SHARP BOUNDS FOR THE DEVIATION OF A FUNCTION FROM THE CHORD GENERATED BY ITS EXTREMITIES AND APPLICATIONS

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Abstract. Sharp bounds for the deviation of a real-valued function f defined on a compact interval $[a,b]$ to the chord generated by its end points $(a, f(a))$ and $(b, f(b))$ under various assumptions for f and f' including absolute continuity, convexity, bounded variation, monotonicity etc., are given. Some applications for weighted means and f-divergence measures in Information Theory are also provided.

1. Introduction

Consider a function $f : [a,b] \to \mathbb{R}$ and assume that it is bounded on $[a,b]$. The chord that connects its end points $A = (a, f(a))$ and $B = (b, f(b))$ has the equation

$$d_f : [a,b] \to \mathbb{R}, \ d_f(t) = \frac{1}{b-a}\left[f(a)(b-t) + f(b)(t-a)\right].$$

We introduce the error in approximating the value of the function $f(t)$ by $d_f(t)$ with $t \in [a,b]$ by $\Phi_f(t)$, *i.e.*, $\Phi_f(t)$ is defined by:

$$\Phi_f(t) := \frac{b-t}{b-a} \cdot f(a) + \frac{t-a}{b-a} \cdot f(b) - f(t). \tag{1.1}$$

The main aim of this paper is to provide sharp upper bounds for the absolute value of the difference $\Phi_f(t)$ in each point $t \in [a,b]$ and under various assumptions on the function f or its derivative f'.

In Section 2, we recall some results in the case that f is bounded below by m and above by M and in the case when f is convex on $[a,b]$.

In Section 3, the case when f is of bounded variation and in particular Lipschitzian or monotonic nondecreasing is analyzed, while in Section 4 the case of absolutely continuous functions is investigated.

Sections 5, 6 and 7 provide sharp bounds for $|\Phi_f(t)|, t \in [a,b]$ when f' is of bounded variation, Lipschitzian or absolutely continuous.

In Section 8 some applications in estimating the weighted mean generated by f, namely

$$M_f(p,x) := \sum_{i=1}^{n} p_i f(x_i), \tag{1.2}$$

Date: 06 August, 2007.

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 94A17.

Key words and phrases. Bounded variation, Absolute continuity, Convexity, Monotonicity, Weighted means, Analytic inequalities, f-Divergence measures, Information Theory.

where $p_i \geq 1$, $\sum_{i=1}^{n} p_i = 1$ and $m \leq x_i \leq M$, $i \in \{1, \ldots, n\}$ for a function f defined on an interval containing m and M are also given.

Finally, applications for the f-*divergence functional*

$$I_f(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{1.3}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{q} = (q_1, \ldots, q_n)$ are positive sequences, that was introduces by Csiszár in [4], as a generalised measure of information, a "distance function" on the set of probability distributions $\mathbb{P}^n$ are also provided.

2. Preliminary Results

The following simple result, which provides a sharp upper bound for the case of bounded functions, has been stated in [7] as an intermediate result needed to obtain a Grüss type inequality.

Theorem 1. *If $f : [a, b] \to \mathbb{R}$ is a bounded function with $-\infty < m \leq f(t) \leq M < \infty$ for any $t \in [a, b]$, then*

$$|\Phi_f(t)| \leq M - m. \tag{2.1}$$

The multiplicative constant 1 *in front of $M - m$ cannot be replaced by a smaller quantity.*

Proof. For the sake of completeness, we present a short proof.

Since f is bounded, we have $m(b-t) \leq (b-t) f(a) \leq (b-t) M$, $m(t-a) \leq (t-a) f(b) \leq (t-a) M$ and $-(b-a) M \leq -(b-a) f(t) \leq -(b-a) m$, which gives, by addition and division with $b - a$ that

$$-(M-m) \leq \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \leq M - m,$$

for each $t \in [a, b]$, i.e., the desired inequality (2.1) holds.

Now, assume that there exists a constant $C > 0$ such that $|\Phi_f(t)| \leq C(M-m)$ for any f as in the statement of the theorem. Then, for $t = \frac{a+b}{2}$, we should have

$$\left|\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq C(M-m). \tag{2.2}$$

If $f : [a, b] \to \mathbb{R}$, $f(t) = \left|t - \frac{a+b}{2}\right|$, then $f(a) = f(b) = \frac{b-a}{2}$, $f\left(\frac{a+b}{2}\right) = 0$, $M = \frac{b-a}{2}$ and $m = 0$ and the inequality (2.2) becomes $\frac{b-a}{2} \leq C \cdot \frac{b-a}{2}$, which implies that $C \geq 1$. ∎

The case of convex functions has been considered in [8] in order to prove another Grüss type inequality. The sharpness of the constant has not been analyzed in the earlier paper.

Theorem 2. *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$, then*

$$0 \leq \Phi_f(t) \leq \frac{(b-t)(t-a)}{b-a}\left[f'_-(b) - f'_+(a)\right] \leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right] \tag{2.3}$$

for any $t \in [a, b]$.

If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $\frac{1}{4}$ are sharp.

Proof. For the sake of completeness, we present a complete proof of (2.3) below.

Since f is convex, then

$$\frac{t-a}{b-a}\cdot f(b)+\frac{b-t}{b-a}\cdot f(a)\geq f\left[\frac{(b-t)a+(t-a)b}{b-a}\right]=f(t)$$

for any $t\in[a,b]$, i.e., $\Phi(t)\geq 0$ for any $t\in[a,b]$.

If either $f'_{-}(b)$ or $f'_{+}(a)$ are infinite, then the last part of (2.3) is obvious.

Suppose that $f'_{-}(b)$ and $f'_{+}(a)$ are finite. Then, by the convexity of f we have $f(t)-f(b)\geq f'_{-}(b)(t-b)$ for any $t\in(a,b)$. If we multiply this inequality with $t-a\geq 0$, we deduce

$$(t-a)f(t)-(t-a)f(b)\geq f'_{-}(b)(t-b)(t-a),\quad t\in(a,b). \tag{2.4}$$

Similarly, we get

$$(b-t)f(t)-(b-t)f(a)\geq f'_{+}(a)(t-a)(b-t),\quad t\in(a,b). \tag{2.5}$$

Adding (2.4) to (2.5) and dividing by $b-a$, we deduce

$$f(t)-\frac{(t-a)f(b)+(b-t)f(a)}{b-a}\geq\frac{(b-t)(t-a)}{b-a}\left[f'_{-}(b)-f'_{+}(a)\right],$$

for any $t\in(a,b)$, which proves the second inequality for $t\in(a,b)$.

If $t=a$ or $t=b$, the inequality also holds.

Now, assume that (2.3) holds with D and E greater than zero, i.e.,

$$\Phi_f(t)\leq D\cdot\frac{(b-t)(t-a)}{b-a}\left[f'_{-}(b)-f'_{+}(a)\right]\leq E(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]$$

for any $t\in[a,b]$. If we choose $t=\frac{a+b}{2}$, then we get

$$\begin{aligned}\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)&\leq\frac{1}{4}D(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]\\&\leq E(b-a)\left[f'_{-}(b)-f'_{+}(a)\right].\end{aligned} \tag{2.6}$$

Consider $f:[a,b]\to\mathbb{R}$, $f(t)=\left|t-\frac{a+b}{2}\right|$. Then f is convex, $f(a)=f(b)=\frac{b-a}{2}$, $f\left(\frac{a+b}{2}\right)=0$, $f'_{-}(b)=1$, $f'_{+}(a)=-1$ and by (2.6) we deduce

$$\frac{b-a}{2}\leq\frac{1}{2}D(b-a)\leq 2E(b-a),$$

which implies that $D\geq 1$ and $E\geq\frac{1}{4}$. ∎

3. The Case when f is of Bounded Variation

We start with the following representation result:

Lemma 1. *If $f:[a,b]\to\mathbb{R}$ is bounded on $[a,b]$ and $Q:[a,b]^2\to\mathbb{R}$ is defined by*

$$Q(t,s):=\begin{cases}t-b & \text{if } a\leq s\leq t\\[2ex] t-a & \text{if } t<s\leq b,\end{cases} \tag{3.1}$$

then we have the representation

$$\Phi_f(t)=\frac{1}{b-a}\int_a^b Q(t,s)\,df(s),\quad t\in[a,b], \tag{3.2}$$

where the integral in (3.2) is taken in the sense of Riemann-Stieltjes.

