# INEQUALITIES FOR THE NORM AND NUMERICAL RADIUS OF COMPOSITE OPERATORS IN HILBERT SPACES

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Abstract. Some new inequalities for the norm and the numerical radius of composite operators generated by a pair of operators are given.

## 1. Introduction

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator $T$ is the subset of the complex numbers $\mathbb{C}$ given by [4, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}. \tag{1.1}$$

It is well known that (see [4]):

(i) The numerical range of an operator is convex;
(ii) The spectrum of an operator is contained in the closure of its numerical range;
(iii) $T$ is self-adjoint if and only if $W(T)$ is real.

The *numerical radius* $w(T)$ of an operator $T$ on $H$ is defined by [4, p. 8]

$$w(T):=\sup\{|\lambda|,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|,\|x\|=1\}. \tag{1.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators acting on $H$ and the following inequality holds true:

$$w(T)\le\|T\|\le 2w(T). \tag{1.3}$$

We recall some classical results involving the numerical radius of two linear operators $A,B$.

The following general result for the product of two operators holds [4, p. 37]:

**Theorem 1.** *If $A,B$ are two bounded linear operators on the Hilbert space $(H,\langle\cdot,\cdot\rangle)$, then*

$$w(AB)\le 4w(A)\,w(B). \tag{1.4}$$

*In the case that $AB=BA$, then*

$$w(AB)\le 2w(A)\,w(B). \tag{1.5}$$

The following results are also well known [4, p. 38].

*Date*: 5 September, 2005.
2000 *Mathematics Subject Classification.* 47A12.
*Key words and phrases.* Numerical range, Numerical radius, Bounded linear operators, Hilbert spaces.

**Theorem 2.** *If $A$ is a unitary operator that commutes with another operator $B$, then*

$$w(AB) \leq w(B). \tag{1.6}$$

*If $A$ is an isometry and $AB = BA$, then (1.6) also holds true.*

We say that $A$ and $B$ *double commute* if $AB = BA$ and $AB^* = B^*A$.

The following result holds [4, p. 38].

**Theorem 3** (Double commute)**.** *If the operators $A$ and $B$ double commute, then*

$$w(AB) \leq w(B)\|A\|. \tag{1.7}$$

As a consequence of the above, we have [4, p. 39]:

**Corollary 1.** *Let $A$ be a normal operator commuting with $B$. Then*

$$w(AB) \leq w(A)\, w(B). \tag{1.8}$$

For other results and historical comments on the above see [4, p. 39–41]. For more results on the numerical radius, see [5].

The main aim of this paper is to establish some new inequalities for composite operators generated by a pair of operators $(A, B)$ under various assumptions. Namely, in one side, several inequalities involving the norm

$$\left\| \frac{A^*A + B^*B}{2} \right\|$$

and the numerical radius $w(B^*A)$ are established. On the other side, upper bounds for the nonnegative quantities

$$\|A\|\,\|B\| - w(B^*A) \text{ and } \|A\|^2\,\|B\|^2 - w^2(B^*A)$$

under special conditions for the operators involved are also given.

## 2. The Results

The following result may be stated:

**Theorem 4.** *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. If $r > 0$ and*

$$\|A - B\| \leq r, \tag{2.1}$$

*then*

$$\left\| \frac{A^*A + B^*B}{2} \right\| \leq w(B^*A) + \frac{1}{2} r^2. \tag{2.2}$$

*Proof.* For any $x \in H$, $\|x\| = 1$, we have from (2.1) that

$$\|Ax\|^2 + \|Bx\|^2 \leq 2\operatorname{Re}\langle Ax, Bx\rangle + r^2. \tag{2.3}$$

However

$$\begin{aligned} \|Ax\|^2 + \|Bx\|^2 &= \langle (A^*A)x, x\rangle + \langle (B^*B)x, x\rangle \\ &= \langle (A^*A + B^*B)x, x\rangle \end{aligned}$$

and by (2.3) we obtain

$$\langle (A^*A + B^*B)x, x\rangle \leq 2\left|\langle (B^*A)x, x\rangle\right| + r^2 \tag{2.4}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (2.4) we get

$$w\left(A^*A + B^*B\right) \leq 2w\left(B^*A\right) + r^2 \tag{2.5}$$

and since the operator $A^*A + B^*B$ is self-adjoint, hence

$$w\left(A^*A + B^*B\right) = \|A^*A + B^*B\|$$

and by (2.5) we deduce the desired inequality (2.2). ∎

**Remark 1.** *We observe that, from the proof of the above theorem, we have the inequalities*

$$0 \leq \left\|\frac{A^*A + B^*B}{2}\right\| - w\left(B^*A\right) \leq \frac{1}{2}\left\|A - B\right\|^2, \tag{2.6}$$

