# ON INTEGRAL VERSION OF ALZER'S INEQUALITY AND MARTINS' INEQUALITY

CHAO-PING CHEN AND FENG QI

Abstract. Let $c > b > a$ and $r$ be real numbers, and let $f$ be a positive, twice differentiable function and satisfy $f'(t) > 0$ and $(\ln f(t))'' \geq 0$ on $(a, +\infty)$. Then

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,c]} f(x)} < \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} < 1 \text{ for all real } r,$$

$$\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} \lesseqgtr \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c \ln f(x)\,\mathrm{d}x\right)} \quad \text{according as } r \gtreqless 0.$$

This solves a recently open problem of B.-N. Guo and F. Qi.

## 1. Introduction

It was shown in [1, 2, 8, 13, 17] that let $n$ be a positive integer, then for $r > 0$,

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} \tag{1}$$

We call the left-hand side of (1) H. Alzer's inequality [1], and the right-hand side of (1) J. S. Martins' inequality [8]. In [3, 14] Alzer's inequality is extended to all real $r$. In [5] it was proved that Martins' inequality is reversed for $r < 0$.

F. Qi and B.-N. Guo [10, 11] presented an integral version of inequality (1) as follows: Let $b > a > 0$ and $\delta > 0$, then for $r > 0$,

$$\frac{b}{b+\delta} < \left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r\,\mathrm{d}x}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{2}$$

We note that the inequality (4) can be written for $r > 0$ as

$$\frac{b}{b+\delta} < \frac{L_r(a,b)}{L_r(a,b+\delta)} < \frac{I(a,b)}{I(a,b+\delta)}, \tag{3}$$

where $L_r(a,b)$ and $I(a,b)$ are respectively the generalized logarithmic mean and the exponential mean of two positive numbers $a, b$, defined in [6, 15, 16] by, for $a = b$ by $L_r(a,b) = a$ and for $a \neq b$ by

$$L_r(a,b) = \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r \neq -1, 0;$$

2000 *Mathematics Subject Classification.* Primary: 26D10, Secondary: 26D15.

*Key words and phrases.* Alzer's inequality, Martins' inequality, mean.

The authors were supported in part by SF of Henan Innovation Talents at Universities, China.

$$L_{-1}(a,b)=\frac{b-a}{\ln b-\ln a}=L(a,b);$$
$$L_0(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}=I(a,b).$$

$L(a,b)$ is the logarithmic mean of two positive numbers $a,b$. When $a\neq b$, $L_r(a,b)$ is a strictly increasing function of $r$. In particular,

$$\lim_{r\to-\infty}L_r(a,b)=\min\{a,b\},\qquad \lim_{r\to+\infty}L_r(a,b)=\max\{a,b\}.$$

In [4], it was indirectly shown that the function $r\mapsto L_r(a,b)/L_r(a,b+\delta)$ is strictly decreasing with $r\in(-\infty,+\infty)$. This yields that

$$\frac{b}{b+\delta}<\left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta}x^r\,\mathrm{d}x}\right)^{1/r}\quad\text{for all real}\quad r,\tag{4}$$

$$\left(\frac{\frac{1}{b-a}\int_a^b x^r\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta}x^r\,\mathrm{d}x}\right)^{1/r}\lesseqgtr\frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}\quad\text{according as}\quad r\gtreqless 0.\tag{5}$$

In [7], B.-N. Guo and F. Qi ask under which conditions the inequality

$$\frac{\sup_{x\in[a,b]}f(x)}{\sup_{x\in[a,b+\delta]}f(x)}<\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta}f^r(x)\,\mathrm{d}x}\right)^{1/r}<\frac{\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{b+\delta-a}\int_a^{b+\delta}\ln f(x)\,\mathrm{d}x\right)}\tag{6}$$

holds for $b>a>0$, $\delta>0$ and $r>0$.

V. Mascioni [9] found the sufficient conditions on the function $f$, and proved the right-hand inequality of (6) for $r>0$. Motivated by the paper of Mascioni [9], we establish the following

**Theorem.** *Let $c>b>a$ and $r$ be real numbers, and let $f$ be a positive, twice differentiable function and satisfy $f'(t)>0$ and $(\ln f(t))''\geq 0$ on $(a,+\infty)$. Then*

$$\frac{\sup_{x\in[a,b]}f(x)}{\sup_{x\in[a,c]}f(x)}<\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r}<1\ \textit{for all real}\ r,\tag{7}$$

$$\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r}\lesseqgtr\frac{\exp\left(\frac{1}{b-a}\int_a^b\ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c\ln f(x)\,\mathrm{d}x\right)}\quad\textit{according as}\ r\gtreqless 0.\tag{8}$$

