A FUNCTION INVOLVING GAMMA FUNCTION AND HAVING LOGARITHMICALLY ABSOLUTE CONVEXITY

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Abstract. In the paper, the logarithmically complete monotonicity, logarithmically absolute monotonicity and logarithmically absolute convexity of the function $\frac{[\Gamma(1+tx)]^s}{[\Gamma(1+sx)]^t}$ for $x,s,t\in\mathbb{R}$ such that $1+sx>0$ and $1+tx>0$ with $s\ne t$ are verified, some known results are generalized.

1. Introduction

By using a geometrical method in [1], the following double inequality was proved:

$$\frac{1}{n!}\le\frac{[\Gamma(1+x)]^n}{\Gamma(1+nx)}\le 1 \tag{1}$$

for $x\in[0,1]$ and $n\in\mathbb{N}$, where $\Gamma(x)$ stands for the classical Euler's gamma function defined for $x>0$ by $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}\,t$.

By analytical arguments in [29], it was presented that the function

$$f(x,y)=\frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \tag{2}$$

for all $y\ge1$ is decreasing in $x\ge0$. From this, it is deduced that

$$\frac{1}{\Gamma(1+y)}\le\frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)}\le 1 \tag{3}$$

for all $y\ge1$ and $x\in[0,1]$, which is a generalization of inequality (1).

Recall [12, 30, 31] that a function f is called completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^kf^{(k)}(x)\ge0$ for all $k\ge0$ on I. The set of completely monotonic functions on I is denoted by $\mathcal{C}[I]$. Recall also [20, 21, 22] that a positive function f is said to be logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^k[\ln f(x)]^{(k)}\ge0$ for all $k\in\mathbb{N}$ on I. The set of logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. Recall [3] that if $f^{(k)}(x)$ for some nonnegative integer k is completely monotonic on an interval I, but $f^{(k-1)}(x)$

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is not completely monotonic on I, then $f(x)$ is called a completely monotonic function of k-th order on an interval I. Among other things, it was proved implicitly or explicitly in [4, 14, 18, 20, 22, 30] that a logarithmically completely monotonic function is always completely monotonic, but not conversely. Motivated or stimulated by the papers [20, 22], among other things, it is further revealed in [4] that $\mathcal{S}\setminus\{0\}\subset\mathcal{L}[(0,\infty)]\subset\mathcal{C}[(0,\infty)]$, where $\mathcal{S}$ denotes the set of Stieltjes transforms. In [4, Theorem 1.1] and [9, 21] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [10, Theorem 4.4]. For more information on the logarithmically completely monotonic functions, please refer to [3, 4, 6, 7, 8, 9, 15, 18, 19, 20, 21, 22, 23, 25, 28, 30] and the references therein.

Definition 1. Let f be a positive function which has derivatives of all orders on an interval I. If $[\ln f(x)]^{(k)}$ for some nonnegative integer k is completely monotonic on I, but $[\ln f(x)]^{(k-1)}$ is not completely monotonic on I, then f is said to be a logarithmically completely monotonic function of k-th order on I.

In [25, 28], the following logarithmically complete monotonicities, as generalizations of the decreasingly monotonic property in [29], are presented:

(1) For given $y>1$, the function $f(x,y)$ defined by (2) is decreasing and logarithmically concave with respect to $x\in(0,\infty)$, and $\frac{1}{f(x,y)}$ is a logarithmically completely monotonic function of second order in $x\in(0,\infty)$.
(2) For given $0<y<1$, the function $f(x,y)$ defined by (2) is increasing and logarithmically convex with respect to $x\in(0,\infty)$, and $f(x,y)$ is a logarithmically completely monotonic function of second order in $x\in(0,\infty)$.
(3) For given $x\in(0,\infty)$, the function $f(x,y)$ defined by (2) is logarithmically concave with respect to $y\in(0,\infty)$, and $\frac{1}{f(x,y)}$ is a logarithmically completely monotonic function of first order in $y\in(0,\infty)$.
(4) For given $x\in(0,\infty)$, let
$$F_x(y)=\frac{\Gamma(1+y)[\Gamma(1+x)]^y}{\Gamma(1+xy)} \tag{4}$$
in $\in(0,\infty)$. If $0<x<1$, then $F_x(y)$ is a logarithmically completely monotonic function of second order in $(0,\infty)$; if $x>1$, then $\frac{1}{F_x(y)}$ is a logarithmically completely monotonic function of second order in $(0,\infty)$.