Proof. We have:

$$\begin{aligned}\int_a^b Q(t,s)\,df(s) &= \int_a^t (t-b)\,df(s) + \int_t^b (t-a)\,df(s) \\ &= (t-b)\int_a^t df(s) + (t-a)\int_t^b df(s) \\ &= (t-b)\,[f(t)-f(a)] + (t-a)\,[f(b)-f(t)] \\ &= (b-a)\,\Phi_f(t)\end{aligned}$$

and the identity is proved. ∎

The following estimation result holds.

Theorem 3. *If $f:[a,b]\to\mathbb{R}$ is of bounded variation, then*

$$(3.3)\qquad |\Phi_f(t)| \le \left(\frac{b-t}{b-a}\right)\cdot\bigvee_a^t(f) + \left(\frac{t-a}{b-a}\right)\cdot\bigvee_t^b(f)$$
$$\le \begin{cases} \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f); \\ \left[\left(\frac{b-t}{b-a}\right)^p+\left(\frac{t-a}{b-a}\right)^p\right]^{\frac{1}{p}}\left[\left(\bigvee_a^t(f)\right)^q+\left(\bigvee_t^b(f)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^t(f)-\bigvee_t^b(f)\right|. \end{cases}$$

The first inequality in (3.3) is sharp. The constant $\frac{1}{2}$ is best possible in the first and third branches.

Proof. We use the fact that for $p:[\alpha,\beta]\to\mathbb{R}$ continuous and $v:[\alpha,\beta]\to\mathbb{R}$ of bounded variation the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and

$$\left|\int_\alpha^\beta p(t)\,dv(t)\right| \le \sup_{t\in[\alpha,\beta]}|p(t)|\bigvee_\alpha^\beta(v).$$

Then, by the identity (3.2), we have

$$\begin{aligned}|\Phi_f(t)| &\le \frac{1}{b-a}\left|(t-b)\int_a^t df(s)+(t-a)\int_t^b df(s)\right| \\ &\le \frac{1}{b-a}\left[(b-t)\left|\int_a^t df(s)\right|+(t-a)\left|\int_t^b df(s)\right|\right] \\ &\le \frac{1}{b-a}\left[(b-t)\bigvee_a^t(f)+(t-a)\bigvee_t^b(f)\right],\end{aligned}$$

and the first inequality in (3.3) is proved.

Now, by the Hölder inequality, we have

$$(b-t)\bigvee_a^t(f)+(t-a)\bigvee_t^b(f)\leq\begin{cases}\max\{b-t,t-a\}\left[\bigvee_a^t(f)+\bigvee_t^b(f)\right]; \\ \left[(b-t)^p+(t-a)^p\right]^{\frac{1}{p}}\left[\left(\bigvee_a^t(f)\right)^q+\left(\bigvee_t^b(f)\right)^q\right]^{\frac{1}{q}} \\ \qquad\qquad\text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (b-t+t-a)\max\left\{\bigvee_a^t(f),\bigvee_t^b(f)\right\},\end{cases}$$

which produces the last part of (3.3).

For $t=\frac{1}{2}(a+b)$, (3.3) becomes

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{2}\bigvee_a^b(f).$$

Assume that there exists a constant $A>0$ such that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right|\leq A\bigvee_a^b(f). \tag{3.4}$$

If in this inequality we choose $f:[a,b]\to\mathbb{R}$, $f(t)=\left|t-\frac{a+b}{2}\right|$, then we deduce $\frac{b-a}{2}\leq A(b-a)$, which implies that $A\geq\frac{1}{2}$. ∎

Corollary 1. *If $f:[a,b]\to\mathbb{R}$ is L_1-Lipschitzian on $[a,t]$ and L_2-Lipschitzian on $[t,b]$, $L_1,L_2>0$, then*

$$|\Phi_f(t)|\leq\frac{(b-t)(t-a)}{b-a}(L_1+L_2)\leq\frac{1}{4}(b-a)(L_1+L_2) \tag{3.5}$$

for any $t\in[a,b]$.

In particular, if f is $L-$Lipschitzian on $[a,b]$, then

$$|\Phi_f(t)|\leq\frac{2(b-t)(t-a)}{b-a}L\leq\frac{1}{2}(b-a)L. \tag{3.6}$$

The constants $\frac{1}{4}$, 2 and $\frac{1}{2}$ are best possible.

The proof is obvious by Theorem 3 on taking into account that any $L-$Lipschitzian function is of bounded variation and $\bigvee_a^b(f)\leq(b-a)L$. The sharpness of the constants follows by choosing the function $f:[a,b]\to\mathbb{R}$, $f(t)=\left|t-\frac{a+b}{2}\right|$ which is Lipschitzian with $L=1$.

Corollary 2. *If $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$, then*

$$|\Phi_f(t)| \le \left(\frac{b-t}{b-a}\right)[f(t)-f(a)] + \left(\frac{t-a}{b-a}\right)[f(b)-f(t)] \tag{3.7}$$

$$\le \begin{cases} \left[\frac{1}{2}+\left|\frac{t-\frac{a+b}{2}}{b-a}\right|\right][f(b)-f(a)]; \\ \left[\left(\frac{b-t}{b-a}\right)^p+\left(\frac{t-a}{b-a}\right)^p\right]^{\frac{1}{p}}\left[[f(t)-f(a)]^q+[f(b)-f(t)]^q\right]^{\frac{1}{q}} \\ \qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}[f(b)-f(a)]+\frac{1}{2}\left|f(t)-\frac{f(a)+f(b)}{2}\right|. \end{cases}$$

The first inequality and the constant $\frac{1}{2}$ in the first branch of the second inequality are sharp.

The inequality is obvious from (3.3). For $t=\frac{a+b}{2}$, we get in (3.7)

$$\left|f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{2}\right| \le \frac{1}{2}[f(b)-f(a)]. \tag{3.8}$$

In (3.8), the constant $\frac{1}{2}$ is sharp since for the monotonic nondecreasing function $f:[a,b]\to\mathbb{R}$

$$f(t)=\begin{cases} 0 & \text{if } t\in\left[a,\frac{a+b}{2}\right]; \\ 1 & \text{if } t\in\left(\frac{a+b}{2},b\right], \end{cases}$$

we obtain in both sides of (3.8) the same quantity $\frac{1}{2}$.

4. The Case when f is Absolutely Continuous

Now, if $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then f is differentiable almost everywhere and $\int_a^b f'(s)\,ds = f(b)-f(a)$, where the integral is taken in the Lebesgue sense, and we can state the following representation result.

Lemma 2. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous, then*

$$\Phi_f(t)=\frac{1}{b-a}\int_a^b Q(t,s)\,f'(s)\,ds, \quad t\in[a,b], \tag{4.1}$$

where the integral is in the Lebesgue sense and Q has been defined in (3.1).