*Both bounds in (7) are best possible.*

## 2. Lemmas

**Lemma 1.** *Let the function $f$ be a positive and twice differentiable on $(a,+\infty)$, where $a$ is a given real number, and let*

$$G(t)=\frac{\frac{1}{t-a}\int_a^t f(x)\,\mathrm{d}x}{f(t)},\quad t>a.$$

*Then we have*

(*i*) *If $f'(t)>0$ and $(\ln f(t))''\geq 0$, then the function $G$ is strictly decreasing on $(a,+\infty)$.*

*(ii) If $f'(t) < 0$ and $(\ln f(t))'' \le 0$, then the function $G$ is strictly increasing on $(a, +\infty)$.*

*Proof.* Easy calculation reveals that

$$\begin{aligned}\frac{[(t-a)f(t)]^2 G'(t)}{f(t)+(t-a)f'(t)} &= \frac{(t-a)f^2(t)}{f(t)+(t-a)f'(t)} - \int_a^t f(x) \triangleq H(t),\\ \frac{[f(t)+(t-a)f'(t)]^2 H'(t)}{(t-a)f^3(t)} &= -(t-a)\frac{f''(t)f(t)-[f'(t)]^2}{f^2(t)} - \frac{f'(t)}{f(t)}\\ &= -[(t-a)(\ln(f(t))'' + (\ln f(t))'].\end{aligned}$$

If $(\ln f(t))' > (<)0$ and $(\ln(f(t))'' \ge (\le)0$ for $t > a$, then $H'(t) < (>)0$ for $t > a$, and then, $H(t) < (>)H(a) = 0$ and $G'(t) < (>)0$ for $t > a$. The proof is complete. $\square$

**Lemma 2** ([12]). *If $\mathcal{F}(t)$ is a strictly increasing (decreasing) integrable function on an interval $I \subseteq \mathbb{R}$, then the arithmetic mean $\mathcal{G}(r,s)$ of function $\mathcal{F}(t)$,*

$$\mathcal{G}(r,s) = \begin{cases} \dfrac{1}{s-r}\displaystyle\int_r^s \mathcal{F}(t)\,\mathrm{d}t, & r \ne s,\\ \mathcal{F}(r), & r = s,\end{cases}$$

*is also strictly increasing (decreasing) with both $r$ and $s$ on $I$.*

## 3. Proof of Theorem

For $r = 0$, (7) can be interpreted as

$$\frac{f(b)}{f(c)} < \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c \ln f(x)\,\mathrm{d}x\right)} < 1. \tag{9}$$

Define for $t > a$,

$$P(t) = \frac{\exp\left(\frac{1}{t-a}\int_a^t \ln f(x)\,\mathrm{d}x\right)}{f(t)}.$$

A simple computation yields

$$\begin{aligned}(t-a)^2\frac{P'(t)}{P(t)} &= (t-a)\ln f(t) - \int_a^t \ln f(x)\,\mathrm{d}x - (t-a)(\ln f(t))' \triangleq Q(t),\\ Q'(t) &= -(t-a)\left[(\ln f(t))' + (t-a)(\ln(f(t))''\right] < 0.\end{aligned}$$

Hence, we have $Q(t) < Q(a) = 0$ and $P'(t) < 0$ for $t > a$. This means the left-hand inequality of (9) holds for $c > b > a$. By Lemma 2, the right-hand inequality of (9) holds clearly.

For $r \ne 0$, (7) is equivalent to

$$\frac{f^r(b)}{f^r(c)} \lessgtr \frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x} \lessgtr 1, \quad \text{according as} \quad r \gtrless 0. \tag{10}$$

Define for $t > a$,

$$G_r(t) = \frac{\frac{1}{t-a}\int_a^t f^r(x)\,\mathrm{d}x}{f^r(t)}.$$

It is easy to see that

$$(\ln f^r(t))'_t \gtrless 0 \quad \text{and} \quad (\ln(f^r(t))''_t \gtreqqless 0, \quad \text{according as} \quad r \gtrless 0. \tag{11}$$

By Lemma 1, the function $t \mapsto G_r(t)$ strictly $\begin{matrix}\text{decreases}\\\text{increases}\end{matrix}$ with respect to $t \in (a, +\infty)$ according as $r \gtrless 0$. This produces the left-hand inequality of (10). By Lemma 2, the right-hand inequality of (10) holds clearly.