*provided that $A, B$ are bounded linear operators in $H$.*

*The second inequality in (2.6) is obvious while the first inequality follows by the fact that*

$$\begin{aligned} \left\langle \left(A^*A + B^*B\right)x, x\right\rangle &= \|Ax\|^2 + \|Bx\|^2 \\ &\geq 2\|Ax\|\|Bx\| \geq 2\left|\left\langle \left(B^*A\right)x, x\right\rangle\right| \end{aligned}$$

*for any $x \in H$.*

The inequality (2.2) is obviously a reach source of particular inequalities of interest.

Indeed, if we assume, for $\lambda \in \mathbb{C}$ and a bounded linear operator $T$, that we have

$$\|T - \lambda T^*\| \leq r, \tag{2.7}$$

for a given positive number $r$, then by (2.6) we deduce the inequality

$$0 \leq \left\|\frac{T^*T + |\lambda|^2 TT^*}{2}\right\| - |\lambda|\, w\left(T^2\right) \leq \frac{1}{2}r^2. \tag{2.8}$$

Now, if we assume that for $\lambda \in \mathbb{C}$ and a bounded linear operator $V$ we have that

$$\|V - \lambda I\| \leq r, \tag{2.9}$$

where $I$ is the identity operator on $H$, then by (2.2) we deduce the inequality

$$0 \leq \left\|\frac{V^*V + |\lambda|^2 I}{2}\right\| - |\lambda|\, w\left(V\right) \leq \frac{1}{2}r^2.$$

As a dual approach, the following result may be noted as well:

**Theorem 5.** *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $H$. Then*

$$\left\|\frac{A + B}{2}\right\|^2 \leq \frac{1}{2}\left[\left\|\frac{A^*A + B^*B}{2}\right\| + w\left(B^*A\right)\right]. \tag{2.10}$$

*Proof.* We obviously have

$$\begin{aligned} \|Ax + Bx\|^2 &= \|Ax\|^2 + 2\operatorname{Re}\langle Ax, Bx\rangle + \|Bx\|^2 \\ &\leq \left\langle \left(A^*A + B^*B\right)x, x\right\rangle + 2\left|\left\langle \left(B^*A\right)x, x\right\rangle\right| \end{aligned}$$

for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get

$$\|A+B\|^2 \leq w(A^*A + B^*B) + 2w(B^*A) = \|A^*A + B^*B\| + 2w(B^*A),$$

from where we get the desired inequality (2.10). ∎

**Remark 2.** *The inequality (2.10) can generate some interesting particular results such as the following inequality*

$$\left\|\frac{T+T^*}{2}\right\|^2 \leq \frac{1}{2}\left[\left\|\frac{T^*T + TT^*}{2}\right\| + w\left(T^2\right)\right], \tag{2.11}$$

*holding for each bounded linear operator $T : H \to H$.*

The following result may be stated as well.

**Theorem 6.** *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $H$ and $p \geq 2$. Then*

$$\left\|\frac{A^*A + B^*B}{2}\right\|^{\frac{p}{2}} \leq \frac{1}{4}\left[\|A-B\|^p + \|A+B\|^p\right]. \tag{2.12}$$

*Proof.* We use the following inequality for vectors in inner product spaces obtained by Dragomir and Sándor in [2]:

$$2\left(\|a\|^p + \|b\|^p\right) \leq \|a+b\|^p + \|a-b\|^p \tag{2.13}$$

for any $a, b \in H$ and $p \geq 2$.

Utilising (2.13) we may write

$$2\left(\|Ax\|^p + \|Bx\|^p\right) \leq \|Ax + Bx\|^p + \|Ax - Bx\|^p \tag{2.14}$$

for any $x \in H$.