The proof is similar to the proof of Lemma 1 and the details are omitted.

We define the Lebesgue $p-$norms as follows:

$$\|g\|_{[\alpha,\beta],s} := \begin{cases} ess\sup\limits_{t\in[\alpha,\beta]}|g(t)| & \text{if } s=\infty, \\ \left(\int_\alpha^\beta |g(t)|^s\,dt\right)^{\frac{1}{s}} & \text{if } s\in[1,\infty). \end{cases}$$

The following estimation holds:

Theorem 4. *If f is absolutely continuous, then*

(4.2)

$$
\begin{aligned}
|\Phi_f(t)| &\leq \left(\frac{b-t}{b-a}\right) \cdot \|f'\|_{[a,t],1} + \left(\frac{t-a}{b-a}\right) \cdot \|f'\|_{[t,b],1} \\
&\leq \begin{cases} \frac{(b-t)(t-a)}{b-a} \|f'\|_{[a,t],\infty} & \text{if } f' \in L_\infty[a,b] \\ \\ \frac{(b-t)(t-a)^{\frac{1}{q}}}{b-a} \|f'\|_{[a,t],p} & \text{if } f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases} \\
&\quad + \begin{cases} \frac{(t-a)(b-t)}{b-a} \|f'\|_{[t,b],\infty} & \text{if } f' \in L_\infty[a,b] \\ \\ \frac{(t-a)(b-t)^{\frac{1}{\beta}}}{b-a} \|f'\|_{[t,b],\alpha} & \text{if } f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1 \end{cases}
\end{aligned}
$$

where the second part should be seen as all four possible combinations.

Proof. The first inequality holds from the representation (4.1) on taking the modulus and applying its properties.

By the integral Hölder inequality, we have

$$
\int_a^t |f'(s)|\,ds \leq \begin{cases} (t-a)\, ess \sup\limits_{s\in[a,t]} |f'(s)| & \text{if } f' \in L_\infty[a,b] \\ (t-a)^{\frac{1}{q}} \left(\int_a^t |f'(s)|^p\,ds\right)^{\frac{1}{p}} & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}
$$

and

$$
\int_t^b |f'(s)|\,ds \leq \begin{cases} (b-t)\, ess \sup\limits_{s\in[t,b]} |f'(s)| & \text{if } f' \in L_\infty[a,b] \\ (b-t)^{\frac{1}{q}} \left(\int_t^b |f'(s)|^p\,ds\right)^{\frac{1}{p}} & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}
$$

which provides the second part of (4.2). ∎

Remark 1. *Some particular inequalities of interest are as follows. If $f' \in L_\infty[a,b]$, then*

$$
\begin{aligned}
|\Phi_f(t)| &\leq \frac{(b-t)(t-a)}{b-a} \left[\|f'\|_{[a,t],\infty} + \|f'\|_{[t,b],\infty}\right] \\
&\leq \frac{2(b-t)(t-a)}{b-a} \|f'\|_{[a,b],\infty} \leq \frac{1}{2}(b-a) \|f'\|_{[a,b],\infty},
\end{aligned} \tag{4.3}
$$

for any $t \in [a,b]$. The first inequality in (4.3) and the constants 2 and $\frac{1}{2}$ are best possible.

If $f' \in L_p[a,b]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then

$$
\begin{aligned}
|\Phi_f(t)| &\leq \left[\frac{(b-t)(t-a)}{b-a}\right]^{\frac{1}{q}} \left[\left(\frac{b-t}{b-a}\right)^{\frac{1}{p}} \|f'\|_{[a,t],p} + \left(\frac{t-a}{b-a}\right)^{\frac{1}{p}} \|f'\|_{[t,b],p}\right] \\
&\leq \left[\frac{(b-t)(t-a)}{b-a}\right]^{\frac{1}{q}} \left[\left(\frac{b-t}{b-a}\right)^{\frac{q}{p}} + \left(\frac{t-a}{b-a}\right)^{\frac{q}{p}}\right]^{\frac{1}{q}} \|f'\|_{[a,b],p}
\end{aligned} \tag{4.4}
$$

for any $t \in [a,b]$.

In particular, for $p = q = 2$, we have

$$|\Phi_f(t)| \leq \sqrt{\frac{(b-t)(t-a)}{b-a}} \left[\sqrt{\frac{b-t}{b-a}} \cdot \|f'\|_{[a,t],2} + \sqrt{\frac{t-a}{b-a}} \cdot \|f'\|_{[t,b],2} \right] \leq \sqrt{\frac{(b-t)(t-a)}{b-a}} \|f'\|_{[a,b],2} \tag{4.5}$$

for any $t \in [a,b]$.

5. The Case when f' is of Bounded Variation

The following representation of the error Φ_f can be stated:

Lemma 3. *If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and such that the derivative f' is Riemann integrable on $[a,b]$, then we have the following representation in terms of the Riemann-Stieltjes integral:*

$$\Phi_f(t) = \frac{1}{b-a} \int_a^b K(t,s)\, df'(s), \quad t \in [a,b], \tag{5.1}$$

where the kernel $K : [a,b]^2 \to \mathbb{R}$ is given by

$$K(t,s) := \begin{cases} (b-t)(s-a) & \text{if } a \leq s \leq t \\ (t-a)(b-s) & \text{if } t < s \leq b. \end{cases} \tag{5.2}$$

Proof. Since f' is Riemann integrable on $[a,b]$, it follows that the Riemann-Stieltjes integrals $\int_a^t (s-a)\, df'(s)$ and $\int_t^b (b-s)\, df'(s)$ exist for each $t \in [a,b]$. Now, integrating by parts in the Riemann-Stieltjes integral, we have:

$$\int_a^b K(t,s)\, df'(s) = (b-t) \int_a^t (s-a)\, df'(s) + (t-a) \int_t^b (b-s)\, df'(s)$$
$$= (b-t) \left[(s-a) f'(s) \Big|_a^t - \int_a^t f'(s)\, ds \right] + (t-a) \left[(b-s) f'(s) \Big|_t^b - \int_t^b f'(s)\, ds \right]$$

$$= (b-t)\left[(t-a) f'(t) - (f(t) - f(a))\right] + (t-a)\left[-(b-t) f'(t) + f(b) - f(t)\right]$$
$$= (t-a)\left[f(b) - f(t)\right] - (b-t)\left[f(t) - f(a)\right] = (b-a)\,\Phi_f(t)$$

for any $t \in [a,b]$, which provides the desired representation (5.1). ∎

Remark 2. *If we define* $\Delta_f : (a,b) \to \mathbb{R}$,

$$\Delta_f(t) = \frac{f(b) - f(t)}{b-t} - \frac{f(t) - f(a)}{t-a},$$

then by the above identity (5.1), we have the representation

$$\Delta_f(t) = \frac{1}{(t-a)(b-t)} \int_a^b K(t,s)\, df'(s) = \int_a^b R(t,s)\, df'(s), \quad t \in (a,b), \tag{5.3}$$

where the new kernel $R : (a,b)^2 \to \mathbb{R}$ *is defined by*

$$R(t,s) := \begin{cases} \frac{s-a}{t-a} & \text{if } a < s \le t \\ \\ \frac{b-s}{b-t} & \text{if } t < s < b. \end{cases}$$

We notice that, for $f(s) := \int_a^s g(z)\, dz$, *the last equality in (5.3) produces the following identity:*

$$\frac{1}{b-t} \int_t^b g(z)\, dz - \frac{1}{t-a} \int_a^t g(z)\, dz = \int_a^b R(t,s)\, dg(s), \tag{5.4}$$

which has been obtained by P. Cerone in [3] *(see eq. (2.12)).*

Notice that, in (5.4), the function g *can be Riemann integrable and not only absolutely continuous as assumed in* [3].