Both bounds in (7) are best possible because of

$$\lim_{r\to+\infty}\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} = \frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,c]} f(x)},$$
$$\lim_{r\to-\infty}\left(\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x}\right)^{1/r} = \frac{\inf_{x\in[a,b]} f(x)}{\inf_{x\in[a,c]} f(x)} = 1.$$

The inequality (8) is equivalent to

$$\frac{\frac{1}{b-a}\int_a^b f^r(x)\,\mathrm{d}x}{\frac{1}{c-a}\int_a^c f^r(x)\,\mathrm{d}x} < \frac{\exp\left(\frac{1}{b-a}\int_a^b \ln f^r(x)\,\mathrm{d}x\right)}{\exp\left(\frac{1}{c-a}\int_a^c \ln f^r(x)\,\mathrm{d}x\right)} \quad \text{for } r \neq 0. \tag{12}$$

Define for $t > a$,

$$F_r(t) = \frac{\frac{1}{t-a}\int_a^t f^r(x)\,\mathrm{d}x}{\exp\left(\frac{1}{t-a}\int_a^t \ln f^r(x)\,\mathrm{d}x\right)}.$$

It is easy to see from the proof of Theorem 1 of [9] that if $f'(t) > 0$ and $(\ln f(t))'' \geq 0$, then the function $F_1$ is strictly increasing on $(a, +\infty)$; If $f'(t) < 0$ and $(\ln f(t))'' \leq 0$, then the function $F_1$ is strictly decreasing on $(a, +\infty)$. Applying this result, together with (11), we obviously imply the function $t \mapsto F_r(t)$ strictly $\begin{matrix}\text{increases}\\\text{decreases}\end{matrix}$ with respect to $t \in (a, +\infty)$ according as $r \gtrless 0$. This produces (12). The proof is complete.

## References

[1] H. Alzer, *On an inequality of H. Minc and L. Sathre*, J. Math. Anal. Appl. **179** (1993), 396–402.

[2] Ch.-P. Chen and F. Qi, *Notes on proofs of Alzer's inequality*, O. Math. Mag. **11** (2003), no. 1, 29–33.

[3] Ch.-P. Chen and F. Qi, *The inequality of Alzer for negative powers*, O. Math. Mag. **11** (2003), no. 2, 442–445.

[4] Ch.-P. Chen and F. Qi, *Monotonicity properties for generalized logarithmic means*, Australian Journal of Mathematical Analysis and Applications **1** (2004), no.2, Article 2. Available online at http://ajmaa.org/

[5] Ch.-P. Chen, F. Qi and S. S. Dragomir, *Reverse of Martins' inequality*, Australian Journal of Mathematical Analysis and Applications, **2** (2005), no.1, Article 2. Available online at http://ajmaa.org/

[6] L. Galvani, *Dei limiti a cui tendono alcune media*, Boll. Un. Mat. Ital. **6** (1927), 173–179.

[7] B.-N. Guo and F. Qi, *An algebraic inequality, II*, RGMIA Res. Rep. Coll. **4** (2001), no.1, Article 8, 55–61. Available online at http://rgmia.vu.edu.au/v4n1.html.

[8] J. S. Martins, *Arithmetic and geometric means, an application to Lorentz sequence spaces*, Math. Nachr. **139** (1988), 281–288.

[9] V. Mascioni, *A sufficient condition for the integral version of Martins' inequality*, J. Inequal. Pure Appl. Math. **5** (2004), no. 2, Artitle 32. Available online at http://jipam.vu.edu.au/issues.php?op=viewissue&issue=78.

[10] F. Qi, *An algebraic inequality*, J. Inequal. Pure Appl. Math. **2** (2001), no. 1, Artitle 13. Available online at http://jipam.vu.edu.au/v2n1/006_00.html.

[11] F. Qi and B.-N. Guo, *An inequality between ratio of the extended logarithmic means and ratio of the exponential means*, Taiwanese J. Math. **7** (2003), no. 2, 229-237.

[12] F. Qi, *Logarithmic convexity of extended mean values*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1787–1796. Available online at http://www.ams.org/journal-getitem?pii=S0002-9939-01-06275-X.

[13] J. Sándor, *On an inequality of Alze*, J. Math. Anal. Appl. **192** (1995), 1034–1035.

[14] J. Sándor, *On an inequality of Alzer, II*, O. Math. Mag. **11** (2003), no. 2, 554–555.

[15] K. B. Stolarsky, *Generalizations of the logarithmic mean*, Math. Mag. **48** (1975), 87–92.

[16] K. B. Stolarsky, *The power and generalized logarithmic means*, Amer. Math. Monthly, **87** (1980), 545–548.

[17] J. S. Ume, *An elementary proof of H. Alzer's inequality*, Math. Japon. **44** (1996), no. 3, 521–522.

(Ch.-P. Chen) Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China

*E-mail address*: chenchaoping@hpu.edu.cn, chenchaoping@sohu.com

(F. Qi) Department of Applied Mathematics and Informatics, Research Institute of Applied Mathematics, Henan Polytechnic University, Jiaozuo City, Henan 454010, China

*E-mail address*: qifeng@hpu.edu.cn, fengqi618@member.ams.org

*URL*: http://rgmia.vu.edu.au/qi.html, http://dami.hpu.edu.cn/qifeng.html