Counting and Computing by e

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Abstract

In this research and review paper, using some inequalities and relations involving e, we count the number of paths and cycles in complete graphs the number of derangements. Connection by e yields some nice formulas for the number of derangements, such as $D_n = \lfloor \frac{n!+1}{e} \rfloor$ and $D_n = \lfloor (e+e^{-1})n! \rfloor - \lfloor en! \rfloor$, and using these relations allow us to compute some incomplete gamma functions and hypergeometric summations; these connections are hidden in the heart of a nice polynomial that we call it derangement function and a simple ordinary differential equation concerning it.

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1 Introduction and Motivation

The initial motivation of writing this paper is hidden in the following combinatorial relations

$$D_n = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} \quad (n \geq 1),$$

and

$$w_n = (n-2)!\sum_{i=0}^{n-2}\frac{1}{i!} \quad (n \geq 2),$$

where D_n is the number of derangements (permutations with no fixed point) of n distinct objects (see [3,4]), and w_n is the number of distinct paths between any pair of vertices in a complete graph on n vertices (see [4,5]). Considering $e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!}$ for $x=-1$ and $x=1$ respectively, we can get the following approximate formulas

$$D_n \approx \frac{n!}{e} \quad \text{and} \quad w_n \approx e(n-2)!.$$

On the other hand, it is well-known that

$$D_n = \left\| \frac{n!}{e} \right\| \quad (\|x\| \text{ denotes the nearest integer to } x),$$

and this can rewritten as follows

$$D_n = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor \quad (\lfloor x \rfloor \text{ denotes the floor of } x).$$

In this paper, we study these kind of combinatorial formulas concerning partial sum of Taylor expansion of e^x for some special x's. Then we use obtained results to calculate some integrals related to the incomplete gamma function (see [1]):

$$\Gamma(a,z) = \int_z^{\infty} e^{-t}t^{a-1}dt \quad (Re(a) > 0).$$

Also, we use them to calculate some hypergeometric summations (see[7]):

$${}_pF_q\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{array}; x\right] = \sum_{k\geq 0} t_k x^k,$$

where

$$\frac{t_{k+1}}{t_k} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x.$$

To do these, we need some properties involving the number e, that we study them in the next section.

2 Two Interesting Formulas Involving e

The obtained relations involving e in this section all are hidden in the proofs of irrationality of it. The first one is from Rudin's analysis [8], as follows

Theorem 1 *For every positive integer $n \geq 1$, we have*

$$\sum_{i=0}^{n} \frac{n!}{i!} = \lfloor en! \rfloor. \tag{1}$$

Proof: Since $n \geq 1$, we have

$$0 < \sum_{i=0}^{\infty} \frac{1}{i!} - \sum_{i=0}^{n} \frac{1}{i!} < \frac{1}{(n+1)!} \sum_{i=0}^{\infty} \frac{1}{(n+1)^i} = \frac{1}{n!n} \leq 1,$$

and so,

$$0 < en! - \sum_{i=0}^{n} \frac{n!}{i!} \leq 1.$$

Since $\sum_{i=0}^{n} \frac{n!}{i!}$ is an integer, the irrationality of e and the truth of the theorem both follow. $\square$

The idea of the next result is hidden in Apostol's analysis [2], where he proved the irrationality of e.

Theorem 2 *Suppose $n \geq 1$ is an integer, we have*

$$n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \begin{cases} \lfloor \frac{n!}{e} + \lambda_1 \rfloor, & n \text{ is odd}, \lambda_1 \in [0, \frac{1}{2}]; \\ \lfloor \frac{n!}{e} + \lambda_2 \rfloor, & n \text{ is even}, \lambda_2 \in [\frac{1}{3}, 1]. \end{cases}$$