Theorem 5. *Assume that* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$. *If* f' *is of bounded variation on* $[a,b]$, *then*

$$|\Phi_f(t)| \le \frac{(t-a)(b-t)}{b-a} \cdot \bigvee_a^b (f') \le \frac{1}{4}(b-a) \bigvee_a^b (f'), \tag{5.5}$$

where $\bigvee_a^b (f')$ *denotes the total variation of* f' *on* $[a,b]$.

The inequalities are sharp and the constant $\frac{1}{4}$ *is best possible.*

Proof. It is well known that, if $p : [\alpha, \beta] \to \mathbb{R}$ is continuous and $v : [\alpha, \beta] \to \mathbb{R}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\, dv(s)$ exists and

$$\left| \int_\alpha^\beta p(s)\, dv(s) \right| \le \sup_{s \in [\alpha,\beta]} |p(s)| \bigvee_\alpha^\beta (v).$$

Now, utilising the representation (5.1) and the above property, we have

$$\begin{aligned} |\Phi_f(t)| &= \frac{1}{b-a} \left| (b-t) \int_a^t (s-a)\, df'(s) + (t-a) \int_t^b (t-s)\, df'(s) \right| \\ &\le \frac{1}{b-a} \left[(b-t) \left| \int_a^t (s-a)\, df'(s) \right| + (t-a) \left| \int_t^b (t-s)\, df'(s) \right| \right] \\ &\le \frac{1}{b-a} \left[(b-t) \bigvee_a^t (f') \sup_{s \in [a,t]} (s-a) + (t-a) \bigvee_t^b (f') \sup_{s \in [t,b]} (t-s) \right] \end{aligned} \tag{5.6}$$

$$= \frac{(t-a)(b-t)}{b-a}\left[\bigvee_a^t (f') + \bigvee_t^b (f')\right] = \frac{(t-a)(b-t)}{b-a}\bigvee_a^b (f').$$

The last part of (5.5) is obvious by the fact that $(t-a)(b-t) \leq \frac{1}{4}(b-a)^2$, $t \in [a,b]$.

For the sharpness of the inequalities in (5.5), assume that there exists $F, G > 0$ such that

$$|\Phi_f(t)| \leq F \cdot \frac{(t-a)(b-t)}{b-a}\bigvee_a^b (f') \leq G(b-a)\bigvee_a^b (f'),$$

with f as in the assumption of the theorem. Then, for $t = \frac{a+b}{2}$, we get

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{4}F(b-a)\bigvee_a^b (f') \leq G(b-a)\bigvee_a^b (f'). \tag{5.7}$$

Consider the function $f : [a,b] \to \mathbb{R}$, $f(t) = \left|t - \frac{a+b}{2}\right|$. This function is absolutely continuous, $f'(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b] \setminus \left\{\frac{a+b}{2}\right\}$ and $\bigvee_a^b (f') = 2$. Thus, (5.7) becomes

$$\frac{b-a}{2} \leq \frac{1}{2}F(b-a) \leq 2G(b-a),$$

which implies that $F \geq 1$ and $G \geq \frac{1}{4}$. ∎

6. The Case when f' is Lipschitzian

The case when the derivative is a Lipschitzian function provides better accuracy in approximating the function f by the straight line d_f as follows:

Theorem 6. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. If f' is K_1-Lipschitzian on $[a,t]$ and K_2-Lipschitzian on $[t,b]$ $(t \in [a,b])$, then*

$$\begin{aligned} |\Phi_f(t)| &\leq \frac{1}{2} \cdot \frac{(t-a)(b-t)}{b-a}\left[(K_1 - K_2)t + K_2 b - K_1 a\right] \\ &\leq \frac{1}{8} \cdot (b-a)\left[(K_1 - K_2)t + K_2 b - K_1 a\right], \quad t \in [a,b]. \end{aligned} \tag{6.1}$$

In particular, if f' is $K-$Lipschitzian on $[a,b]$, then

$$|\Phi_f(t)| \leq \frac{1}{2}(b-t)(t-a)K \leq \frac{1}{8}(b-a)^2 K, \quad t \in [a,b]. \tag{6.2}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

Proof. We utilize the fact that for an $L-$Lipschitzian function, $p : [\alpha,\beta] \to \mathbb{R}$ and a Riemann integrable function $v : [\alpha,\beta] \to \mathbb{R}$, the Riemann-Stieltjes integral $\int_\alpha^\beta p(s)\,dv(s)$ exists and

$$\left|\int_\alpha^\beta p(s)\,dv(s)\right| \leq L\int_\alpha^\beta |p(s)|\,ds.$$

Then we have

$$\left|\int_a^t (s-a)\,df'(s)\right| \leq K_1 \cdot \int_a^t (s-a)\,ds = \frac{1}{2}K_1(t-a)^2 \tag{6.3}$$

and

$$\left|\int_t^b (t-s)\,df'(s)\right| \leq K_2 \cdot \int_t^b (t-s)\,ds = \frac{1}{2} K_2 (b-t)^2 . \tag{6.4}$$

Now, on making use of the inequality (5.6) we have, by (6.3) and (6.4), that

$$\begin{aligned} |\Phi_f(t)| &\leq \frac{1}{b-a}\left[\frac{1}{2}(b-t)(t-a)^2 \cdot K_1 + \frac{1}{2}(t-a)(b-t)^2 \cdot K_2\right] \\ &= \frac{1}{2} \cdot \frac{(t-a)(b-t)}{b-a}\left[L_1(t-a) + L_2(b-t)\right], \end{aligned}$$

which produces the first inequality in (6.1). The other inequalities are obvious.

To prove the sharpness of the constants in (6.2), let us assume that there exist $H, K > 0$ so that

$$|\Phi_f(t)| \leq H(b-t)(t-a)L \leq K(b-a)^2 L \tag{6.5}$$

for any $t \in [a,b]$ and f an $L-$Lipschitzian function on $[a,b]$. For $t = \frac{a+b}{2}$ we get from (6.5) that

$$\left|\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{4} HL(b-a)^2 \leq LK(b-a)^2 . \tag{6.6}$$

Consider $f : [a,b] \to \mathbb{R}$, $f(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$. Then $f'(t) = t - \frac{a+b}{2}$ is Lipschitzian with the constant $L = 1$ and (6.6) becomes

$$\frac{1}{8}(b-a)^2 \leq \frac{1}{4} H(b-a)^2 \leq K(b-a)^2 ,$$

which implies that $H \geq \frac{1}{2}$ and $K \geq \frac{1}{8}$. ∎

7. The Case when f' is Absolutely Continuous

The following representation result also holds.

Lemma 4. *If $f : [a,b] \to \mathbb{R}$ is differentiable and the derivative f' is absolutely continuous, then*

$$\Phi_f(t) = \frac{1}{b-a}\int_a^b K(t,s) f''(s)\,ds \tag{7.1}$$

for any $t \in [a,b]$, where the integral in (7.1) is considered in the Lebesgue sense.

The proof is similar to the one in Lemma 3 on integrating by parts in the Lebesgue integral $\int_a^b K(t,s) f''(s)\,ds$. The details are omitted.

