INTEGRAL CHARACTERIZATIONS FOR EXPONENTIAL STABILITY OF SEMIGROUPS AND EVOLUTION FAMILIES ON BANACH SPACES

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Abstract. Let X be a real or complex Banach space and $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be a strongly continuous and exponentially bounded evolution family on X. Let J be a non-negative functional on the positive cone of the space of all real-valued locally bounded functions on $\mathbb{R}_+ := [0,\infty)$. We suppose that J satisfies some extra-assumptions. Then the family $\mathcal{U}$ is uniformly exponentially stable provided that for every $x \in X$ we have:

$$\sup_{s\geq 0} J(||U(s+\cdot,s)x||) < \infty.$$

This result is connected to the uniform asymptotic stability of the well-posed linear and non-autonomous abstract Cauchy problem

$$\begin{cases} \dot{u}(t) &= A(t)u(t), \quad t \geq s \geq 0, \\ u(s) &= x \quad x \in X. \end{cases}$$

In the autonomous case, i.e. when $U(t,s) = T(t-s)$ for some strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ we obtain the well-known theorems of Datko, Littman, Neerven, Pazy and Rolewicz.

1. Introduction

Let X be a real or complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norm of vectors in X and operators in $\mathcal{L}(X)$ will be denoted by $||\cdot||$. Let $\mathbf{T} := \{T(t)\}_{t\geq 0}$ be a semigroup of operators acting on X, that is, $T(t) \in \mathcal{L}(X)$ for every $t \geq 0$, $T(0) = I$ the identity operator in $\mathcal{L}(X)$ and $T(t+s) = T(t)\circ T(s)$ for every $t \geq 0$ and $s \geq 0$. The semigroup $\mathbf{T}$ is called strongly continuous if for each $x \in X$ the map $t \mapsto T(t)x : [0,\infty) \to X$ is continuous. Every strongly continuous semigroup is locally bounded, that is, there exist $h > 0$ and $M \geq 1$ such that $||T(t)|| \leq M$ for all $t \in [0,h]$. It is easy to see that every locally bounded semigroup is exponentially bounded, that is, there exist $\omega \in \mathbb{R}_+$ and $M \geq 1$ such that

$$||T(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

It is well-known that if $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on a Banach space X and there exists $p \in [1,\infty)$ such that for each $x \in X$ one has

$$\int_0^\infty ||T(t)x||^p dt = M(p,x) < \infty, \tag{1.1}$$

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then $\mathbf{T}$ is exponentially stable, that is, its uniform growth bound

$$\omega_0(\mathbf{T}) := \inf_{t>0} \frac{\ln ||T(t)||}{t},$$

is negative. This result is usually referred to as the Datko-Pazy theorem, see [6, 12]. An important application of the Datko-Pazy theorem can be found in [16]. A quantitative version of this theorem states that if $M(p, x)$ from (1.1) is equal to $C||x||^p$, where C is some positive constant, then $\omega_0(\mathbf{T}) < -\frac{1}{pC}$. See [10] Theorem 3.1.8 for details. An important generalization of the Datko-Pazy theorem was given by S. Rolewicz, [13]. In the autonomous case the Rolewicz theorem reads as follows. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X. If there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) > 0$ for each $t > 0$ and if*

$$\int_0^\infty \phi(||T(t)x||)dt := M_\phi(x) < \infty \text{ for each } x \in X, \tag{1.2}$$

then the semigroup $\mathbf{T}$ is exponentially stable. The same result was obtained independently by Littman [8]. In particular, from Rolewicz's theorem it follows that the Datko-Pazy theorem remains valid for $p \in (0, 1)$. The condition (1.1) indicates that for each $x \in X$ the map $t \mapsto T(t)x$ belongs to $L^p(\mathbb{R}_+)$. Jan van Neerven has shown in [9] that a strongly continuous semigroup $\mathbf{T}$ on X is uniformly exponentially stable if there exists a Banach function space over $\mathbb{R}_+ := [0, \infty)$ with the property that

$$\lim_{t\to\infty} ||1_{[0,t]}||_E = \infty, \tag{1.3}$$

such that

$$||T(\cdot)x|| \in E \text{ for every } x \in X. \tag{1.4}$$

He has also shown that the autonomous variant of the Rolewicz theorem can be derived from his result by taking for E a suitable Orlicz space over $\mathbb{R}_+$. In another paper, [11], Jan van Neerven has come to the same conclusion by replacing either (1.1), (1.2) or (1.4) by the hypothesis that the set of all $x \in X$ for which the following inequality holds

$$J(||T(\cdot)x||) < \infty,$$

is of the second category in X. Here J is a certain lower semi-continuous functional as defined in Theorem 2 from [11]. The proof of this latter result is based on a non-trivial result from operator theory given by V. Müler, see Lemma 1 from [11], for further details. We give here a surprisingly simple proof for a result of the same type, moreover, we do not require the lower semi-continuity of J.