Proof: Suppose $k \geq 1$ be an integer, we have

$$0 < \frac{1}{e} - \sum_{i=0}^{2k-1} \frac{(-1)^i}{i!} < \frac{1}{(2k)!},$$

so, for every λ_1 and λ_2, we have

$$\lambda_1 < \frac{(2k-1)!}{e} + \lambda_1 - \sum_{i=0}^{2k-1} \frac{(-1)^i (2k-1)!}{i!} < \lambda_1 + \frac{1}{2},$$

and

$$\lambda_2 - 1 < \frac{(2k)!}{e} + \lambda_2 - \sum_{i=0}^{2k} \frac{(-1)^i (2k)!}{i!} < \lambda_2.$$

If $0 \leq \lambda_1 \leq \frac{1}{2}$, then

$$\sum_{i=0}^{2k-1} \frac{(-1)^i (2k-1)!}{i!} = \left\lfloor \frac{(2k-1)!}{e} + \lambda_1 \right\rfloor .$$

Also, if $\lambda_2 \geq \frac{1}{3}$, we obtain

$$0 < \frac{(2k)!}{e} + \lambda_2 - \sum_{i=0}^{2k} \frac{(-1)^i (2k)!}{i!},$$

and with $\frac{1}{3} \leq \lambda_2 \leq 1$, we yield

$$\sum_{i=0}^{2k} \frac{(-1)^i (2k)!}{i!} = \left\lfloor \frac{(2k)!}{e} + \lambda_2 \right\rfloor .$$

This completes the proof. $\square$

Note and Problem 1 *From Theorems 1 and 2, we obtain*

$$\begin{aligned} n! \sum_{i=0}^{n} \frac{1^i}{i!} &= \lfloor en! \rfloor \\ n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} &= \lfloor e^{-1} n! + \lambda \rfloor, \qquad \lambda \in \left[\frac{1}{3}, \frac{1}{2} \right]. \end{aligned}$$

Now, suppose x is a real number, is there the set $\Lambda_x \subseteq \mathbb{R}$, such that

$$n! \sum_{i=0}^{n} \frac{x^i}{i!} = \lfloor e^x n! + \lambda \rfloor, \qquad \lambda \in \Lambda_x ?$$

for example we know $0 \in \Lambda_1$ and $[\frac{1}{3}, \frac{1}{2}] \subseteq \Lambda_{-1}$.

Now, we use obtained results, to study of paths and cycles in complete graphs and then in computing the number of derangements.

3 Paths and Cycles in Complete Graphs

We remember that a path of length i between two vertices u and v is a sequence of distinct vertices as follows (see [3]),

$$u\overbrace{\bigcirc\bigcirc\bigcirc\bigcirc\cdots\bigcirc\bigcirc\bigcirc\bigcirc}^{(i-1)-\text{distindt vertices}}v.$$

If $u = v$, then we have a cycle through u.

Theorem 3 *The number of paths between every pair of vertices in a complete graph on n vertices with $n > 2$, is*

$$w_n = \lfloor e(n-2)! \rfloor,$$

and sum of their lengths is

$$L_w(n) = 1 + (n-2)\lfloor e(n-2)! \rfloor.$$

Proof: Suppose u and v are two vertices in a complete graph on n vertices. For counting number of paths between u and v, we classify them according their length; the number of paths of length i is

$$w(i) = \frac{(n-2)!}{(n-1-i)!},$$

and since $1 \leq i \leq n-1$, applying Theorem 1 with $n > 2$, we obtain

$$w_n = \sum_{i=1}^{n-1} w(i) = \sum_{i=1}^{n-1} \frac{(n-2)!}{(n-1-i)!} = \sum_{i=0}^{n-2} \frac{(n-2)!}{i!} = \lfloor e(n-2)! \rfloor.$$

Also,

$$L_w(n) = \sum_{i=1}^{n-1} iw(i) = \sum_{i=1}^{n-1} \frac{i(n-2)!}{(n-1-i)!} = 1+(n-2)\sum_{i=0}^{n-2} \frac{(n-2)!}{i!} = 1+(n-2)\lfloor e(n-2)! \rfloor.$$

This completes the proof. □

Corollary 1 *The average of the length of paths in a complete graph on n vertices is*

$$\frac{L_w(n)}{w_n} = n - 2 + O\left(\frac{1}{(n-2)!}\right),$$

and the maximum number of paths occurs around this average.