Theorem 7. *If f is as in Lemma 4, then*

$$|\Phi_f(t)| \leq \frac{(b-t)(t-a)}{b-a} \cdot K(t), \quad t \in [a,b], \tag{7.2}$$

where

$$K(t) := \begin{cases} \|f''\|_{[a,t],1}\,; & \\ \frac{(t-a)^{1/q}}{(q+1)^{1/q}} \|f''\|_{[a,t],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ & f'' \in L_p[a,b]\,; \\ \frac{1}{2}(t-a)\|f''\|_{[a,t],\infty} & \text{if } f''\in L_\infty[a,b]\,; \end{cases} + \begin{cases} \|f''\|_{[t,b],1} & \\ \frac{(b-t)^{1/\beta}}{(\beta+1)^{1/\beta}} \|f''\|_{[t,b],\alpha} & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & f''\in L_\alpha[a,b]\,; \\ \frac{1}{2}(b-t)\|f''\|_{[t,b],\infty} & \text{if } f''\in L_\infty[a,b]\,; \end{cases} \tag{7.3}$$

and the definition of K should be seen as all 9 possible combinations.

Proof. We have, by (5.6) that

$$|\Phi_f(t)| \le \frac{1}{b-a}\left[(b-t)\left|\int_a^t (s-a) f''(s)\,ds\right| + (t-a)\left|\int_t^b (t-s) f''(s)\,ds\right|\right] \tag{7.4}$$

for any $t\in[a,b]$.

Utilising Hölder's inequality, we have

$$\left|\int_a^t (s-a) f''(s)\,ds\right| \le \begin{cases} \sup\limits_{s\in[a,t]} (s-a)\int_a^t |f''(s)|\,ds; & \\ \left(\int_a^t (s-a)^q\,ds\right)^{1/q}\left(\int_a^t |f''(s)|^p\,ds\right)^{1/p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ & f''\in L_p[a,b]\,; \\ ess\sup\limits_{s\in[a,t]} |f''(s)|\int_a^t (s-a)\,ds & \text{if } f''\in L_\infty[a,b]\,; \end{cases}$$

$$= \begin{cases} (t-a)\|f''\|_{[a,t],1}\,, & \\ \frac{(t-a)^{1+1/q}}{(q+1)^{1/q}}\|f''\|_{[a,t],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ & f''\in L_p[a,b]\,; \\ \dfrac{1}{2}(t-a)^2\|f''\|_{[a,t],\infty} & \text{if } f''\in L_\infty[a,b]\,; \end{cases} \tag{7.5}$$

and, similarly,

$$\left|\int_t^b (b-s) f''(s)\,ds\right| \le \begin{cases} (b-t)\|f''\|_{[t,b],1} & \\ \frac{(b-t)^{1+1/\beta}}{(\beta+1)^{1/\beta}}\|f''\|_{[t,b],\alpha} & \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ & f''\in L_\alpha[a,b]\,; \\ \dfrac{1}{2}(b-t)^2\|f''\|_{[t,b],\infty} & \text{if } f''\in L_\infty[a,b]\,, \end{cases} \tag{7.6}$$

for any $t\in[a,b]$.

Finally, on making use of (7.4) – (7.6), we deduce the desired inequality (7.2). ∎

Remark 3. *The inequalities in (7.2) have some instances of interest that are useful in applications. For example, in terms of the sup-norm we have:*

$$|\Phi_f(t)| \le \frac{1}{2} \cdot \frac{(b-t)(t-a)}{b-a} \left[(t-a) \|f''\|_{[a,t],\infty} + (b-t) \|f''\|_{[t,b],\infty} \right] \le \frac{1}{2} \cdot (b-t)(t-a) \|f''\|_{[a,b],\infty}, \quad t \in [a,b], \tag{7.7}$$

where $f'' \in L_\infty [a,b]$. *The constant* $\frac{1}{2}$ *is best possible in both inequalities. The function* $f(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2$ *produces an equality in (7.7) for* $t = \frac{a+b}{2}$.

If we assume that $\alpha = p$, $\beta = q$ *in (7.2), then we also have:*

$$|\Phi_f(t)| \le \frac{(b-t)(t-a)}{(q+1)^{1/q}(b-a)} \left[(t-a)^{1/q} \|f''\|_{[a,t],p} + (b-t)^{1/q} \|f''\|_{[t,b],p} \right] \le \frac{(b-t)(t-a)}{(q+1)^{1/q}(b-a)^{1/p}} \|f''\|_{[a,b],p}, \quad t \in [a,b], \tag{7.8}$$

for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f'' \in L_p [a,b]$, *since, by Hölder's inequality, we have*

$$\begin{aligned} & (t-a)^{1/q} \|f''\|_{[a,t],p} + (b-t)^{1/q} \|f''\|_{[t,b],p} \\ & \le \left[(t-a)^{q/q} + (b-t)^{q/q} \right]^{1/q} \left[\|f''\|^p_{[a,t],p} + \|f''\|^p_{[t,b],p} \right]^{1/p} \\ & = (b-a)^{1/p} \|f''\|_{[a,b],p}. \end{aligned}$$

In the case that $p = q = 2$, *we get the following inequality for the Euclidean norm* $\|f''\|_{[a,b],2}$:

$$|\Phi_f(t)| \le \frac{\sqrt{3}}{3} \cdot \frac{(b-t)(t-a)}{b-a} \left[\sqrt{t-a} \|f''\|_{[a,t],2} + \sqrt{b-t} \|f''\|_{[t,b],2} \right] \le \frac{\sqrt{3}}{3} \cdot \frac{(b-t)(t-a)}{b-a} \|f''\|_{[a,b],2}, \quad t \in [a,b]. \tag{7.9}$$

It is an open question whether or not the constant $\frac{\sqrt{3}}{3}$ *is best possible in (7.9).*

Finally, from (7.2) we also have:

$$|\Phi_f(t)| \le \frac{(b-t)(t-a)}{b-a} \|f''\|_{[a,b],1} \le \frac{1}{4}(b-a) \|f''\|_{[a,b],1} \tag{7.10}$$

for any $t \in [a,b]$.

8. Applications for Weighted Means

For a function $f : [a,b] \to \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n) \in [a,b]^n$ and $\mathbf{p} = (p_1, \dots, p_n)$ a probability sequence, i.e., $p_i \ge 0$, $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n p_i = 1$, we define the mean:

$$M_f(\mathbf{p}; \mathbf{x}) := \sum_{i=1}^n p_i f(x_i). \tag{8.1}$$

If $f(t) = t$, $t \in [a,b]$, then

$$M_f(\mathbf{p}; \mathbf{x}) = A(\mathbf{p}; \mathbf{x}) = \sum_{i=1}^n p_i x_i,$$

which is the *arithmetic mean* of $\mathbf{x}$ with the weights $\mathbf{p}$.

The main aim of the present section is to provide sharp bounds for the error in approximating $\mathcal{M}_f(\mathbf{p};\mathbf{x})$ in terms of the simpler quantity

$$f(a)\cdot\frac{b-A(\mathbf{p};\mathbf{x})}{b-a}+f(b)\cdot\frac{A(\mathbf{p};\mathbf{x})-a}{b-a}. \tag{8.2}$$

The following proposition contains some results of this type.

Proposition 1. *Let $f:[a,b]\to\mathbb{R}$ be a bounded function on $[a,b]$, $\mathbf{x}=(x_1,\dots,x_n)\in[a,b]^n$ and $\mathbf{p}$ a probability sequence. Define the error functional $\mathcal{E}_f(\mathbf{p};\mathbf{x})$ by:*

$$\mathcal{E}_f(\mathbf{p};\mathbf{x}):=f(a)\cdot\frac{b-A(\mathbf{p};\mathbf{x})}{b-a}+f(b)\cdot\frac{A(\mathbf{p};\mathbf{x})-a}{b-a}-\mathcal{M}_f(\mathbf{p};\mathbf{x}). \tag{8.3}$$

(i) *If $-\infty<m\le f(t)\le M<\infty$ for any $t\in[a,b]$, then*

$$|\mathcal{E}_f(\mathbf{p};\mathbf{x})|\le M-m. \tag{8.4}$$

The inequality is sharp.