Now, observe that

$$\|Ax\|^p + \|Bx\|^p = \left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}}$$

and by the elementary inequality:

$$\frac{\alpha^q + \beta^q}{2} \geq \left(\frac{\alpha+\beta}{2}\right)^q, \quad \alpha, \beta \geq 0 \text{ and } q \geq 1$$

we have

$$\left(\|Ax\|^2\right)^{\frac{p}{2}} + \left(\|Bx\|^2\right)^{\frac{p}{2}} \geq 2^{1-\frac{p}{2}}\left(\|Ax\|^2 + \|Bx\|^2\right)^{\frac{p}{2}} = 2^{1-\frac{p}{2}}\left[\langle(A^*A + B^*B)x, x\rangle\right]^{\frac{p}{2}}. \tag{2.15}$$

Combining (2.14) with (2.15) we get

$$\frac{1}{4}\left[\|Ax - Bx\|^p + \|Ax + Bx\|^p\right] \geq \left|\left\langle\left(\frac{A^*A + B^*B}{2}\right)x, x\right\rangle\right|^{\frac{p}{2}} \tag{2.16}$$

for any $x \in H$, $\|x\| = 1$. Taking the supremum over $x \in H$, $\|x\| = 1$, and taking into account that

$$w\left(\frac{A^*A + B^*B}{2}\right) = \left\|\frac{A^*A + B^*B}{2}\right\|,$$

we deduce the desired result (2.12). ∎

**Remark 3.** *If $p = 2$, then we have the inequality:*

$$\left\| \frac{A^*A + B^*B}{2} \right\| \leq \left\| \frac{A - B}{2} \right\|^2 + \left\| \frac{A + B}{2} \right\|^2, \tag{2.17}$$

*for any $A, B$ bounded linear operators. This result can also be obtained directly on utilising the parallelogram identity.*

*We also should observe that for $A = T$ and $B = T^*$, $T$ a normal operator, the inequality (2.12) becomes*

$$\|T\|^p \leq \frac{1}{4} \left[ \|T - T^*\|^p + \|T + T^*\|^p \right],$$

*where $p \geq 2$.*

The following result may be stated as well.

**Theorem 7.** *Let $A, B : H \to H$ be two bounded linear operators on the Hilbert space $H$ and $r \geq 1$. If $A^*A \geq B^*B$ in the operator order or, equivalently, $\|Ax\| \geq \|Bx\|$ for any $x \in H$, then:*

$$\left\| \frac{A^*A + B^*B}{2} \right\|^r \leq \|A\|^{r-1} \|B\|^{r-1} w\left(B^*A\right) + \frac{1}{2} r^2 \|A\|^{2r-2} \|A - B\|^2. \tag{2.18}$$

*Proof.* We use the following inequality for vectors in inner product spaces due to Goldstein, Ryff and Clarke [3]:

$$\|a\|^{2r} + \|b\|^{2r} \leq 2 \|a\|^{r-1} \|b\|^{r-1} \operatorname{Re}\langle a, b\rangle + r^2 \|a\|^{2r-2} \|a - b\|^2, \tag{2.19}$$

where $r \geq 1$, $a, b \in H$ and $\|a\| \geq \|b\|$.

Utilising (2.19) we can state that:

$$\begin{aligned} &\|Ax\|^{2r} + \|Bx\|^{2r} \\ &\qquad \leq 2 \|Ax\|^{r-1} \|Bx\|^{r-1} |\langle Ax, Bx\rangle| + r^2 \|Ax\|^{2r-2} \|Ax - Bx\|^2, \end{aligned} \tag{2.20}$$

for any $x \in H$.

As in the proof of Theorem 6, we also have

$$2^{1-r} \left[ \langle (A^*A + B^*B) x, x\rangle \right]^r \leq \|Ax\|^{2r} + \|Bx\|^{2r}, \tag{2.21}$$

for any $x \in H$.

Therefore, by (2.20) and (2.21) we deduce

$$\begin{aligned} &\left[ \left\langle \left( \frac{A^*A + B^*B}{2} \right) x, x \right\rangle \right]^r \\ &\qquad \leq \|Ax\|^{r-1} \|Bx\|^{r-1} |\langle Ax, Bx\rangle| + \frac{1}{2} r^2 \|A\|^{2r-2} \|Ax - Bx\|^2 \end{aligned} \tag{2.22}$$

for any $x \in H$.