Proof: According to Theorem 3, we have

$$\frac{L_w(n)}{w_n} = n - 2 + \frac{1}{w_n} = n - 2 + O\left(\frac{1}{(n-2)!}\right).$$

For the next assertion we have two proofs:
Method 1. As we mentioned in the proof of Theorem 3, the number of paths of length i is

$$w(i) = \frac{(n-2)!}{(n-i-1)!} \quad (1 \leq i \leq n-1),$$

and $w(n-1) = w(n-2) = (n-2)!$; beside, for $1 \leq i \leq n-3$ we have $1 \leq w(i) \leq \frac{(n-2)!}{2}$, and this yields that the maximum of $w(i)$ occurs at $i = n+2$ and $i = n+1$; i.e, around the average.

Method 2. Expand $w(i)$ as a real function with putting $w(x) = \frac{(n-2)!}{\Gamma(n-x)}$, in which $x \in [1, n-1]$ is real. Now, $\frac{d}{dx}w(x) = \frac{(n-2)!\Gamma'(n-x)}{\Gamma(n-x)^2}$ and so, the maximum of $w(x)$ for $1 \leq x \leq n-1$, occurs exactly at $x_{\max} = n - \gamma_0$ in which $\gamma_0 \simeq 1.461632145$ is the unique zero of $\Gamma'(|x|) = 0$ and therefore $\frac{3}{2} - \gamma_0 < x_{\max} - \frac{L_w(n)}{w_n} < 2 - \gamma_0$; so, maximum number of paths occurs at the near of the average. $\square$

The proof of the next theorem is similar to the proof Theorem 3.

Theorem 4 *The number of cycles through every vertex in a complete graph on n vertices with $n > 2$, is*

$$c_n = \lfloor e(n-1)! \rfloor - n,$$

and sum of their lengths is

$$L_c(n) = \lfloor en! \rfloor - \lfloor e(n-1)! \rfloor - 2n + 1.$$

Note and Problem 2 *We know that a complete graph on n vertices is regular of degree $n-1$. Can we have similar formulas about k-regular graphs?*

4 Computing the Number of Derangements

Theorem 2, immediately yields the following family of formulas for D_n, which is an extension of $D_n = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$;

Theorem 5 *For every positive integer $n \geq 1$ and $\lambda \in [\frac{1}{3}, \frac{1}{2}]$, we have*

$$D_n = \left\lfloor \frac{n!}{e} + \lambda \right\rfloor.$$

Proof: We give two proofs:
Method 1. Using theorem 2, we have

$$D_n = \begin{cases} \lfloor \frac{n!}{e} + \lambda_1 \rfloor, & n \text{ is odd}, \lambda_1 \in [0, \frac{1}{2}]; \\ \lfloor \frac{n!}{e} + \lambda_2 \rfloor, & n \text{ is even}, \lambda_2 \in [\frac{1}{3}, 1], \end{cases}$$

and this yields the result.
Method 2. We know

$$D_n = n! \left(1 - \frac{1}{1!} + \cdots + \frac{(-1)^n}{n!}\right) = \frac{n!}{e} + (-1)^n \left(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \cdots \right),$$

so, for every $n \in \mathbb{N}$ we have

$$\left| D_n - \frac{n!}{e} \right| < \frac{1}{n+1}.$$

If n is even, $D_n > \frac{n!}{e}$ and $D_n = \lfloor \frac{n!}{e} + \lambda \rfloor$ provided $\frac{1}{n+1} \leq \lambda \leq 1$. If n is odd, $D_n < \frac{n!}{e}$ and $D_n = \lfloor \frac{n!}{e} + \lambda \rfloor$ provided $0 < \frac{1}{n+1} + \lambda \leq 1$. So we require $\frac{1}{3} \leq \lambda \leq 1$ and $0 \leq \lambda \leq \frac{1}{2}$. This completes the second proof. $\square$