(ii) *If $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$|\mathcal{E}_f(\mathbf{p};\mathbf{x})|\le\left[\frac{1}{2}+\sum_{i=1}^n p_i\left|\frac{x_i-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f). \tag{8.5}$$

The constant $\frac{1}{2}$ is best possible in (8.5).

(iii) *If $f:[a,b]\to\mathbb{R}$ is $L-$Lipschitzian on $[a,b]$, then*

$$\begin{aligned}|\mathcal{E}_f(\mathbf{p};\mathbf{x})|&\le\frac{2L}{b-a}\sum_{i=1}^n p_i(b-x_i)(x_i-a)\\&\le\frac{2L}{b-a}[b-A(\mathbf{p};\mathbf{x})][A(\mathbf{p};\mathbf{x})-a]\le\frac{1}{2}L(b-a).\end{aligned} \tag{8.6}$$

All the inequalities in (8.6) are sharp.

Proof. Let us prove only the inequality (8.6). The other inequalities follow likewise. Applying the inequality (3.6) for $t=x_i$, $i\in\{1,\dots,n\}$, we have

$$\left|f(x_i)-\frac{f(a)(b-x_i)+f(b)(x_i-a)}{b-a}\right|\le\frac{2L}{b-a}(b-x_i)(x_i-a), \tag{8.7}$$

for any $i\in\{1,\dots,n\}$. Multiplying (8.7) with p_i, summing over i from 1 to n and utilising the generalised triangle inequality $\sum_{i=1}^n|\alpha_i|\ge\left|\sum_{i=1}^n\alpha_i\right|$, we deduce the first inequality in (8.6).

Further, we use the following Čebyšev inequality:

$$\sum_{i=1}^n p_i\alpha_i\beta_i\le\sum_{i=1}^n p_i\alpha_i\sum_{i=1}^n p_i\beta_i, \tag{8.8}$$

provided that $p_i\ge0$, $\sum_{i=1}^n p_i=1$ and $(\alpha_i)_{i=\overline{1,n}}$, $(\beta_i)_{i=\overline{1,n}}$ are *asynchronous,* i.e.,

$$(\alpha_i-\alpha_j)(\beta_i-\beta_j)\le0\quad\text{for any } i,j\in\{1,\dots,n\}.$$

Then we have from (8.8)

$$\begin{aligned}\sum_{i=1}^n p_i(b-x_i)(x_i-a)&\le\sum_{i=1}^n p_i(b-x_i)\sum_{i=1}^n p_i(x_i-a)\\&=[b-A(\mathbf{p};\mathbf{x})][A(\mathbf{p};\mathbf{x})-a]\end{aligned}$$

and the second inequality in (8.6) is proved. The last part is obvious.

The sharpness of the inequality follows from the case $n = 1$. The details are omitted. ∎

If $f : [a,b] \to \mathbb{R}$ is a convex function, then the following result is known in the literature as the (discrete) *Lah-Ribarić inequality*:

$$\sum_{i=1}^{n} p_i f(x_i) \leq \frac{1}{b-a} \{ f(a) [b - A(\mathbf{p};\mathbf{x})] + f(b) [A(\mathbf{p};\mathbf{x}) - a] \} . \tag{8.9}$$

For a generalisation to positive linear functional that incorporates both the original Lah-Ribarić integral inequality and the discrete version of it due to Beesack and Pečaric [2], see [12, p. 98].

In terms of the error functional $\mathcal{E}_f(\mathbf{p};\mathbf{x})$, we then have $\mathcal{E}_f(\mathbf{p};\mathbf{x}) \geq 0$, when f is convex and $\mathbf{p}, \mathbf{x}$ are as above. Now, on utilising Theorem 2, we can state the following reverse of the Lah-Ribarić inequality (8.9).

Proposition 2. *If $f : [a,b] \to \mathbb{R}$ is convex on $[a,b]$ and the lateral derivatives $f'_-(b)$, $f'_+(a)$ are finite, then*

$$\begin{aligned} (0 \leq) \mathcal{E}_f(\mathbf{p};\mathbf{x}) &\leq \frac{f'_-(b) - f'_+(a)}{b-a} \sum_{i=1}^{n} p_i (b - x_i)(x_i - a) \\ &\leq \frac{f'_-(b) - f'_+(a)}{b-a} [b - A(\mathbf{p};\mathbf{x})] [A(\mathbf{p};\mathbf{x}) - a] \\ &\leq \frac{1}{4} (b-a) \left[f'_-(b) - f'_+(a) \right] . \end{aligned} \tag{8.10}$$

The inequalities are sharp and $\frac{1}{4}$ is best possible.

The following results in terms of the derivative of a function f can be stated as well.

Proposition 3. *Assume that $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.*

(i) *If f' is of bounded variation on $[a,b]$, then*

$$\begin{aligned} |\mathcal{E}_f(\mathbf{p};\mathbf{x})| &\leq \frac{1}{b-a} \bigvee_a^b (f') \sum_{i=1}^{n} p_i (b - x_i)(x_i - a) \\ &\leq \frac{1}{b-a} \bigvee_a^b (f') [A(\mathbf{p};\mathbf{x}) - a] [b - A(\mathbf{p};\mathbf{x})] \leq \frac{1}{4} (b-a) \bigvee_a^b (f') . \end{aligned} \tag{8.11}$$

All inequalities in (8.11) are sharp. The constant $\frac{1}{4}$ is best possible.

(ii) *If f' is $K-$Lipschitzian on $[a,b]$ $(K > 0)$, then*

$$\begin{aligned} |\mathcal{E}_f(\mathbf{p};\mathbf{x})| &\leq \frac{1}{2} K \sum_{i=1}^{n} p_i (b - x_i)(x_i - a) \\ &\leq \frac{1}{2} K [b - A(\mathbf{p};\mathbf{x})] [A(\mathbf{p};\mathbf{x}) - a] \leq \frac{1}{8} (b-a)^2 K. \end{aligned} \tag{8.12}$$

The constants $\frac{1}{2}$ and $\frac{1}{8}$ are best possible.

The proof is obvious by Theorem 5 and Theorem 6 and the details are omitted.

The above results can be useful in providing various inequalities between means. For instance, if we denote by $G(\mathbf{p},\mathbf{x})$ the *geometric mean* $\prod_{i=1}^{n} x_i^{p_i}$, then for the convex function $f(t) = -\ln t$, we have for $0 < m \le x_i \le M < \infty$, $i \in \{1, \dots, n\}$ that:

$$\mathcal{E}_f(\mathbf{p};\mathbf{x}) = \ln G(\mathbf{p};\mathbf{x}) - \ln \left[m^{\frac{M-A(\mathbf{p};\mathbf{x})}{M-m}} \cdot M^{\frac{A(\mathbf{p};\mathbf{x})-m}{M-m}} \right] = \ln \left[\frac{G(\mathbf{p};\mathbf{x})}{m^{\frac{M-A(\mathbf{p};\mathbf{x})}{M-m}} \cdot M^{\frac{A(\mathbf{p};\mathbf{x})-m}{M-m}}} \right],$$

$$\bigvee_m^M (f) = \ln\left(\frac{M}{m}\right),$$

f is $L-$Lipschitzian with the constant $L = \|f'\|_{\infty,[m,M]} = \frac{1}{m}$ and

$$\frac{f'(M) - f'(m)}{M-m} = \frac{1}{mM}, \qquad \bigvee_m^M (f') = \frac{M-m}{mM}.$$

Also, f' is $K-$Lipschitzian with the constant $K = \|f''\|_{\infty,[m,M]} = \frac{1}{m^2}$.