Taking the supremum in (2.22) we obtain the desired result (2.18). ∎

**Remark 4.** *Following* [4, p. 156]*, we recall that the bounded linear operator $V$ is hyponormal, if*

$$\|V^*x\| \leq \|Vx\| \text{ for all } x \in H.$$

*Now, if we choose in (2.18) $A = V$ and $B = V^*$, then, on taking into account that for hyponormal operators $w\left(V^2\right) = \|V\|^2$, we get the inequality*

$$\left\| \frac{V^*V + VV^*}{2} \right\|^r \leq \|V\|^{2r-2} \left[ \|V\|^2 + \frac{1}{2} r^2 \|V - V^*\|^2 \right], \tag{2.23}$$

*holding for any hyponormal operator $V$ and any $r \geq 1$.*

## 3. Further Inequalities for an Invertible Operator

In this section we assume that $B : H \to H$ is an invertible bounded linear operator and let $B^{-1} : H \to H$ be its inverse. Then, obviously,

$$\|Bx\| \geq \frac{1}{\|B^{-1}\|} \|x\| \quad \text{for any } \; x \in H, \tag{3.1}$$

where $\left\|B^{-1}\right\|$ denotes the norm of the inverse $B^{-1}$.

The following result holds true:

**Theorem 8.** *Let $A, B : H \to H$ be two bounded linear operators on $H$ and $B$ is invertible such that, for a given $r > 0$,*

$$\|A - B\| \leq r. \tag{3.2}$$

*Then:*

$$\|A\| \leq \left\|B^{-1}\right\| \left[w\left(B^* A\right) + \frac{1}{2} r^2\right]. \tag{3.3}$$

*Proof.* The condition (3.2) is obviously equivalent to:

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \left\langle \left(B^* A\right) x, x \right\rangle + r^2 \tag{3.4}$$

for any $x \in H$, $\|x\| = 1$.

Since, by (3.1),

$$\|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2, \quad x \in H$$

and $\operatorname{Re} \left\langle \left(B^* A\right) x, x \right\rangle \leq \left|\left\langle \left(B^* A\right) x, x \right\rangle\right|,$ hence by (3.4) we get

$$\|Ax\|^2 + \frac{\|x\|^2}{\|B^{-1}\|^2} \leq 2 \left|\left\langle \left(B^* A\right) x, x \right\rangle\right| + r^2 \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.5), we have

$$\|A\|^2 + \frac{1}{\|B^{-1}\|^2} \leq 2w\left(B^* A\right) + r^2. \tag{3.6}$$

By the elementary inequality

$$\frac{2\|A\|}{\|B^{-1}\|} \leq \|A\|^2 + \frac{1}{\|B^{-1}\|^2} \tag{3.7}$$

and by (3.6) we then deduce the desired result (3.3). ∎

**Remark 5.** *If we choose above $B = \lambda I$, $\lambda \neq 0$, then we get the inequality*

$$(0 \leq)\, \|A\| - w(A) \leq \frac{1}{2|\lambda|} r^2, \tag{3.8}$$

*provided $\|A - \lambda I\| \leq r$. This result has been obtained in the earlier paper* [1].

*Also, if we assume that $B = \lambda A^*$, $A$ is invertible, then we obtain*

$$\|A\| \leq \left\|A^{-1}\right\| \left[w\left(A^2\right) + \frac{1}{2|\lambda|} r^2\right], \tag{3.9}$$

*provided $\|A - \lambda A^*\| \leq r$, $\lambda \neq 0$.*

The following result may be stated as well:

**Theorem 9.** *Let $A, B : H \to H$ be two bounded linear operators on $H$. If $B$ is invertible and for $r > 0$,*

$$\|A - B\| \leq r, \tag{3.10}$$

*then*

$$(0 \leq) \|A\| \|B\| - w(B^*A) \leq \frac{1}{2}r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2}. \tag{3.11}$$

*Proof.* The condition (3.10) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 \leq 2 \operatorname{Re} \langle B^*Ax, x \rangle + r^2 + \|B\|^2 - \|Bx\|^2. \tag{3.12}$$

Since

$$\operatorname{Re} \langle B^*Ax, x \rangle \leq |\langle B^*Ax, x \rangle|, \quad \|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2} \|x\|^2$$

and

$$\|Ax\|^2 + \|B\|^2 \geq 2 \|B\| \|Ax\|$$

for any $x \in H$, hence by (3.12) we get

$$2 \|B\| \|Ax\| \leq 2 |\langle B^*Ax, x \rangle| + r^2 + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{\|B^{-1}\|^2} \tag{3.13}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ we deduce the desired result (3.11). ∎

**Remark 6.** *If we choose in Theorem 9, $B = \lambda A^*$, $\lambda \neq 0$, $A$ is invertible, then we get the inequality:*

$$(0 \leq) \|A\|^2 - w(A^2) \leq \frac{1}{2|\lambda|}r^2 + |\lambda| \cdot \frac{\|A\|^2 \|A^{-1}\|^2 - 1}{\|A^{-1}\|^2} \tag{3.14}$$

*provided $\|A - \lambda A^*\| \leq r$.*

The following result may be stated as well.