Corollary 2 *For every positive integer $n \geq 1$, we have*

$$D_n = \left\lfloor \frac{n!+1}{e} \right\rfloor. \tag{2}$$

Proof: Use Theorem 5 with $\lambda = \frac{1}{e}$. $\square$

Now, consider the idea of proving the relation $D_n = \|\frac{n!}{e}\|$, which is the following trivial inequality

$$\left|\frac{n!}{e} - D_n\right| \leq \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots,$$

and let $M(n)$ denote the right side of above inequality. We have

$$M(n) < \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \cdots = \frac{1}{n},$$

and therefore,

$$D_n = \left\lfloor \frac{n!}{e} + \frac{1}{n} \right\rfloor \qquad (n \geq 2). \tag{3}$$

Also, we can get a better bound for $M(n)$ as follows

$$M(n) < \frac{1}{n+1}\left(1 + \frac{1}{(n+2)} + \frac{1}{(n+2)^2} + \cdots\right) = \frac{n+2}{(n+1)^2},$$

and similarly,

$$D_n = \left\lfloor \frac{n!}{e} + \frac{n+2}{(n+1)^2} \right\rfloor \qquad (n \geq 2). \tag{4}$$

The above idea is extensible and several approximations of $M(n)$ lead us to other families of formulas for D_n;

Theorem 6 *Suppose m is an integer and $m \geq 3$. The number of derangements of n distinct objects with $n \geq 2$ is*

$$D_n = \left\lfloor \left(\frac{\lfloor e(n+m-2)! \rfloor}{(n+m-2)!} + \frac{n+m}{(n+m-1)(n+m-1)!} + e^{-1} \right) n! \right\rfloor - \lfloor en! \rfloor. \tag{5}$$

Proof: For $m \geq 3$ we have

$$\left|\frac{n!}{e} - D_n\right| < \frac{1}{n+1}\left(1 + \frac{1}{n+2}(\cdots 1 + \frac{1}{n+m-1}(\frac{n+m}{n+m-1})\cdots)\right).$$

Let $M_m(n)$ denote the right side of the above inequality; we have

$$M_m(n)\prod_{i=1}^{m-1}(n+i) = \frac{n+m}{n+m-1} + \sum_{j=2}^{m-1}\prod_{i=j}^{m-1}(n+i),$$

and dividing by $\prod_{i=1}^{m-1}(n+i)$ we obtain

$$M_m(n) = n!\left(\frac{n+m}{(n+m-1)(n+m-1)!} + \sum_{i=n+1}^{n+m-2}\frac{1}{i!}\right).$$

Therefore,

$$D_n = \left\lfloor \frac{n!}{e} + n!\left(\frac{n+m}{(n+m-1)(n+m-1)!} + \sum_{i=n+1}^{n+m-2}\frac{1}{i!}\right)\right\rfloor,$$

and reforming this by using $\sum\limits_{i=n+1}^{n+m-2}\frac{1}{i!} = \sum\limits_{i=0}^{n+m-2}\frac{1}{i!} - \sum\limits_{i=0}^{n}\frac{1}{i!}$, completes the proof. □

Corollary 3 *For $n \geq 2$, we have*

$$D_n = \lfloor (e+e^{-1})n! \rfloor - \lfloor en! \rfloor. \tag{6}$$

Proof: We give two proofs:
Method 1. Because (5) holds for all $m \geq 3$, we have

$$\begin{aligned} D_n &= \lim_{m\to\infty}\left\lfloor \left(\frac{\lfloor e(n+m-2)! \rfloor}{(n+m-2)!} + \frac{n+m}{(n+m-1)(n+m-1)!} + e^{-1}\right) n! \right\rfloor - \lfloor en! \rfloor \\ &= \lfloor (e+e^{-1})n! \rfloor - \lfloor en! \rfloor. \end{aligned}$$