In [13], It was shown that if f is a differentiable and logarithmically convex function in $[0,\infty)$, then the function $\frac{[f(x)]^a}{f(ax)}$ for $a\ge1$ (or $0<a\le1$ respectively) is decreasing (or increasing respectively) in $[0,\infty)$. As one of applications to inequalities involving gamma function, Riemann's zeta function and the complete elliptic integrals of the first kind, inequalities (1) and (3) were deduced.

In [11], an inequality involving a positive linear operator acting on the composition of two continuous functions is presented and, as applications of this inequality, some new inequalities involving the beta, gamma and Riemann's zeta functions and a large family of functions which are Mellin transforms are produced. In particular, for $\beta > \delta > 0$, $\alpha\beta > -1$ and $\alpha\delta > -1$, if either $\alpha < 0$ or $\alpha > 1$, then inequality

$$\frac{[\Gamma(1+\delta)]^\alpha}{\Gamma(1+\alpha\delta)} > \frac{[\Gamma(1+\beta)]^\alpha}{\Gamma(1+\alpha\beta)} \tag{5}$$

holds true; if $0 < \alpha \le 1$, inequality (5) is reversed. It is not difficult to see that the left hand sides in inequalities (1) and (3) are special cases of inequality (5).

In [5, Theorem 2.1], it was proved that the function

$$G_{s,t}(x) = \frac{[\Gamma(1+tx)]^s}{[\Gamma(1+sx)]^t} \tag{6}$$

is decreasing (or increasing respectively) in $x \in [0,\infty)$ if either $s \ge t > 0$ or $0 > s \ge t$ (or both $s > 0$ and $t < 0$ respectively) such that $1 + sx > 0$ and $1 + tx > 0$. This result generalized and extended the corresponding conclusions in [1, 11, 29].

Recall [12, 15, 16, 27, 30, 31] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t) \ge 0$ for $t \in I$ and $k \in \mathbb{N}$. Recall also [14, 26] that a positive function f is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(t)]^{(k)} \ge 0$ for $t \in I$ and $k \in \mathbb{N}$. In [14, 26] it was proved that a logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I, but not conversely.

Definition 2. Let f be a positive function which has derivatives of all orders on an interval I. If $[\ln f(x)]^{(k)}$ for some nonnegative integer k is absolutely monotonic on I, but $[\ln f(x)]^{(k-1)}$ is not absolutely monotonic on I, then f is said to be a logarithmically absolutely monotonic function of k-th order on I.

Recall [12, p. 375, Definition 3] and [16, 17, 30, 31] that a function f which has derivatives of all orders on an interval I is said to be absolutely convex on I if $f^{(2k)}(x) \ge 0$ on I for any nonnegative integer k.

Definition 3. A positive function f which has derivatives of all orders on an interval I is said to be logarithmically absolutely convex on I if $[\ln f(x)]^{(2k)} \ge 0$ on I for $k \in \mathbb{N}$.

The main aim of this paper is to generalize and extend some results obtained in [1, 5, 11, 13, 25, 28, 29].

Our main results can be stated as the following theorem.

Theorem 1. *The function $G_{s,t}(x)$ defined by (6) for $x, s, t \in \mathbb{R}$ such that $1+sx > 0$ and $1 + tx > 0$ with $s \ne t$ has the following properties:*

(1) $G_{s,t}(x) = \frac{1}{G_{t,s}(x)}$;
(2) *For $t > s > 0$ and $x \in (0,\infty)$, $G_{s,t}(x)$ is an increasing function and a logarithmically completely monotonic function of second order in x;*

(3) *For $t > s > 0$ and $x \in \left(-\frac{1}{t}, 0\right)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x;*
(4) *For $s < t < 0$ and $x \in (-\infty, 0)$, $G_{s,t}(x)$ is a decreasing function and a logarithmically absolutely monotonic function of second order in x;*
(5) *For $s < t < 0$ and $x \in \left(0, -\frac{1}{s}\right)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x;*
(6) *For $s < 0 < t$ and $x \in \left(-\frac{1}{t}, 0\right)$, $G_{t,s}(x)$ is an increasing function and a logarithmically absolutely convex function in x;*
(7) *For $s < 0 < t$ and $x \in \left(0, -\frac{1}{s}\right)$, $G_{t,s}(x)$ is a decreasing function and a logarithmically absolutely convex function in x.*

2. Proof of Theorem 1

It is clear that $G_{s,t}(x) = \frac{1}{G_{t,s}(x)}$. Therefore, it is sufficient to show Theorem 1 only for $s < t$.