In order to introduce some non-autonomous results of this type we recall the notion of an evolution family.

A family $\mathcal{U} = \{U(t, s)\}_{t\geq s\geq 0}$ of bounded linear operators on a Banach space X is a strongly continuous evolution family if

(1) $U(t, t) = I$ and $U(r, s) = U(t, s)$ for $t \geq r \geq s \geq 0$.
(2) The map $t \mapsto U(t, s)x : [s, \infty) \to X$ is continuous for every $s \geq 0$ and every $x \in X$.

The family $\mathcal{U}$ is exponentially bounded if there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$||U(t, s)|| \leq M_\omega e^{\omega(t-s)} \text{ for } t \geq s \geq 0. \tag{1.5}$$

Then $\omega(\mathcal{U}) := \inf\{\omega \in \mathbb{R} : \text{there is } M_\omega \geq 0 \text{ such that (1.5) holds}\}$ is called the growth bound of $\mathcal{U}$. The family $\mathcal{U}$ is uniformly exponentially stable if its growth bound is negative.

In [1] it is proved that an exponentially bounded evolution family $\mathcal{U}$ is uniformly exponentially stable if there exists a solid space E satisfying (1.3) such that for each $s \geq 0$ and each $x \in X$ the map $||U(s + \cdot, s)x||$ belongs to E and

$$\sup_{s\geq 0} |||U(s + \cdot, s)x|| := K(x) < \infty.$$

The non-autonomous Datko theorem, [7], follows from this by taking $E = L^p(\mathbb{R}_+)$. The theorem of Rolewicz, [14], can be derived as well by taking for E a suitable Orlicz space over $\mathbb{R}_+$, see Theorem 2.10 from [1]. New guidelines about the proof of the Datko theorem can be found in [5] and [15]. In this paper we propose a more natural generalization of the theorems of Datko and Rolewicz which can also be extended to the general non-autonomous case. For some recently obtained autonomous or periodic versions of the above; see [4], [11].

2. A Generalization of the Datko-Pazy Theorem

We begin by stating and proving two lemmas which are useful later.

Lemma 1. *Let $\mathbf{T} = \{T(t) : t \geq 0\}$ be a locally bounded semigroup on a Banach space X. If for each $x \in X$ there exists $t(x) > 0$ such that $T(t(x))x = 0$, then $\mathbf{T}$ is uniformly exponentially stable.*

Proof. It is easy to see that $\mathbf{T}$ is uniformly bounded. Indeed, if not, then there exists a sequence (t_n) of positive real numbers with $t_n \to \infty$ such that $||T(t_n)|| \to \infty$. By the Uniform Boundedness Theorem it follows that there exists $x \in X$ such that $||T(t_n)x|| \to \infty$. This is in contradiction to the hypothesis. Now let $\nu > 0$. The semigroup $\{e^{\nu t}T(t)\}$ verifies the hypothesis of the present Lemma and it is uniformly bounded. Finally, we deduce that $\mathbf{T}$ is uniformly exponentially stable. ∎

Lemma 2. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a locally bounded semigroup such that for each $x \in X$ the map $t \mapsto ||T(t)x||$ is continuous on $(0, \infty)$. If there exist a positive h and $0 < q < 1$ such that for all $x \in X$ there exists $t(x) \in (0, h]$ with*

$$||T(t(x))x|| \leq q||x||, \tag{2.1}$$

then the semigroup $\mathbf{T}$ is uniformly exponentially stable.

Proof. Let $x \in X$ be fixed and $t_1 \in (0, h]$ such that $||T(t_1)x|| \leq q||x||$, then there exists $t_2 \in (0, h]$ such that

$$||T(t_2 + t_1)x|| \leq q||T(t_1)x|| \leq q^2||x||.$$

By mathematical induction it is easy to see that there exists a sequence (t_n), with $0 < t_n \leq h$ such that $||T(s_n)x|| \leq q^n||x||$, where $s_n := t_1 + t_2 + \cdots + t_n$.