**Theorem 10.** *Let $A, B : H \to H$ be two bounded linear operators on $H$. If $B$ is invertible and for $r > 0$ we have*

$$\|A - B\| \leq r < \|B\|, \tag{3.15}$$

*then*

$$\|A\| \leq \frac{1}{\sqrt{\|B\|^2 - r^2}} \left( w(B^*A) + \frac{\|B\|^2 \|B^{-1}\|^2 - 1}{2\|B^{-1}\|^2} \right). \tag{3.16}$$

*Proof.* The first part of condition (3.15) is obviously equivalent to

$$\|Ax\|^2 + \|Bx\|^2 \leq 2 \operatorname{Re} \langle Ax, Bx \rangle + r^2$$

for any $x \in H$, which is clearly equivalent to

$$\|Ax\|^2 + \|B\|^2 - r^2 \leq 2 \operatorname{Re} \langle B^*Ax, x \rangle + \|B\|^2 - \|Bx\|^2. \tag{3.17}$$

Since

$$\operatorname{Re}\langle B^*Ax, x\rangle \leq |\langle B^*Ax, x\rangle|,$$
$$\|Bx\|^2 \geq \frac{1}{\|B^{-1}\|^2}\|x\|^2$$

and, by the second part of (3.15),

$$\|Ax\|^2 + \|B\|^2 - r^2 \geq 2\sqrt{\|B\|^2 - r^2}\,\|Ax\|,$$

for any $x \in H$, hence by (3.17) we get

$$2\|Ax\|\sqrt{\|B\|^2 - r^2} \leq 2|\langle B^*Ax, x\rangle| + \frac{\|B\|^2\|B^{-1}\|^2 - 1}{\|B^{-1}\|^2} \tag{3.18}$$

for any $x \in H$, $\|x\| = 1$.

Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.18), we deduce the desired inequality (3.16). ∎

**Remark 7.** *The above Theorem 10 has some particular cases of interest. For instance, if we choose $B = \lambda I$, with $|\lambda| > r$, then (3.15) is obviously fulfilled and by (3.16) we get*

$$\|A\| \leq \frac{w(A)}{\sqrt{1 - \left(\frac{r}{|\lambda|}\right)^2}}, \tag{3.19}$$

*provided $\|A - \lambda I\| \leq r$. This result has been obtained in the earlier paper* [1].

*On the other hand, if in the above we choose $B = \lambda A^*$ with $\|A\| \geq \frac{r}{|\lambda|}$ $(\lambda \neq 0)$, then by (3.16) we get*

$$\|A\| \leq \frac{1}{\sqrt{\|A\|^2 - \left(\frac{r}{|\lambda|}\right)^2}}\left[w(A^2) + |\lambda| \cdot \frac{\|A\|^2\|A^{-1}\|^2 - 1}{2\|A^{-1}\|^2}\right], \tag{3.20}$$

*provided $\|A - \lambda A^*\| \leq r$.*

The following result may be stated as well.

**Theorem 11.** *Let $A, B$ and $r$ be as in Theorem 8. Moreover, if*

$$\|B^{-1}\| < \frac{1}{r}, \tag{3.21}$$

*then*

$$\|A\| \leq \frac{\|B^{-1}\|}{\sqrt{1 - r^2\|B^{-1}\|^2}}w(B^*A). \tag{3.22}$$

*Proof.* Observe that, by (3.6) we have

$$\|A\|^2 + \frac{1 - r^2\|B^{-1}\|^2}{\|B^{-1}\|^2} \leq 2w(B^*A). \tag{3.23}$$

Utilising the elementary inequality

$$2\frac{\|A\|}{\|B^{-1}\|}\sqrt{1 - r^2\|B^{-1}\|^2} \leq \|A\|^2 + \frac{1 - r^2\|B^{-1}\|^2}{\|B^{-1}\|^2}, \tag{3.24}$$

which can be stated since (3.21) is assumed to be true, hence by (3.23) and (3.24) we deduce the desired result (3.22). ∎

**Remark 8.** *If we assume that $B = \lambda A^*$ with $\lambda \neq 0$ and $A$ an invertible operator, then, by applying Theorem 11, we get the inequality:*

$$\|A\| \leq \frac{\left\|A^{-1}\right\| w\left(A^2\right)}{\sqrt{|\lambda|^2 - r^2 \left\|A^{-1}\right\|^2}}, \tag{3.25}$$

*provided $\|A - \lambda A^*\| \leq r$ and $\left\|A^{-1}\right\| \leq \frac{|\lambda|}{r}$.*

The following result may be stated as well.

