Primes in the Quadratic Intervals

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Abstract

In this note, we prove that for $n \geq 30$, there exists at lest a prime number in the interval $\Big(n^2, \big(n+f(n)\big)^2\Big]$ in which $f(n)$ is a function with the order of $O(\frac{n}{\ln^2 n})$, and we count the number of primes in this interval. By using the result of this counting, we estimate the probability that a prime exists in the interval $\big(n^2, (n+1)^2\big)$. Also, we show that there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, the interval $\Big[\big(n-g(n)\big)^2, n^2\Big)$, in which $g(n) = O(n^{\frac{1}{20}})$.

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1 Introduction

It seems that the story of studying intervals containing primes becomes to Bertrand, where he conjectured that for all $n \in \mathbb{N}$, $\mathbb{P} \cap (n, 2n] \neq \phi$. Note that, as usual, we let $\mathbb{P}$ be the set of all prime numbers. In this area, many activities have done until now, for example two of more recent of them are as follows:

▶ In 1999, P. Dusart [2] showed that for every $x \geq 3275$, we have

$$\mathbb{P} \cap \left(x, x\left(1 + \frac{1}{2\ln^2 x}\right)\right] \neq \phi.$$

▶ In 2001, Baker, Harman and Pintz [1] proved that there exists real x_0 such that for all $x > x_0$ the interval $[x - x^{0.525}, x]$ contains a prime.

Ok! What we are going to do? Before answering, we introduce some notations: We denote the interval $\left(n^2, \left(n + f(n)\right)^2\right]$, by $I^2_{+f(n)}$. So, we have $I^2_{+1} = \left(n^2, (n+1)^2\right]$. Also, by $I^2_{-g(n)}$ we will mean the interval $\left[\left(n - g(n)\right)^2, n^2\right)$.
In this note, we study existence of primes in the intervals $I^2_{+f(n)}$ and $I^2_{-g(n)}$. Also, we consider the following open problem:

$$\mathbb{P} \cap I^2_1 \neq \phi \qquad (n \in \mathbb{N}).$$

We study the probabilistic existence of primes in the interval I^2_{+1}. For estimate above probability, we will need the following sharp bounds for the function $\pi(x) = \#\mathbb{P} \cap [1, x]$:

$$L(x) = \frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x}\right) \leq \pi(x) \qquad (x \geq 32299),$$

and

$$\pi(x) \leq U(x) = \frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{2.51}{\ln^2 x}\right) \qquad (x \geq 355991).$$

These results are due to P. Dusart [2].

2 Existence of Primes in $I^2_{+f(n)}$ and $I^2_{-g(n)}$

2.1 Study of $I^2_{+f(n)}$

In this part, we prove the following theorem:

Theorem 1 *If $n \geq 30$, then we have*

$$\mathbb{P} \cap I^2_{+f(n)} \neq \phi,$$

in which,

$$f(n) = \frac{n}{8\ln^2 n + \sqrt{8}\ln n\sqrt{1 + 8\ln^2 n}} = O\left(\frac{n}{\ln^2 n}\right).$$

For prove this theorem, we need the following lemma which help us to change the form of intervals.

Lemma 1 *Suppose* $a, b > 0$. *We have,*

$$a^2 + b^2 = \left(a + \frac{b^2}{a + \sqrt{a^2 + b^2}}\right)^2.$$

Proof. Let $a^2 + b^2 = (a + \frac{b}{M})^2$, with $M > 0$. Solving this quadratic equation with respect to M, we have

$$M = \frac{a}{b} + \sqrt{1 + \left(\frac{a}{b}\right)^2},$$

and this yields the result. □

Now, proof of theorem 1:

Proof. For $30 \leq n \leq 57$ we can check the result by computer. For $n \geq 58 = \left\lceil \sqrt{3275} \right\rceil$, according to P. Dusart [2], there exists at least a prime p such that

$$n^2 < p \leq n^2 \left(1 + \frac{1}{2\ln^2(n^2)}\right).$$

Now, by lemma 1, we have

$$n^2 \left(1 + \frac{1}{2\ln^2(n^2)}\right) = n^2 + \left(\frac{n}{\sqrt{8}\ln n}\right)^2 = \big(n + f(n)\big)^2,$$

such that,

$$f(n) = \frac{n}{8\ln^2 n + \sqrt{8}\ln n \sqrt{1 + 8\ln^2 n}} = O\Big(\frac{n}{\ln^2 n}\Big).$$