Applying Proposition 1, we get

$$0 \le \ln \left[\frac{G(\mathbf{p};\mathbf{x})}{m^{\frac{M-A(\mathbf{p};\mathbf{x})}{M-m}} \cdot M^{\frac{A(\mathbf{p};\mathbf{x})-m}{M-m}}} \right] \le \left[\frac{1}{2} + \sum_{i=1}^n p_i \left| \frac{x_i - \frac{m+M}{2}}{M-m} \right| \right] \ln\left(\frac{M}{m}\right),$$

while from Propositions $2-3$ we get

$$\begin{aligned}
0 &\le \ln \left[\frac{G(\mathbf{p};\mathbf{x})}{m^{\frac{M-A(\mathbf{p};\mathbf{x})}{M-m}} \cdot M^{\frac{A(\mathbf{p};\mathbf{x})-m}{M-m}}} \right] \\
&\le \min\left\{ \frac{2}{m(M-m)}, \frac{1}{mM}, \frac{1}{2m^2} \right\} \cdot \sum_{i=1}^n p_i (M - x_i)(x_i - m) \\
&\le \min\left\{ \frac{2}{m(M-m)}, \frac{1}{mM}, \frac{1}{2m^2} \right\} \cdot [M - A(\mathbf{p};\mathbf{x})][A(\mathbf{p};\mathbf{x}) - m] \\
&\le \min\left\{ \frac{M-m}{2m}, \frac{(M-m)^2}{4mM}, \frac{(M-m)^2}{8m^2} \right\}.
\end{aligned}$$

Remark 4. *All the results in this section can be stated for positive linear functionals defined on linear spaces of functions. Applications for Lebesgue integrals in the general setting of measurable spaces can be provided as well. However, for the sake of brevity, we do not state them here.*

9. Applications for f-Divergences

Given a convex function $f : [0, \infty) \to \mathbb{R}$, the *$f$-divergence functional*

$$I_f(\mathbf{p}, \mathbf{q}) := \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right), \tag{9.1}$$

where $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_1, \dots, q_n)$ are positive sequences was introduces by Csiszár in [4], as a generalised measure of information, a "distance function" on the set of probability distributions $\mathbb{P}^n$. As in [4], we interpret undefined expressions by

$$f(0) = \lim_{t\to 0+} f(t), \qquad 0f\left(\frac{0}{0}\right) = 0, \tag{9.2}$$

$$0f\left(\frac{a}{0}\right) = \lim_{q\to 0+} f\left(\frac{a}{q}\right) = a \lim_{t\to\infty} \frac{f(t)}{t}, \quad a > 0.$$

The following results were essentially given by Csiszár and Körner [5]:

(i) *If f is convex, then $I_f(\mathbf{p},\mathbf{q})$ is jointly convex in p and q;*
(ii) *For every $p,q \in R_+^n$, we have*

$$I_f(\mathbf{p},\mathbf{q}) \geq \sum_{j=1}^{n} q_j f\left(\frac{\sum_{j=1}^{n} p_j}{\sum_{j=1}^{n} q_j}\right). \tag{9.3}$$

If f is strictly convex, equality holds in (9.3) iff

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}.$$

If f is normalised, i.e., $f(1) = 0$, then for every $p,q \in R_+^n$ with $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i$, we have the inequality

$$I_f(\mathbf{p},\mathbf{q}) \geq 0. \tag{9.4}$$

In particular, if $\mathbf{p},\mathbf{q} \in \mathbb{P}^n$, then (9.4) holds. This is the well-known *positive property* of the f-divergence.

We now give some examples of divergence measures in Information Theory which are particular cases of $f-$divergences.

(1) **Kullback-Leibler distance** ([10]). The *Kullback-Leibler distance* $D(\cdot,\cdot)$ is defined by

$$D(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

If we choose $f(t) = t\ln t$, $t > 0$, then obviously

$$I_f(\mathbf{p},\mathbf{q}) = D(\mathbf{p},\mathbf{q}).$$

(2) **Variational distance** (l_1-distance). The *variational distance* $V(\cdot,\cdot)$ is defined by

$$V(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} |p_i - q_i|.$$

If we choose $f(t) = |t-1|$, $t \in [0,\infty)$, then we have

$$I_f(\mathbf{p},\mathbf{q}) = V(\mathbf{p},\mathbf{q}).$$

(3) **Hellinger discrimination** ([1]). The *Hellinger discrimination* is defined by $\sqrt{2h^2(\cdot,\cdot)}$, where $h^2(\cdot,\cdot)$ is given by

$$h^2(\mathbf{p},\mathbf{q}) := \frac{1}{2}\sum_{i=1}^{n}\left(\sqrt{p_i} - \sqrt{q_i}\right)^2.$$

It is obvious that if $f(t) = \frac{1}{2}\left(\sqrt{t}-1\right)^2$, then

$$I_f(\mathbf{p},\mathbf{q}) = h^2(\mathbf{p},\mathbf{q}).$$

(4) **Triangular discrimination** ([14]). We define *triangular discrimination* between $\mathbf{p}$ and $\mathbf{q}$ by

$$\Delta(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{n} \frac{|p_i - q_i|^2}{p_i + q_i}.$$

It is obvious that if $f(t) = \frac{(t-1)^2}{t+1}$, $t \in (0,\infty)$, then

$$I_f(\mathbf{p},\mathbf{q}) = \Delta(\mathbf{p},\mathbf{q}).$$

Note that $\sqrt{\Delta(\mathbf{p},\mathbf{q})}$ is known in the literature as the Le Cam distance.

(5) χ^2-**distance.** We define the χ^2-*distance* (chi-square distance) by

$$\chi^2(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i}.$$

It is clear that if $f(t) = (t-1)^2$, $t \in [0,\infty)$, then

$$I_f(\mathbf{p},\mathbf{q}) = \chi^2(\mathbf{p},\mathbf{q}).$$

(6) **Rényi's divergences** ([13]). For $\alpha \in \mathbb{R}\backslash\{0,1\}$, consider

$$\rho_\alpha(\mathbf{p},\mathbf{q}) := \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}.$$

It is obvious that if $f(t) = t^\alpha$ $(t \in (0,\infty))$, then

$$I_f(\mathbf{p},\mathbf{q}) = \rho_\alpha(\mathbf{p},\mathbf{q}).$$

Rényi's divergences $R_\alpha(\mathbf{p},\mathbf{q}) := \frac{1}{\alpha(\alpha-1)} \ln[\rho_\alpha(\mathbf{p},\mathbf{q})]$ have been introduced for all real orders $\alpha \neq 0$, $\alpha \neq 1$ (and continuously extended for $\alpha = 0$ and $\alpha = 1$) in [11], where the reader may find many inequalities valid for these divergences, without, as well as with, some restrictions for $\mathbf{p}$ and $\mathbf{q}$.

For other examples of divergence measures, see the paper [9] and the books [11] and [15], where further references are given.

Now, for $0 < r < 1 < R < \infty$ we consider the expression

$$\frac{1}{R-r}[(R-1)f(r) + (1-r)f(R)]$$

and are interested to compare it with the f-divergence $I_f(\mathbf{p},\mathbf{q})$ which can be extended for larger classes than convex functions with the same definition (9.1) and the same conventions as those from (9.2).