If $s_n \to \infty$, then for each $t \in [s_n, s_{n+1}]$ we have that $t < (n + 1)h$ and

$$||T(t)x|| \leq Mq^n||x|| \leq Me^{-\ln(q)}e^{\frac{\ln(q)}{T}t}||x||,$$

that is, $\mathbf{T}$ is exponentially stable.

If the sequence (s_n) is bounded, let $t(x)$ be the limit of (s_n). By the assumption of continuity it follows that $T(t(x)) = 0$ and then application of Lemma 1 completes the proof. ∎

We can now state the main result of this section.

Theorem 1. *Let $\mathcal{M}_{loc}([0,\infty))$ be the space of all real valued locally bounded functions on $\mathbb{R}_+ = [0,\infty)$ endowed with the topology of uniform convergence on bounded sets and $\mathcal{M}^+_{loc}(\mathbb{R}_+)$ its positive cone.*

Let $J : \mathcal{M}^+_{loc}(\mathbb{R}_+) \to [0,\infty]$ be a map with the following properties:

1. *J is nondecreasing.*

2. *For each positive real number ρ,*

$$\lim_{t\to\infty} J(\rho \cdot 1_{[0,t]}) = \infty.$$

If $\mathbf{T}$ is a semigroup on a Banach space X as in Lemma 2 such that

$$\sup_{||x||\le 1} J(||T(\cdot)x||) := K_J < \infty, \tag{2.2}$$

then $\mathbf{T}$ is exponentially stable.

Proof. Suppose that $\mathbf{T}$ is not exponentially stable. For all $h > 0$ and all $0 < q < 1$ then there exists $x_0 \in X$ of norm one such that

$$||T(t)x_0|| > q \text{ for every } t \in [0,h],$$

as proved in Lemma 2. It follows then that

$$K_J \ge J(||T(\cdot)x_0||) \ge J(q \cdot 1_{[0,h]})$$

which contradicts (2.2). ∎

Corollary 1. *Let $\mathbf{T} = \{T(t)\}_{t\ge 0}$ be a semigroup on a Banach space X as in Lemma 2 and $1 \le p < \infty$. If (1.1) holds for all $x \in X$ then the semigroup $\mathbf{T}$ is exponentially stable.*

Proof. For each fixed positive h consider the bounded linear operator

$$x \mapsto T_h x : X \to L^p(\mathbb{R}_+, X)$$

defined by

$$(T_h x)(t) = \begin{cases} T(t)x, & \text{if} \quad 0 \le t \le h \\ 0, & \text{if} \quad t > h. \end{cases}$$

For each $x \in X$ we have:

$$||T_h x||_{L^p(\mathbb{R}_+,X)} = \left(\int_0^h ||T(t)x||^p dt \right)^{\frac{1}{p}} \le M(p,x)^{\frac{1}{p}}.$$

From the Uniform Boundedness Theorem it follows that there exists a positive constant C_p such that

$$||T_h x||_{L^p(\mathbb{R}_+,X)} \le C_p ||x|| \text{ for every } x \in X.$$

Now it is easy to derive the inequality

$$\sup_{||x||\le 1} \int_0^\infty ||T(t)x||^p dt \le K_p < \infty,$$

where K_p is a positive constant. Choose $J(f) := \int_0^\infty f(t)^p dt$, apply Theorem 1 and the proof is complete. ∎

Corollary 2. *Let $\mathbf{T} = \{T(t)\}_{t\ge 0}$ be a semigroup on a Banach space X as in the above Lemma 2. If there exists a non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ for each $t > 0$ and (1.2) holds then the semigroup $\mathbf{T}$ is exponentially stable.*

Proof. Seemingly we could proceed as in the proof of Corollary 1, but, however, we cannot directly apply the Uniform Boundedness Theorem. First we prove that the semigroup $\mathbf{T}$ is uniformly bounded. In fact, this has been done in [2] in the general framework of the evolution families. For the sake of completeness we mention some steps of that proof for this particular case. We may assume that $\phi(0) = 0$, $\phi(1) = 1$ and that ϕ is strictly increasing on $\mathbb{R}_+$, if not, we replace ϕ by some multiple of the function

$$t \mapsto \bar{\phi}(t) := \begin{cases} \int_0^t \phi(u)du, & \text{if} \quad 0 \le t \le 1 \\ \frac{at}{at+1-a}, & \text{if} \quad t > 1, \end{cases}$$

where $a := \int_0^1 \phi(u)du$.