Method 2. By using (1), for every $n \geq 1$ we obtain

$$M(n) = n!(e - \sum_{i=0}^{n}\frac{1}{i!}) = en! - \lfloor en! \rfloor = \{en!\} \quad (\{x\} \text{ denotes the fractional part of } x),$$

and the proof follows. □

Now,

$$\lim_{m\to\infty} M_m(n) = M(n),$$

and if we put $M_1(n) = \frac{1}{n}$ and $M_2(n) = \frac{n+2}{(n+1)^2}$ (see formulas (3) and (4)), then

$$M_{m+1}(n) < M_m(n) \quad (n \geq 1).$$

Now, we find bounds sharper than $\{en!\}$ for $e^{-1}n! - D_n$ and consequently another family of formulas for D_n. This family is an extension of (6).

Theorem 7 *Suppose m is an integer and $m \geq 1$. The number of derangements of n distinct objects with $n \geq 2$ is*

$$D_n = \left\lfloor \left(\frac{\{e(n+2m)!\}}{(n+2m)!} + \sum_{i=1}^{m}\frac{n+2i-1}{(n+2i)!} + e^{-1}\right) n! \right\rfloor.$$

Proof: Since $m \geq 1$, we have

$$\frac{e^{-1}n! - D_n}{(-1)^{n+1}} = n! \sum_{i=1}^{\infty} \left(\frac{1}{(n+2i-1)!} - \frac{1}{(n+2i)!} \right) < n! \left(\sum_{i=1}^{m} \frac{n+2i-1}{(n+2i)!} + \sum_{i=2m+1}^{\infty} \frac{1}{(n+i)!} \right).$$

Let $N_m(n)$ denote the right member of above inequality. Considering (1), we obtain

$$N_m(n) = n! \sum_{i=1}^{m} \left(\frac{n+2i-1}{(n+2i)!} + \frac{\{e(n+2m)!\}}{(n+2m)!} \right),$$

and for $n \geq 2$, we yield that $D_n = \lfloor e^{-1}n! + N_m(n) \rfloor$. This completes the proof. $\square$

Corollary 4 *For all integers $m, n \geq 1$, we have*

$$N_{m+1}(n) < N_m(n), \quad N_1(n) < \{en!\}.$$

Therefore we have the following chain of bounds for $|\frac{n!}{e} - D_n|$

$$\left| \frac{n!}{e} - D_n \right| < \cdots < N_2(n) < N_1(n) < \{en!\} < \cdots < M_2(n) < M_1(n) < 1 \quad (n \geq 2).$$

In the next section you will see that how searching other formulas for D_n lead us to connections between D_n, incomplete gamma functions and hypergeometric summations.

5 Applications: Derangement Function, Incomplete Gamma Function and Hypergeometric Function

Let's find other formulas for D_n. The computer algebra program MAPLE yields that

$$D_n = (-1)^n \texttt{hypergeom}([1, -n], [\ \ ], 1),$$

and

$$D_n = e^{-1}\Gamma(n+1, -1),$$

where $\texttt{hypergeom}([1, -n], [\ \ ], 1)$ is MAPLE's notation for a hypergeometric function; more generally,

$$\texttt{hypergeom}([a_1 \ \ a_2 \ \ \cdots \ \ a_p], [b_1 \ \ b_2 \ \ \cdots \ \ b_q], x) = {}_pF_q \left[\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix} ; x \right].$$

Now, because we know the value of D_n, we can estimate some summations and integrals. To do this, we define the *derangement function*, a natural generalization of the number of derangements, denoted by $D_n(x)$, for every integer $n \geq 0$ and every real x as follows

$$D_n(x) = \begin{cases} n! \sum_{i=0}^{n} \frac{x^i}{i!}, & x \neq 0; \\ n!, & x = 0. \end{cases}$$

It is easy to obtain the following generalized recursive relations

$$D_n(x) = (x+n)D_{n-1}(x) - x(n-1)D_{n-2}(x) = x^n + nD_{n-1}(x), \quad (D_0(x) = 1, D_1(x) = x+1).$$

The function $D_n(x)$ connects proven results in two previous sections with incomplete gamma functions and hyperbolic summations. Also, these connection has some consequences, which we collected some of them in the next theorem and corollaries.

