# UNIVERSAL MULTIPLICATION TABLE 

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Abstract. In this paper, considering the concept of Universal Multiplication Table, we show that for every $n \geq 2$, the inequality:

$$
M(n)=\#\{i j \mid 1 \leq i, j \leq n\} \geq \frac{n^{2}}{\mathfrak{N}\left(n^{2}\right)}
$$

holds true with:

$$
\left.\mathfrak{N}(n)=n^{\frac{\log 2}{\log \log n}\left(1+\frac{387}{200 \log \log n}\right.}\right) .
$$

## 1. Introduction

Consider the following $n \times n$ Multiplication Table, which we denote it by $M T_{n \times n}$ :

| 1 | 2 | 3 | $\cdots$ | $n$ |
| :---: | :---: | :---: | :--- | :---: |
| 2 | 4 | 6 | $\cdots$ | $2 n$ |
| 3 | 6 | 9 | $\cdots$ | $3 n$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $n$ | $2 n$ | $3 n$ | $\cdots$ | $n^{2}$ |

Let $\mathfrak{M}(n ; k)$ be the number of $k$ 's, which appear in $M T_{n \times n}$; i.e.

$$
\begin{equation*}
\mathfrak{M}(n ; k)=\#\left\{(a, b) \in \mathbb{N}_{n}^{2} \mid a b=k\right\} \tag{1.1}
\end{equation*}
$$

where $\mathbb{N}_{n}=\mathbb{N} \cap[1, n]$. For example, we have:

$$
\mathfrak{M}(2 ; 2)=2, \mathfrak{M}(7 ; 6)=4, \mathfrak{M}(10 ; 9)=3, \mathfrak{M}(100 ; 810)=10, \mathfrak{M}(100 ; 9900)=2
$$

In this paper first we study some elementary properties of the function $\mathfrak{M}(n ; k)$, for a fixed $n \in \mathbb{N}$. Then we try to connect $\mathfrak{M}(n ; k)$ by the famous Multiplication Table Function ${ }^{1} ; M(n)=\#\left\{i j \mid(i, j) \in \mathbb{N}_{n}^{2}\right\}$ in order to get some lower bounds for it. To do this, we introduce the concept of Universal Multiplication Table, which is an infinite array generated by multiplying the components of points in the infinite lattice $\mathbb{N}^{2}$. Let $D(n)=\{d: d>0, d \mid n\}$. To get above mentioned bounds for the function $M(n)$, we will need some upper bounds for the Divisor Function $d(n)=\# D(n)$, which we recall best known, due to J.L. Nicolas [5]:

$$
\frac{\log d(n)}{\log 2} \leq \frac{\log n}{\log \log n}\left(1+\frac{1.9349 \cdots}{\log \log n}\right) \quad(n \geq 3)
$$

or

$$
\begin{equation*}
d(n) \leq \mathfrak{N}(n) \tag{1.2}
\end{equation*}
$$

[^0]for $n \geq 3$, with
$$
\mathfrak{N}(n)=n^{\frac{\log 2}{\log \log n}\left(1+\frac{387}{200 \log \log n}\right)}
$$

## 2. Some Elementary Properties of the Function $\mathfrak{M}(n ; k)$

Considering (1.1), for every $s \in \mathbb{C}$, we have:

$$
\begin{equation*}
\sum_{1 \leq i, j \leq n} \frac{1}{(i j)^{s}}=\sum_{k=1}^{n^{2}} \frac{\mathfrak{M}(n ; k)}{k^{s}}=\sum_{k=1}^{\infty} \frac{\mathfrak{M}(n ; k)}{k^{s}} \tag{2.1}
\end{equation*}
$$

The left hand side of above identity is equal to $\zeta_{n}^{2}(s)$, in which $\zeta_{n}(s)=\sum_{i=1}^{n} \frac{1}{i^{s}}$, and the number of summands in the right hand side of above identity, is equal to $M(n)$. Also, summing and counting all numbers in $M T_{n \times n}$, we obtain respectively:

$$
\sum_{k=1}^{n^{2}} k \mathfrak{M}(n ; k)=\left(\frac{n(n+1)}{2}\right)^{2}
$$

and

$$
\sum_{k=1}^{n^{2}} \mathfrak{M}(n ; k)=n^{2}
$$

which both of them are special cases of (2.1) for $s=-1$ and $s=0$, respectively. To have some formulas for the function $\mathfrak{M}(n ; k)$, we define Incomplete Divisor Function to be $d(k ; x)=\# D(k) \cap[1, x]$. This function has some properties, which we list some of them:

1. It is trivial that for every $x \geq 1$ we have:

$$
1 \leq d(k ; x) \leq \min \{x, d(k)\}
$$

So, $d(k ; x)=O(x)$ and naturally we ask: What is the exact order of $d(k ; x)$ ? The next property, maybe useful to find answer.
2. If we let $D(k)=\left\{1=d_{1}, d_{2}, \cdots, d_{d(k)}=k\right\}$, then we have:

$$
\begin{aligned}
\int_{1}^{k} d(k ; x) d x & =\sum_{i=1}^{d(k)-1}\left(d_{i+1}-d_{i}\right) i=\sum_{i=1}^{d(k)-1}(i+1) d_{i+1}-i d_{i}-\sum_{i=1}^{d(k)-1} d_{i+1} \\
& =d(k) d_{d(k)}-1 d_{1}-\sum_{d \mid k, d>1} d=k d(k)-\sigma(k)
\end{aligned}
$$

where $\sigma(k)=\sum_{a \in D(k)} a$, and we have the following bound due to G. Robin [7]:

$$
\begin{equation*}
\sigma(n)<\mathfrak{R}(n) \quad(n \geq 3) \tag{2.2}
\end{equation*}
$$

with

$$
\mathfrak{R}(n)=e^{\gamma} n \log \log n+\frac{3241 n}{5000 \log \log n},
$$

where $\gamma \approx 0.5772156649$ is Euler's constant. Considering (1.2) and (2.2), we obtain the following inequality for every $k \geq 3$ :

$$
2 k-\mathfrak{R}(k)<\int_{1}^{k} d(k ; x) d x<k \mathfrak{N}(k)-k-1 .
$$

In general, every knowledge about $d(k ; x)$ is useful, because:

Proposition 2.1. For every positive integers $k$ and $n$, we have:

$$
\mathfrak{M}(n ; k)=d(k ; n)-d\left(k ; \frac{k}{n}\right)+R(n ; k),
$$

where

$$
R(n ; k)=\left\lfloor\frac{k}{n}\right\rfloor-\left\lfloor\frac{k-1}{n}\right\rfloor= \begin{cases}1, & n \mid k \\ 0, & \text { other wise }\end{cases}
$$

Proof. Considering (1.1), we have:

$$
\mathfrak{M}(n ; k)=\#\left\{(a, b) \in \mathbb{N}_{n}^{2} \mid a b=k\right\}=\sum_{d \mid k, d \leq n, \frac{k}{d} \leq n} 1=\sum_{d \mid k, \frac{k}{n} \leq d \leq n} 1
$$

Applying the definition of $d(k ; x)$, completes the proof.

## 3. Universal Multiplication Table Function

We define the Universal Multiplication Table Function $\mathfrak{M}(k)$ to be the number of $k$ 's, which appear in the universal multiplication table.
Proposition 3.1. For every positive integer $k$, we have:

$$
\mathfrak{M}(k)=d(k) .
$$

Proof. Here we have two proofs:
Elementary Method. Considering the definition of universal multiplication table, we have:

$$
\mathfrak{M}(k)=\lim _{n \rightarrow \infty} \mathfrak{M}(n ; k)=\lim _{n \rightarrow \infty} \sum_{d \mid k, \frac{k}{n} \leq d \leq n} 1=\sum_{d \mid k, 0<d<\infty} 1=d(k) .
$$

Analytic Method. Considering (2.1) for $\Re(s)>1$ and taking limit both sides of it, when $n$ tends to infinity, we obtain:

$$
\begin{equation*}
\sum_{k=1}^{\infty} \frac{\mathfrak{M}(k)}{k^{s}}=\zeta^{2}(s) \tag{3.1}
\end{equation*}
$$

in which $\zeta(s)$ is the Riemann zeta-function. According to the Theorem 11.17 of [1], we obtain:

$$
\mathfrak{M}(k)=\lim _{T \rightarrow \infty} \frac{1}{2 T} \int_{-T}^{T} \zeta^{2}(\sigma+i t) k^{\sigma+i t} d t, \quad(\sigma>1)
$$

Since $\zeta^{2}(s)=\sum_{m=1}^{\infty} d(m) m^{-s}$, we have:

$$
\begin{aligned}
\mathfrak{M}(k) & =\lim _{T \rightarrow \infty} \frac{1}{2 T} \int_{-T}^{T} \zeta^{2}(\sigma+i t) k^{\sigma+i t} d t \\
& =\sum_{m=1}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim _{T \rightarrow \infty} \frac{1}{2 T} \int_{-T}^{T}\left(\frac{k}{m}\right)^{i t} d t \\
& =\sum_{m=1, m \neq k}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim _{T \rightarrow \infty} \frac{1}{2 T} \int_{-T}^{T}\left(\frac{k}{m}\right)^{i t} d t+d(k) \\
& =\sum_{m=1, m \neq k}^{\infty} d(m) m^{-\sigma} k^{\sigma} \lim _{T \rightarrow \infty} \frac{1}{T} \sin \left(T \log \left(\frac{k}{m}\right)\right)+d(k)=d(k) .
\end{aligned}
$$

This completes the proof.