Let $x \in X$ be fixed, N be a positive integer such that $M_\phi(x) < N$ and let $t \ge N$. For each $\tau \in [t-N, t]$ and all $u \ge 0$ we have:

$$e^{-\omega N} 1_{[t-N,t]}(u)||T(t)x|| \le e^{-\omega(t-\tau)} 1_{[t-N,t]}(u)||T(t-\tau)T(\tau)x|| \le M||T(u)x||$$

and then

$$N\phi\left(\frac{||T(t)x||}{Me^{\omega N}}\right) \le \int_{t-N}^{t} \phi\left(\frac{||T(t)x||}{Me^{\omega N}}\right) du \le M_\phi(x).$$

Hence $||T(t)x|| \le Me^{\omega N} M_\phi(x)$ for every $t \ge N$, and so the semigroup $\mathbf{T}$ is uniformly bounded.

From [11] Lemma 3.2.1 it follows that there exists an Orlicz's space E satisfying (1.3) such that for each $x \in X$ which satisfies (1.2), the map $t \mapsto T(t)x$ belongs to E. For each non-negative, bounded and measurable real-valued function f we put $J(f) := \sup_{t \ge 0} |1_{[0,t]} f|_E$, giving,

$$J(||T(\cdot)x||) = \sup_{t\ge 0} |1_{[0,t]}||T(\cdot)x|||_E \le |||T(\cdot)x|||_E < \infty,$$

for every $x \in X$.

Arguing as in Corollary 1 it follows that there exists a positive constant K_ϕ, independent of x, such that

$$\sup_{||x||\le 1} J(||T(\cdot)x||) < K_\phi < \infty.$$

Application of Theorem 1 completes the proof. ∎

3. The Non-autonomous Case

We state and prove two lemmas that will be used in the sequel.

Lemma 3. *Let $\mathcal{U} = \{U(t,s)\}_{t \ge s \ge 0}$ be an exponentially bounded evolution family on a Banach space X. If for each $x \in X$ there exists $t(x) > 0$ such that $U(s+t(x), s)x = 0$ for every $s \ge 0$ then the family $\mathcal{U}$ is uniformly exponentially stable.*

Proof. First we prove that there exists $M > 0$ such that

$$\sup_{s\ge 0} ||U(s+t,s)|| \le M \text{ for all } t \ge 0.$$

Indeed, if we suppose the contrary then there exists a sequence (t_n) of positive real numbers with $t_n \to \infty$ such that $\lim_{n\to\infty} ||U(s+t_n, s)|| = \infty$. From the Uniform Boundedness Theorem it follows that there exists $x \in X$ such that $||U(s+t_n, s)x|| \to \infty$ when $n \to \infty$ which is in contradiction to the hypothesis. We now observe that the family $\{e^{\nu(t-s)}U(t,s)\}_{t\geq s\geq 0}$ verifies the hypothesis of the present lemma and then

$$||U(t,s)|| \leq Me^{-\nu(t-s)} \text{ for all } t \geq s,$$

i.e. the assertion holds. ∎

Lemma 4. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an exponentially bounded evolution family on a Banach space X such that for each $y \in X$ and each $s \geq 0$ the map*

$$t \mapsto ||U(s+t,s)y|| : \mathbb{R}_+ \to \mathbb{R}_+$$

is continuous on $(0,\infty)$. If there exist positive real numbers h and $q < 1$ such that for every $x \in X$ there exists $t(x) \in (0,h]$ with the property that

$$\sup_{s\geq 0} ||U(s+t(x),s)x|| \leq q||x||,$$

then the family $\mathcal{U}$ is exponentially stable.

Proof. Is similar to that of Lemma 2 and so we omit the details. ∎

Theorem 2. *Let $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ be an evolution family on a Banach space X as in the above Lemma 4 and let J be a functional as in Theorem 1. If there exists $r > 0$ such that*

$$\sup_{s\geq 0} \sup_{||x||\leq r} J(||U(s+\cdot,s)x||) := L(J,r) < \infty, \tag{3.1}$$

then the evolution family $\mathcal{U}$ is uniformly exponentially stable.