Taking logarithm of $G_{s,t}(x)$ yields

$$\ln G_{s,t}(x) = s \ln \Gamma(1 + tx) - t \ln \Gamma(1 + sx) \tag{7}$$

and

$$\begin{aligned} \frac{\partial^k [\ln G_{s,t}(x)]}{\partial x^k} &= st^k \psi^{(k-1)}(1 + tx) - ts^k \psi^{(k-1)}(1 + sx) \\ &= \frac{st}{x^{k-1}} \left[(tx)^{k-1} \psi^{(k-1)}(1 + tx) - (sx)^{k-1} \psi^{(k-1)}(1 + sx)\right] \end{aligned} \tag{8}$$

for $k \in \mathbb{N}$.

In [2, Lemma 2.2], it was obtained that the function $x^\alpha \left|\psi^{(i)}(1 + x)\right|$ is strictly increasing in $(0, \infty)$ if and only if $\alpha \ge i$, where $i \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. This was generalized in [24, Theorem 1] as follows: For $\beta \ge \frac{1}{2}$, the function $x^\alpha \left|\psi^{(i)}(x + \beta)\right|$ is strictly increasing in $[0, \infty)$ if and only if $\alpha \ge i \in \mathbb{N}$. In particular, the functions $x^{2i} \psi^{(2i)}(1 + x)$ are decreasing and the functions $x^{2i-1} \psi^{(2i-1)}(1 + x)$ are increasing in $[0, \infty)$ for $i \in \mathbb{N}$. From this, it is readily obtained for $i \in \mathbb{N}$, $t > s > 0$ and $x > 0$ that

$$\frac{\partial^{2i} [\ln G_{s,t}(x)]}{\partial x^{2i}} > 0 \qquad \text{and} \qquad \frac{\partial^{2i+1} [\ln G_{s,t}(x)]}{\partial x^{2i+1}} < 0.$$

Since

$$\frac{\partial [\ln G_{s,t}(x)]}{\partial x} = st[\psi(1 + tx) - \psi(1 + sx)] \tag{9}$$

and the psi function ψ is increasing in $(0, \infty)$, it is is easy to see

$$\frac{\partial [\ln G_{s,t}(x)]}{\partial x} > 0.$$

Consequently, for $t > s > 0$ and $x > 0$, $G_{s,t}(x)$ is an increasing function in $x \in (0, \infty)$ and a logarithmically completely monotonic function of second order in $x \in (0, \infty)$.

If $t > s > 0$ and $0 > x > -\frac{1}{t}$, then

$$(-1)^{k-1}\frac{\partial^k[\ln G_{s,t}(x)]}{\partial x^k} = \frac{st}{(-x)^{k-1}}\bigl[(tx)^{k-1}\psi^{(k-1)}(1+tx) - (sx)^{k-1}\psi^{(k-1)}(1+sx)\bigr]$$

for $k \in \mathbb{N}$. Let $h_k(u) = u^k\psi^{(k)}(1+u)$ for $u \in (-1,0)$ and $k \in \mathbb{N}$. Straightforward computation gives

$$\begin{aligned} h_k'(u) &= u^{k-1}\bigl[k\psi^{(k)}(1+u) + u\psi^{(k+1)}(1+u)\bigr] \\ &= (-u)^{k-1}\bigl\{k\bigl[(-1)^{k+1}\psi^{(k)}(1+u)\bigr] + (-u)\bigl[(-1)^{k+2}\psi^{(k+1)}(1+u)\bigr]\bigr\} > 0. \end{aligned}$$

This means that the function $h_k(u)$ for $u \in (-1,0)$ and $k \in \mathbb{N}$ is increasing. Then it is concluded that $(-1)^{k-1}\frac{\partial^k[\ln G_{s,t}(x)]}{\partial x^k} < 0$ for $k \in \mathbb{N}$. As a result, the function $G_{s,t}(x)$ is logarithmically completely monotonic in $x \in \bigl(-\frac{1}{t},0\bigr)$ for $t > s > 0$.