**Theorem 12.** *Let $A, B : H \to H$ be two bounded linear operators. If $r > 0$ and $B$ is invertible with the property that $\|A - B\| \leq r$ and*

$$\frac{1}{\sqrt{r^2+1}} \leq \left\|B^{-1}\right\| < \frac{1}{r}, \tag{3.26}$$

*then*

$$\|A\|^2 \leq w^2\left(B^*A\right) + 2w\left(B^*A\right) \cdot \frac{\left\|B^{-1}\right\| - \sqrt{1 - r^2\left\|B^{-1}\right\|^2}}{\left\|B^{-1}\right\|}. \tag{3.27}$$

*Proof.* Let $x \in H$, $\|x\| = 1$. Then by (3.5) we have

$$\|Ax\|^2 + \frac{1}{\left\|B^{-1}\right\|^2} \leq 2\left|\langle B^*Ax, x\rangle\right| + r^2, \tag{3.28}$$

and since

$$\frac{1}{\left\|B^{-1}\right\|^2} - r^2 > 0,$$

we can conclude that $|\langle B^*Ax, x\rangle| > 0$ for any $x \in H$, $\|x\| = 1$.

Dividing in (3.28) with $|\langle B^*Ax, x\rangle| > 0$, we obtain

$$\frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} \leq 2 + \frac{r^2}{|\langle B^*Ax, x\rangle|} - \frac{1}{\left\|B^{-1}\right\|^2 |\langle B^*Ax, x\rangle|}. \tag{3.29}$$

Subtracting $|\langle B^*Ax, x\rangle|$ from both sides of (3.29), we get

$$\begin{aligned} &\frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} - |\langle B^*Ax, x\rangle| \\ &\leq 2 - |\langle B^*Ax, x\rangle| - \frac{1 - r^2\left\|B^{-1}\right\|^2}{|\langle B^*Ax, x\rangle| \left\|B^{-1}\right\|^2} \\ &= 2 - \frac{2\sqrt{1 - r^2\left\|B^{-1}\right\|^2}}{\left\|B^{-1}\right\|} - \left(\sqrt{|\langle B^*Ax, x\rangle|} - \frac{\sqrt{1 - r^2\left\|B^{-1}\right\|^2}}{\left\|B^{-1}\right\| \sqrt{|\langle B^*Ax, x\rangle|}}\right)^2 \\ &\leq 2\left(\frac{\left\|B^{-1}\right\| - \sqrt{1 - r^2\left\|B^{-1}\right\|^2}}{\left\|B^{-1}\right\|}\right), \end{aligned} \tag{3.30}$$

which gives:

$$\|Ax\|^2 \leq |\langle B^*Ax, x\rangle|^2 + 2\left|\langle B^*Ax, x\rangle\right| \frac{\left\|B^{-1}\right\| - \sqrt{1 - r^2\left\|B^{-1}\right\|^2}}{\left\|B^{-1}\right\|}. \tag{3.31}$$

We also remark that, by (3.26) the quantity

$$\left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2} \geq 0,$$

hence, on taking the supremum in (3.31) over $x \in H$, $\|x\| = 1$, we deduce the desired inequality. ∎

**Remark 9.** *It is interesting to remark that if we assume $\lambda \in \mathbb{C}$ with $0 < r \leq |\lambda| \leq \sqrt{r^2+1}$ and $\|A - \lambda I\| \leq r$, then by (3.2) we can state the following inequality:*

$$\|A\|^2 \leq |\lambda|^2 w\left(A^2\right) + 2|\lambda| \left(1 - \sqrt{|\lambda|^2 - r^2}\right) w(A). \tag{3.32}$$

*Also, if $\|A - A^*\| \leq r$, $A$ is invertible and $\frac{1}{\sqrt{r^2+1}} \leq \left\|A^{-1}\right\| \leq \frac{1}{r}$, then, by (3.27) we also have*

$$\|A\|^2 \leq w^2\left(A^2\right) + 2w\left(A^2\right) \cdot \frac{\left\|A^{-1}\right\| - \sqrt{1 - r^2 \left\|A^{-1}\right\|^2}}{\left\|A^{-1}\right\|}. \tag{3.33}$$

One can also prove the following result.