Now, fix positive integer $k$ and consider $\mathfrak{M}(n ; k)$, as an arithmetic function of the variable $n$. Clearly, $\mathfrak{M}(n ; k)$ is increasing, and for $n>k$, we have $\mathfrak{M}(n ; k)=\mathfrak{M}(k)$. Thus considering Proposition 3.1, we obtain:

$$
\begin{equation*}
\mathfrak{M}(n ; k) \leq d(k), \tag{3.2}
\end{equation*}
$$

and if $k \geq 3$, considering (1.2) yields that:

$$
\mathfrak{M}(n ; k) \leq \mathfrak{N}(k)
$$

## 4. Statistical Study of $\mathfrak{M}(n ; k)$ 's

Consider $S=\left[\mathfrak{M}(n ; k) \mid 1 \leq k \leq n^{2}\right]$ as a list of statistical data and suppose $\overline{\mathfrak{M}}(n)$ is the average of above list, then we have:

$$
\overline{\mathfrak{M}}(n)=\frac{\sum_{k=1}^{n^{2}} \mathfrak{M}(n ; k)}{\#\left\{i j \mid(i, j) \in \mathbb{N}_{n}^{2}\right\}}=\frac{n^{2}}{M(n)}
$$

Thus, we have:

$$
\begin{equation*}
M(n)=\frac{n^{2}}{\overline{\mathfrak{M}}(n)} \tag{4.1}
\end{equation*}
$$

Considering (3.2), it is clear that:

$$
\overline{\mathfrak{M}}(n) \leq \max \{\mathfrak{M}(n ; k)\}_{k=1}^{n^{2}} \leq \max \{d(k)\}_{k=1}^{n^{2}}
$$

To use (1.2), we observe that the function $\mathfrak{N}(n)$ is increasing for $n \geq 114$. So, we have:

$$
\overline{\mathfrak{M}}(n) \leq \max \left\{d(1), d(2), \cdots, d(114), d\left(n^{2}\right)\right\} \leq \max \left\{12, \mathfrak{N}\left(n^{2}\right)\right\} \quad(n \geq \sqrt{3})
$$

and since $\mathfrak{N}(n)>114.1$ holds for every $n>0$, we obtain:

$$
\overline{\mathfrak{M}}(n) \leq \mathfrak{N}\left(n^{2}\right) \quad(n \geq 2)
$$

Therefore, we have proved the following result.
Theorem 4.1. For every $n \geq 2$, we have:

$$
M(n) \geq \frac{n^{2}}{\mathfrak{N}\left(n^{2}\right)}
$$

Remark 4.2. One of the wonderful results about $M T_{n \times n}$ is Erdös Multiplication Table Theorem [6], which asserts:

$$
\lim _{n \rightarrow \infty} \frac{M(n)}{n^{2}}=0
$$

Above theorem yields that in the Erdös's theorem, however the ratio $\frac{M(n)}{n^{2}}$ tends to zero, but it doesn't faster than $\frac{1}{\mathfrak{N}\left(n^{2}\right)}$. More precisely, Erdös showed that $M(n)=n^{2}(\log n)^{-c+o(1)}$ for $c=1+\frac{\log \log 2}{\log 2}[2,3]$. The following table includes some computational results about $M(n)$ by the Maple software.

| $n$ | $M(n)$ | $M(n) / n^{2} \approx$ | $n$ | $M(n)$ | $M(n) / n^{2} \approx$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 42 | 0.4200000000 | 2000 | 959759 | 0.2399397500 |
| 50 | 800 | 0.3200000000 | 3000 | 2121063 | 0.2356736667 |
| 100 | 2906 | 0.2906000000 | 4000 | 3723723 | 0.2327326875 |
| 1000 | 248083 | 0.2480830000 | 5000 | 5770205 | 0.2308082000 |

Note that, the true order of $M(n)$ is $n^{2}(\log n)^{-c}(\log \log n)^{-3 / 2}$ [3].

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## References

1] Tom. M. Apostol, Introduction to Analytic Number Theory, Springer-Verlag, New York, 1976.
[2] Paul Erdös, An asymptotic inequality in the theory of numbers (Russian), Vestnik Leningrad Univ. Mat. Mekh. i Astr., 13, (1960), 41-49.
[3] Kevin Ford, personal comments.
[4] Aleksandar Ivic, The Riemann Zeta Function, John Wiley \& sons, 1985.
[5] Jean-Louis Nicolas, On highly composite numbers. Ramanujan revisited (Urbana-Champaign, Ill., 1987), 215-244, Academic Press, Boston, MA, 1988.
[6] László Babai, Carl Pomerance, and Péter Vértesi, The Mathematics of Paul Erdös, Notices of Amer. Math. Soc., vol. 45, no. 1, 1998, 19-23.
[7] Guy Robin, Grandes valeurs de la fonction somme des diviseurs et hypothse de Riemann, $J$. Math. Pures Appl. (9) 63 (1984), no. 2, 187-213.

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[^0]:    1991 Mathematics Subject Classification. 65A05, 03G10, 11S40.
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    ${ }^{1}$ This sequence, has been indexed in "The On-Line Encyclopedia of Integer Sequences" data base with ID A027424. Web page of above data base is: http://www.research.att.com/njas/sequences/index.html

