log-convex and $(2i)$-log-concave in x for $i\in\mathbb{N}$.*

Theorem 5. *For $b>a>0$ and $\beta\ge\frac{1}{2}$, the function $\frac{[\Gamma(bx+\beta)]^a}{[\Gamma(ax+\beta)]^b}$ is $(2i+1)$-log-concave and $(2i)$-log-convex in $x>0$ for $i\in\mathbb{N}$.*

3. Proofs of theorems

Now we are in a position to prove our theorems.

Proof of Theorem 1. If $f(x)$ is 2-log-convex (or 2-log-concave, respectively) on I, then $[\ln f(x)]'' \ge 0$ and $[\ln f(x)]' = \frac{f'(x)}{f(x)}$ is increasing (or decreasing, respectively).

Logarithmic differentiation of (3) yields

$$[\ln g_{a,b}(x)]' = ab\biggl[\frac{f'(bx)}{f(bx)} - \frac{f'(ax)}{f(ax)}\biggr] = ab\int_{ax}^{bx} [\ln f(u)]'' \,\mathrm{d} u. \tag{18}$$

(1) When $f(x)$ is 2-log-convex on I,
   (a) if either $ab > 0$ and $bx > ax$ or $ab < 0$ and $bx < ax$, then $[\ln g_{a,b}(x)]' \ge 0$, and then the function $g_{a,b}(x)$ is increasing in x;
   (b) if either $ab > 0$ and $bx < ax$ or $ab < 0$ and $bx > ax$, then $[\ln g_{a,b}(x)]' \le 0$, and then the function $g_{a,b}(x)$ is decreasing in x.
(2) When $f(x)$ is 2-log-concave on I,
   (a) if either $ab > 0$ and $bx > ax$ or $ab < 0$ and $bx < ax$, then $[\ln g_{a,b}(x)]' \le 0$, and then the function $g_{a,b}(x)$ is decreasing in x;
   (b) if either $ab > 0$ and $bx < ax$ or $ab < 0$ and $bx > ax$, then $[\ln g_{a,b}(x)]' \ge 0$, and then the function $g_{a,b}(x)$ is increasing in x.

The proof of Theorem 1 is finished. □

Proof of Theorem 2. It is easy to see that (18) can be rewritten as

$$[\ln g_{a,b}(x)]' = ab\Bigl\{[\ln f(u)]'\big|_{u=bx} - [\ln f(u)]'\big|_{u=ax}\Bigr\}. \tag{19}$$

Differentiating on both sides of (19) consecutively gives

$$\begin{aligned}[\ln g_{a,b}(x)]^{(i)} &= ab\Bigl\{b^{i-1}[\ln f(u)]^{(i)}\big|_{u=bx} - a^{i-1}[\ln f(u)]^{(i)}\big|_{u=ax}\Bigr\}\\ &= \frac{ab}{x^{i-1}}\Bigl\{(bx)^{i-1}[\ln f(u)]^{(i)}\big|_{u=bx} - (ax)^{i-1}[\ln f(u)]^{(i)}\big|_{u=ax}\Bigr\}\end{aligned} \tag{20}$$

for $i \in \mathbb{N}$. Therefore, the following conclusions can be concluded:

(1) For either $b > a > 0$ and $x > 0$ or $0 > b > a$ and $x > 0$,
   (a) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is increasing on I, then $[\ln g_{a,b}(x)]^{(i)} \ge 0$ and the function $g_{a,b}(x)$ is i-log-convex;
   (b) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is decreasing on I, then $[\ln g_{a,b}(x)]^{(i)} \le 0$ and the function $g_{a,b}(x)$ is i-log-concave.
(2) For $b > 0 > a$ and $x > 0$,
   (a) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is increasing on I, then $[\ln g_{a,b}(x)]^{(i)} \le 0$ and the function $g_{a,b}(x)$ is i-log-concave;
   (b) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is decreasing on I, then $[\ln g_{a,b}(x)]^{(i)} \ge 0$ and the function $g_{a,b}(x)$ is i-log-convex.
(3) For either $b > a > 0$ and $x > 0$ or $0 > b > a$ and $x < 0$,
   (a) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is increasing on I, then $[\ln g_{a,b}(x)]^{(2i-1)} \le 0$ and $[\ln g_{a,b}(x)]^{(2i)} \ge 0$, and the function $g_{a,b}(x)$ is $(2i-1)$-log-concave and $(2i)$-log-convex;
   (b) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is decreasing on I, then $[\ln g_{a,b}(x)]^{(2i-1)} \ge 0$ and $[\ln g_{a,b}(x)]^{(2i)} \le 0$, and the function $g_{a,b}(x)$ is $(2i-1)$-log-convex and $(2i)$-log-concave.
(4) For $b > 0 > a$ and $x < 0$,

(a) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is increasing on I, then $[\ln g_{a,b}(x)]^{(2i-1)} \geq 0$ and $[\ln g_{a,b}(x)]^{(2i)} \leq 0$, and the function $g_{a,b}(x)$ is $(2i-1)$-log-convex and $(2i)$-log-concave;

(b) if the function $u^{i-1}[\ln f(u)]^{(i)}$ is decreasing on I, then $[\ln g_{a,b}(x)]^{(2i-1)} \leq 0$ and $[\ln g_{a,b}(x)]^{(2i)} \geq 0$, and the function $g_{a,b}(x)$ is $(2i-1)$-log-concave and $(2i)$-log-convex.

The proof of Theorem 2 is complete. □

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