**Theorem 13.** *Let $A, B : H \to H$ be two bounded linear operators. If $r > 0$ and $B$ is invertible with the property that $\|A - B\| \leq r$ and $\left\|B^{-1}\right\| \leq \frac{1}{r}$, then*

$$\begin{aligned} (0 \leq) \|A\|^2 \|B\|^2 - w^2(B^*A) \\ \leq 2w(B^*A) \cdot \frac{\|B\|}{\|B^{-1}\|} \left(\|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2}\right). \end{aligned} \tag{3.34}$$

*Proof.* We subtract the quantity $\frac{|\langle B^*Ax, x\rangle|}{\|B\|^2}$ from both sides of (3.29) to obtain

$$\begin{aligned} 0 &\leq \frac{\|Ax\|^2}{|\langle B^*Ax, x\rangle|} - \frac{|\langle B^*Ax, x\rangle|}{\|B\|^2} \\ &\leq 2 - \frac{|\langle B^*Ax, x\rangle|}{\|B\|^2} - \frac{1 - r^2 \left\|B^{-1}\right\|^2}{|\langle B^*Ax, x\rangle| \left\|B^{-1}\right\|^2} \\ &= 2 - 2 \cdot \frac{\sqrt{1 - r^2 \left\|B^{-1}\right\|^2}}{\|B\| \left\|B^{-1}\right\|} - \left(\frac{\sqrt{|\langle B^*Ax, x\rangle|}}{\|B\|} - \frac{\sqrt{1 - r^2 \left\|B^{-1}\right\|^2}}{\sqrt{|\langle B^*Ax, x\rangle|} \left\|B^{-1}\right\|}\right)^2 \\ &\leq 2 \cdot \frac{\left(\|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2}\right)}{\|B\| \left\|B^{-1}\right\|}, \end{aligned} \tag{3.35}$$

which is equivalent with

$$\begin{aligned} (0 \leq) \|Ax\|^2 \|B\|^2 - |\langle B^*Ax, x\rangle|^2 \\ \leq 2\frac{\|B\|}{\|B^{-1}\|} |\langle B^*Ax, x\rangle| \left(\|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2}\right) \end{aligned} \tag{3.36}$$

for any $x \in H$, $\|x\| = 1$.

The inequality (3.36) also shows that $\|B\| \left\|B^{-1}\right\| \geq \sqrt{1 - r^2 \left\|B^{-1}\right\|^2}$ and then, by (3.36), we get

$$\|Ax\|^2 \|B\|^2 \leq |\langle B^* Ax, x\rangle|^2 + 2\frac{\|B\|}{\|B^{-1}\|} |\langle B^* Ax, x\rangle| \left( \|B\| \left\|B^{-1}\right\| - \sqrt{1 - r^2 \left\|B^{-1}\right\|^2} \right) \tag{3.37}$$

for any $x \in X$, $\|x\| = 1$.

Taking the supremum in (3.37) we deduce the desired inequality (3.34). ∎

**Remark 10.** *The above Theorem 13 has some particular instances of interest as follows. If, for instance, we choose $B = \lambda I$ with $|\lambda| \geq r > 0$ and $\|A - \lambda I\| \leq r$, then by (3.34) we obtain the inequality*

$$(0 \leq) \|A\|^2 - w^2(A) \leq 2|\lambda| w(A) \left(1 - \sqrt{1 - \frac{r^2}{|\lambda|^2}}\right). \tag{3.38}$$

*Also, if $A$ is invertible, $\|A - \lambda A^*\| \leq r$ and $\left\|A^{-1}\right\| \leq \frac{|\lambda|}{r}$, then by (3.34) we can state:*

$$(0 \leq) \|A\|^4 - w^2\left(A^2\right) \leq 2|\lambda| w\left(A^2\right) \cdot \frac{\|A\|}{\|A^{-1}\|} \left( \|A\| \left\|A^{-1}\right\| - \sqrt{1 - \frac{r^2}{|\lambda|^2} \left\|A^{-1}\right\|^2} \right). \tag{3.39}$$

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