This completes the proof. □

Note 1 *The truth of theorem 1, holds also for*

$$n = 2, 4, 6, 9, 10, 14, 15, 16, 17, 20, 21, 22, 24, 25, 26, 27, 28.$$

2.2 Study of $I^2_{-g(n)}$

Theorem 2 *There exists* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$ *we have*

$$\mathbb{P} \cap I^2_{-g(n)} \neq \phi,$$

in which,

$$g(n) = n - \sqrt{n^2 - n^{1.05}} = O(n^{\frac{1}{20}}).$$

Proof. We know that [1], there exists real x_0 such that for all $x > x_0$ we have

$$\mathbb{P} \cap [x - x^{0.525}, x] \neq \phi.$$

Let $x = n^2$. So, for $n > n_0 = \lceil \sqrt{x_0} \rceil$ we have $\mathbb{P} \cap [n^2 - n^{1.05}, n^2] \neq \phi$. Now, let $n^2 - n^{1.05} = \big(n - g(n)\big)^2$. This completes the proof. □

Note 2 *We can see that* $g(n) \sim \frac{1}{2} n^{\frac{1}{20}}$. *Beside, we have the following bounds for* $g(n)$:

$$\frac{1}{2} n^{\frac{1}{20}} < g(n) < n^{\frac{1}{20}}.$$

which hold for all $n > 1$. *Also, for every* $\epsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$ *we have*

$$\frac{1}{2} n^{\frac{1}{20}} < g(n) < \frac{1}{2 - \epsilon} n^{\frac{1}{20}}.$$

3 How Many Primes?

3.1 Counting Primes in $I^2_{+f(n)}$

Let $F(n)$ is the number of primes in $I^2_{+f(n)}$; i.e.

$$F(n) = \#\mathbb{P} \cap I^2_{f(n)} = \pi\Big(\big(n + f(n)\big)^2\Big) - \pi(n^2).$$

By using Prime Number Theorem we can see that:

$$F(n) \sim \frac{1}{2}\Big(\frac{\big(n + f(n)\big)^2}{\ln\big(n + f(n)\big)} - \frac{n^2}{\ln n}\Big) \qquad (n \to \infty).$$

Beside, by considering asymptotic behavior of $f(n)$ we yield:

$$F(n) \sim \frac{1}{32}\Big(\frac{\big(n + \frac{n}{\ln^2 n}\big)^2}{\ln\big(n + \frac{n}{\ln^2 n}\big)} - \frac{n^2}{\ln n}\Big) \qquad (n \to \infty).$$

Theorem 1, asserts that for $n \geq 58$ we have $F(n) > 0$. By using P. Dusart's bounds on $\pi(x)$ we can yield the following bounds for $F(n)$:

$$L\Big(\big(n + f(n)\big)^2\Big) - U(n^2) \leq F(n) \leq U\Big(\big(n + f(n)\big)^2\Big) - L(n^2),$$

which holds for all $n \geq \max\Big\{\Big\lceil\sqrt{355991}\Big\rceil, \Big\lceil\sqrt{32299}\Big\rceil\Big\} = 597$.

But, since $\lim_{n\to\infty} L\Big(\big(n + f(n)\big)^2\Big) - U(n^2) = -\infty$, we replace above lower bound by trivial one, 1. So, we have

$$1 \leq F(n) \leq U\Big(\big(n + f(n)\big)^2\Big) - L(n^2) \qquad (n \geq 597).$$

About sharp lower and upper bounds for $F(n)$, we have the following conjecture which supported by some computational evidences:

Conjecture 1 *For every $\epsilon > 0$, there exists $n_0(\epsilon) \in \mathbb{N}$, such that for every $n \geq n_0$ we have*

$$\frac{1}{32-\epsilon}\left(\frac{\left(n+\frac{n}{\ln^2 n}\right)^2}{\ln\left(n+\frac{n}{\ln^2 n}\right)} - \frac{n^2}{\ln n}\right) \leq F(n) \leq \frac{1}{32+\epsilon}\left(\frac{\left(n+\frac{n}{\ln^2 n}\right)^2}{\ln\left(n+\frac{n}{\ln^2 n}\right)} - \frac{n^2}{\ln n}\right).$$

3.2 Probabilistic Existence of Primes in I_1^2

Estimating of $F(n)$ can be useful in the following theorem.

Theorem 3 *The probability that the interval I_1^2 contains a prime is*

$$1 - \left(\frac{\big(n+f(n)\big)^2 - (n+1)^2}{\big(n+f(n)\big)^2 - n^2}\right)^{F(n)}.$$

Note that $f(n)$ and $F(n)$ are defined in above.

Proof. There are $F(n)$ primes between n^2 and $\big(n+f(n)\big)^2$. Since $n^2 < (n+1)^2 < \big(n+f(n)\big)^2$, and because these primes distributed randomly, the probability that all of these primes are between $(n+1)^2$ and $\big(n+f(n)\big)^2$ is equal to

$$\left(\frac{\big(n+f(n)\big)^2 - (n+1)^2}{\big(n+f(n)\big)^2 - n^2}\right)^{F(n)},$$

and this yields the result. □

References

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