Proof. Suppose that the family $\mathcal{U}$ is not uniformly exponentially stable. Under such circumstances as proved in Lemma 4, for every positive real number h and every $q \in (0,1)$ there exist $x_0 \in X$ of norm one and $s_0 \geq 0$ such that

$$||U(s_0+t,s_0)x_0|| > q \text{ for all } t \in [0,h].$$

Thus

$$L(J,r) \geq J(||U(s_0+t,s_0)rx_0||) \geq J(rq \cdot 1_{[0,h]})$$

for each $h > 0$, which contradicts (3.1). ∎

Theorem 3. *Let J be as in the above Theorem 1. We suppose, in addition, that J is lower semi-continuous and convex in the sense of Jensen (or sub-additive, that is, $J(f+g) \leq J(f) + J(g)$ for every f and g in $\mathcal{M}_{loc}(\mathbb{R}_+)$). Let $\mathcal{U}$ be an evolution family as in the Lemma 4. If the set $\mathcal{X}$ of all $x \in X$ for which*

$$\sup_{s\geq 0} J(||U(s+\cdot,s)x||) < \infty$$

is of the second category in X, then the family $\mathcal{U}$ is uniformly exponentially stable.

Proof. Let $s \geq 0$, be fixed. The map $x \mapsto ||U(s+\cdot,s)x|| : X \to \mathcal{M}_{loc}(\mathbb{R}_+)$ is continuous. As a consequence, the map

$$x \mapsto \Phi_s(x) := J(||U(s+\cdot,s)x||) : X \to [0,\infty]$$

is lower semi-continuous as well. For each positive integer k, the set

$$X_k(s) := \{x \in X : J(||U(s+\cdot,s)x||) \leq k\}$$

is closed, because it is the reverse image of the real closed interval $[0, k]$ by the map Φ_s. It is clear that the set

$$X_k := \left\{ x \in X : \sup_{s \geq 0} J(||U(s+\cdot, s)x||) \leq k \right\} = \cap_{s \geq 0} X_k(s)$$

is also closed and moreover that $\mathcal{X}$ is the union of all sets X_k. Because $\mathcal{X}$ is of the second category in X, there exists a set X_{k_0} whose interior is non empty. Let $x_0 \in X$ and $r_0 > 0$ such that $B(x_0, r_0)$ belongs to X_{k_0}. It is easy to see that $B\left(0, \frac{1}{2}r_0\right)$ belongs to X_{k_0}, that is,

$$\sup_{s \geq 0} \sup_{||x|| \leq \frac{1}{2}r_0} J(||U(s+\cdot, s)x||) \leq k_0.$$

Indeed for every $x \in X$ with $||x|| \leq r_0$ we have:

$$\begin{aligned} J\left(\left\|U(s+\cdot, s)\left(\frac{1}{2}x\right)\right\|\right) &= J\left(\frac{1}{2}||U(s+\cdot, s)[(x+x_0) - x_0]||\right) \\ &\leq J\left(\frac{1}{2}\left[||U(s+\cdot, s)(x+x_0) + ||U(s+\cdot, s)x_0||\right]\right) \\ &\leq \frac{1}{2}J(||U(s+\cdot, s)(x+x_0)||) + \frac{1}{2}J(||U(s+\cdot, s)x_0||) \\ &\leq k_0. \end{aligned}$$

Application of Theorem 2 completes the proof. ∎

Corollary 3. *Let $\mathcal{U} = \{U(t,s)\}_{t \geq s \geq 0}$ be an exponentially bounded evolution family on a Banach space X such that for each $x \in X$ the map $t \mapsto ||U(s+t, s)x||$ is continuous on $(0, \infty)$ for every $s \geq 0$. Consider the following three inequalities:*

1. *There exists $p \in [1, \infty)$ such that*

$$\sup_{s \geq 0} \int_0^\infty ||U(s+t, s)x||^p dt < \infty$$

for every $x \in X$.

2. *There exists a Banach function space E satisfying (1.3) such that for each $s \geq 0$ and each $x \in X$ the map $U(s+\cdot, s)x$ belongs to E and for every $x \in X$ we have*

$$\sup_{s \geq 0} |||U(s+\cdot, s)x|||_E < \infty.$$

3. *There exists a non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for each $t > 0$ such that*

$$\sup_{s \geq o} \int_0^\infty \phi(||U(s+t, s)x||) dt < \infty$$

for every $x \in X$.

If any one of these statements is true then the family $\mathcal{U}$ is exponentially stable.

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