Theorem 8 *Suppose x is a real number, we have*

$$D_n(x) = x^n \ {}_2F_0 \left[\begin{matrix} 1 & -n \\ - & \end{matrix} ; -\frac{1}{x} \right] \quad (x \neq 0), \tag{7}$$

and

$$D_n(x) = e^x \Gamma(n+1, x). \tag{8}$$

Proof: The function $D_n(x)$ satisfies in the following differential equation

$$D_n(x) - \frac{d}{dx} D_n(x) = x^n, \quad D_n(0) = n!,$$

solving this equation by summations yields (7) and solving it by integrals yields (8). □

Corollary 5 *For every $n \in \mathbb{N}$, we have*

$${}_2F_0 \left[\begin{matrix} 1 & -n \\ - & \end{matrix} ; -1 \right] = \lfloor en! \rfloor,$$

and

$${}_2F_0 \left[\begin{matrix} 1 & -n \\ - & \end{matrix} ; 1 \right] = (-1)^n \left\lfloor \frac{n!+1}{e} \right\rfloor.$$

Proof: Consider (7) with $x = -1$ and $x = 1$ respectively, and use the relations (1) and (2). □

Corollary 6 *For every real $x \neq 0$, we have*

$${}_1F_1 \left[\begin{matrix} n+1 \\ n+2 \end{matrix} ; -x \right] = \frac{(n+1)(n! - e^{-x} D_n(x))}{x^{n+1}}.$$

Proof: Obvious. □

Corollary 7 *For every integer $n \geq 1$ we have*

$$\int_{-1}^{\infty} e^{-t} t^n dt = e \left\lfloor \frac{n!+1}{e} \right\rfloor,$$

$$\int_0^\infty e^{-t}t^n dt = n!,$$

$$\int_1^\infty e^{-t}t^n dt = \frac{\lfloor en! \rfloor}{e},$$

and

$$\int_0^1 e^{-t}t^n dt = \frac{\{en!\}}{e},$$

$$\int_{-1}^0 e^{-t}t^n dt = \begin{cases} -e\{\frac{n!}{e}\} & n \text{ is odd,} \\ e - e\{\frac{n!}{e}\} & n \text{ is even,} \end{cases}$$

$$\int_{-1}^1 e^{-t}t^n dt = e\lfloor (e+e^{-1})n! \rfloor - (e+e^{-1})\lfloor en! \rfloor,$$

Proof: Consider the definition of incomplete gamma function and use relation (8) with $x=-1$, $x=0$ and $x=1$ respectively, and use (1), (5) and (6). □

Note and Problem 3 *Note that $D_n(x)$ is a nice polynomial. Its value for $x=-1$ is D_n, for $x=0$ is the number of permutations of n distinct objects and for $x=1$ is w_{n+2} = the number of distinct paths between every pair of vertices in a complete graph on $n+2$ vertices. Is there any combinatorial meaning for the value of $D_n(x)$ for other values of x?*

Note and Problem 4 *Since, $n! = \Gamma(n+1)$ in which $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, we can generalize the derangement function as follows*

$$D_\lambda(x) = \begin{cases} \int_0^\lambda \frac{x^t dt}{\Gamma(t+1)}, & x \neq 0; \\ \Gamma(\lambda - 1), & x = 0, \end{cases}$$

in which $\lambda > 0$ is a real number. Here our famous e can be replace by $E = \int_0^\infty \frac{dt}{\Gamma(t+1)} \simeq$ 2.266534508. Can we study $D_\lambda(x)$ and yield some new nice results?

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