If $s < t < 0$ and $x < 0$, then

$$(-1)^{k}\frac{\partial^k[\ln G_{s,t}(x)]}{\partial x^k} = \frac{st}{(-x)^{k-1}}\bigl[(sx)^{k-1}\psi^{(k-1)}(1+sx) - (tx)^{k-1}\psi^{(k-1)}(1+tx)\bigr]$$

for $k \in \mathbb{N}$, accordingly

$$(-1)^{2i}\frac{\partial^{2i}[\ln G_{s,t}(x)]}{\partial x^{2i}} > 0 \qquad \text{and} \qquad (-1)^{2i+1}\frac{\partial^{2i+1}[\ln G_{s,t}(x)]}{\partial x^{2i+1}} < 0$$

for $i \in \mathbb{N}$. From (9), it follows that $\frac{\partial[\ln G_{s,t}(x)]}{\partial x} < 0$. Hence, $G_{s,t}(x)$ is a logarithmically absolutely monotonic function of second order in $x \in (-\infty,0)$ for $s < t < 0$.

If $s < t < 0$ and $0 < x < -\frac{1}{s}$, then formula (8) is valid. In virtue of the increasingly monotonic property of the function $h_k(u)$ for $u \in (-1,0)$ and $k \in \mathbb{N}$, it is deduced that $\frac{\partial^k[\ln G_{s,t}(x)]}{\partial x^k} > 0$ for $k \in \mathbb{N}$. This means that the function $G_{s,t}(x)$ for $s < t < 0$ is logarithmically completely monotonic in $x \in \bigl(0,-\frac{1}{s}\bigr)$.

If $s < 0 < t$ and $-\frac{1}{t} < x < 0$, then

$$\begin{aligned} \frac{\partial^k[\ln G_{s,t}(x)]}{\partial x^k} = \frac{-st}{(-x)^{k-1}}\bigl\{&(-1)^{k-1}(-tx)^{k-1}\bigl[(-1)^k\psi^{(k-1)}(1+tx)\bigr] \\ &- (sx)^{k-1}\bigl[(-1)^k\psi^{(k-1)}(1+sx)\bigr]\bigr\} \end{aligned}$$

for $k \in \mathbb{N}$. Accordingly, it is obtained readily that $\frac{\partial^{2i}[\ln G_{s,t}(x)]}{\partial x^{2i}} < 0$ for $i \in \mathbb{N}$. This implies that the function $\frac{1}{G_{s,t}(x)}$ is logarithmically absolutely convex in $x \in \bigl(-\frac{1}{t},0\bigr)$ for $s < 0 < t$. Formula (9) implies that the function $G_{s,t}(x)$ is increasing in $x \in \bigl(-\frac{1}{t},0\bigr)$ for $s < 0 < t$.

If $s < 0 < t$ and $0 < x < -\frac{1}{s}$, then

$$\begin{aligned} \frac{\partial^k[\ln G_{s,t}(x)]}{\partial x^k} = \frac{-st}{x^{k-1}}(-1)^k\bigl\{&(sx)^{k-1}\bigl[(-1)^k\psi^{(k-1)}(1+sx)\bigr] \\ &- (tx)^{k-1}\bigl[(-1)^k\psi^{(k-1)}(1+tx)\bigr]\bigr\} \end{aligned}$$

for $k \in \mathbb{N}$. Hence, it is deduced apparently that $\frac{\partial^{2i}[\ln G_{s,t}(x)]}{\partial x^{2i}} < 0$ for $i \in \mathbb{N}$. This implies that the function $\frac{1}{G_{s,t}(x)}$ is logarithmically absolutely convex in $x \in \bigl(0,-\frac{1}{s}\bigr)$

for $s < 0 < t$. Formula (9) implies that the function $G_{s,t}(x)$ is decreasing in $x \in \left(0, -\frac{1}{s}\right)$ for $s < 0 < t$. The proof of Theorem 1 is finished.

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