Proposition 4. *Let $f : [r,R] \to \mathbb{R}$ be a bounded function on the interval $[r,R]$ with $0 < r < 1 < R < \infty$. Assume that $\mathbf{p},\mathbf{q} \in \mathbb{P}^n$ are such that*

$$r \leq \frac{p_i}{q_i} \leq R \text{ for each } i \in \{1,...,n\} \tag{9.5}$$

and define the error functional

$$\delta_f(\mathbf{p},\mathbf{q};r,R) := \frac{1}{R-r}[(R-1)f(r) + (1-r)f(R)] - I_f(\mathbf{p},\mathbf{q}).$$

(i) *If $-\infty < m \leq f(t) \leq M < \infty$ for any $t \in [r,R]$, then*

$$|\delta_f(\mathbf{p},\mathbf{q};r,R)| \leq M - m. \tag{9.6}$$

The inequality is sharp.

(ii) *If $f : [r,R] \to \mathbb{R}$ is of bounded variation on $[r,R]$, then*

$$|\delta_f(\mathbf{p},\mathbf{q};r,R)| \leq \left[\frac{1}{2} + \frac{1}{R-r}\sum_{i=1}^{n}\left|p_i - \frac{r+R}{2}\cdot q_i\right|\right] \bigvee_r^R (f). \tag{9.7}$$

The constant $\frac{1}{2}$ is best possible in (8.5).

(iii) *If* $f:[r,R]\to\mathbb{R}$ *is* $L-$*Lipschitzian on* $[r,R]$, *then*

$$|\delta_f(\mathbf{p},\mathbf{q};r,R)| \leq \frac{2L}{R-r}\left[(R-1)(1-r)-\chi^2(p,q)\right] \leq \frac{2L}{R-r}(R-1)(1-r) \leq \frac{1}{2}L(R-r), \tag{9.8}$$

where the K. Pearson χ^2*-divergence is obtained from (9.1) for the convex function* $f(t)=(1-t)^2$, $t\in\mathbb{R}$ *and given by the equivalent expressions:*

$$\chi^2(p,q) := \sum_{j=1}^n q_j\left(\frac{p_j}{q_j}-1\right)^2 = \sum_{j=1}^n \frac{(p_j-q_j)^2}{q_j} = \sum_{j=1}^n \frac{p_j^2}{q_j}-1. \tag{9.9}$$

Proof. The proof follows in a similar manner with the one from Proposition 1 on choosing $a=r, b=R, p_i=q_i$ and $x_i=\frac{p_i}{q_i}$ with $i\in\{1,...,n\}$ and the details are omitted. ∎

In the case of convex functions we have

Proposition 5. *If* $f:[r,R]\to\mathbb{R}$ *is convex on* $[r,R]$ *and the lateral derivatives* $f'_-(R)$, $f'_+(r)$ *are finite, then*

$$(0\leq)\,\delta_f(\mathbf{p},\mathbf{q};r,R) \leq \frac{f'_-(R)-f'_+(r)}{R-r}\left[(R-1)(1-r)-\chi^2(p,q)\right] \leq \frac{f'_-(R)-f'_+(r)}{R-r}(R-1)(1-r) \leq \frac{1}{4}(R-r)\left[f'_-(R)-f'_+(r)\right], \tag{9.10}$$

provided that $\mathbf{p},\mathbf{q}\in\mathbb{P}^n$ *are such that (9.5) holds.*

The inequalities are sharp and $\frac{1}{4}$ *is best possible.*

We notice that the result from Proposition 5 has been firstly obtained by the author in the paper [6].

Finally, we can state:

Proposition 6. *Assume that* $f:[r,R]\to\mathbb{R}$ *is absolutely continuous on* $[r,R]$ *and that* $\mathbf{p},\mathbf{q}\in\mathbb{P}^n$ *are such that (9.5) holds.*

(i) *If* f' *is of bounded variation on* $[r,R]$, *then*

$$|\delta_f(\mathbf{p},\mathbf{q};r,R)| \leq \frac{1}{R-r}\bigvee_r^R(f')\left[(R-1)(1-r)-\chi^2(p,q)\right] \leq \frac{1}{R-r}(R-1)(1-r)\bigvee_r^R(f') \leq \frac{1}{4}(R-r)\bigvee_r^R(f'). \tag{9.11}$$

All inequalities in (9.11) are sharp. The constant $\frac{1}{4}$ *is best possible.*

(ii) *If* f' *is* $K-$*Lipschitzian on* $[r,R]$ $(K>0)$, *then*

$$|\delta_f(\mathbf{p},\mathbf{q};r,R)| \leq \frac{1}{2}K\left[(R-1)(1-r)-\chi^2(p,q)\right] \leq \frac{1}{2}K(R-1)(1-r) \leq \frac{1}{8}(R-r)^2K. \tag{9.12}$$

The constants $\frac{1}{2}$ *and* $\frac{1}{8}$ *are best possible.*

References

[1] R. BERAN, Minimum Hellinger distance estimates for parametric models, *Ann. Statist.,* **5** (1977), 445-463.

[2] P.R. BEESACK and J.E. PEČARIĆ, On the Jessen's inequality for convex functions, *J. Math. Anal. Appl.*, **110**(1985), 536-552.

[3] P. CERONE, On an identity for the Chebychev functional and some ramifications, *J. Ineq. Pure & Appl. Math.*, **3**(1)(2002), Art. 4. [ONLINE: http://jipam.vu.edu.au/article.php?sid=157].

[4] I. CSISZÁR, Information-type measures of differences of probability distributions and indirect observations, *Studia Sci. Math. Hung.*, **2** (1967), 299-318.

[5] I. CSISZÁR and J. KÖRNER, *Information Theory: Coding Theorems for Discrete Memoryless Systems*, Academic Press, New York, 1981.

[6] S.S. DRAGOMIR, Other inequalities for Csiszár divergence and applications, Preprint, RGMIA Monographs, Victoria University, 2000 [ONLINE http://sci.vu.edu.au/~rgmia/Csiszar/ICDApp.pdf].

[7] S.S. DRAGOMIR, A generalisation of Cerone's identity and applications, *Oxford Tamsui J. Math. Sci.* **23**(2007), No. 1, 79-90. Preprint *RGMIA Res. Rep. Coll.*, **8**(2) (2005), Art. 19. [ONLINE: http://rgmia.vu.edu.au/v8n2.html].

[8] S.S. DRAGOMIR, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.*, **20**(2007), 123-130.

[9] J.N. KAPUR, A comparative assessment of various measures of directed divergence, *Advances in Management Studies,* **3** (1984), No. 1, 1-16.

[10] S. KULLBACK and R.A. LEIBLER, On information and sufficiency, *Annals Math. Statist.,* **22** (1951), 79-86.

[11] F. LIESE and I. VAJDA, *Convex Statistical Distances,* Teubner Verlag, Leipzig, 1987.

[12] J.E. PEČARIĆ, F. PROSCHAN and Y.L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc., 1992.

[13] A. RENYI, On measures of entropy and information, *Proc. Fourth Berkeley Symp. Math. Statist. Prob., Vol. 1, University of California Press, Berkeley, 1961.*

[14] F. TOPSØE, Some inequalities for information divergence and related measures of discrimination, *Res. Rep. Coll. RGMIA*, **2** (1) (1999), 85-98.

[15] I. VAJDA, *Theory of Statistical Inference and Information,* Kluwer, Boston, 1989.

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