SOME JENSEN'S TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S.S. DRAGOMIR

Abstract. Some Jensen's type inequalities for twice differentiable functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for particular cases of interest are also provided.

1. Introduction

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [3, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(\bar{f})=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)}\qquad f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [3] and the references therein. For other results, see [9], [4], [8] and [6]. For recent results, see [1] and [2].

Date: September 30, 2008.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Selfadjoint operators, Positive operators, Jensen's inequality, Convex functions, Functions of selfadjoint operators.

2. Some Jensen's Type Inequalities for Operators

The following result that provides an operator version for the Jensen inequality is due to Mond & Pečarić [7] (see also [3, p. 5]):

Theorem 1 (Mond-Pečarić, 1993, [7]). *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

Theorem 2 (Hölder-McCarthy, 1967, [5]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r > 1$ and $x \in H$ with $\|x\| = 1$;

(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ for all $0 < r < 1$ and $x \in H$ with $\|x\| = 1$;

(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ for all $r < 0$ and $x \in H$ with $\|x\| = 1$.

The following theorem is a multiple operator version of Theorem 1 (see for instance [3, p. 5]):

Theorem 3. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ for some scalars $m < M$ and $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. If f is a convex function on $[m, M]$, then*

$$f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right) \leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle . \tag{2.1}$$

The following particular case is of interest:

Corollary 1. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, ..., n\}$ for some scalars $m < M$. If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$f\left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle\right) \leq \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle , \tag{2.2}$$

for any $x \in H$ with $\|x\| = 1$.

3. Jensen's Inequality for Twice Differentiable Functions

The following result may be stated:

Theorem 4. *Let A be a positive definite operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If f is a twice differentiable function on (m, M) and for $p \in (-\infty, 0) \cup (1, \infty)$ we have for some $\gamma < \Gamma$ that*

$$\gamma \leq \frac{t^{2-p}}{p(p-1)} \cdot f''(t) \leq \Gamma \text{ for any } t \in (m, M), \tag{3.1}$$

then

$$\begin{aligned}(3.2)\quad \gamma\left(\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\right) &\le \langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\\ &\le \Gamma\left(\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\right)\end{aligned}$$

for each $x\in H$ *with* $\|x\|=1$.

If

$$\delta\le\frac{t^{2-p}}{p(1-p)}\cdot f''(t)\le\Delta \text{ for any } t\in(m,M) \tag{3.3}$$

and for some $\delta<\Delta$, *where* $p\in(0,1)$, *then*

$$\begin{aligned}(3.4)\quad \delta\left(\langle Ax,x\rangle^{p}-\langle A^{p}x,x\rangle\right) &\le \langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\\ &\le \Delta\left(\langle Ax,x\rangle^{p}-\langle A^{p}x,x\rangle\right)\end{aligned}$$

for each $x\in H$ *with* $\|x\|=1$.

Proof. Consider the function $g_{\gamma,p}:(m,M)\to\mathbb{R}$ given by $g_{\gamma,p}(t)=f(t)-\gamma t^{p}$ where $p\in(-\infty,0)\cup(1,\infty)$. The function $g_{\gamma,p}$ is twice differentiable,

$$g''_{\gamma,p}(t)=f''(t)-\gamma p(p-1)t^{p-2}$$

for any $t\in(m,M)$ and by (3.1) we deduce that $g_{\gamma,p}$ is convex on (m,M). Now, applying the Mond & Pečarić inequality for $g_{\gamma,p}$ we have

$$\begin{aligned}0 &\le \langle (f(A)-\gamma A^{p})x,x\rangle-\left[f(\langle Ax,x\rangle)-\gamma\langle Ax,x\rangle^{p}\right]\\ &= \langle f(A)x,x\rangle-f(\langle Ax,x\rangle)-\gamma\left[\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\right]\end{aligned}$$

which is equivalent with the first inequality in (3.2).

By defining the function $g_{\Gamma,p}:(m,M)\to\mathbb{R}$ given by $g_{\Gamma,p}(t)=\Gamma t^{p}-f(t)$ and applying the same argument we deduce the second part of (3.2).

The rest goes likewise and the details are omitted. ∎

Remark 1. *We observe that if* f *is a twice differentiable function on* (m,M) *and* $\varphi:=\inf_{t\in(m,M)}f''(t)$, $\Phi:=\sup_{t\in(m,M)}f''(t)$, *then by (3.2) we get the inequality*

$$\begin{aligned}(3.5)\quad \frac{1}{2}\varphi\left[\langle A^{2}x,x\rangle-\langle Ax,x\rangle^{2}\right] &\le \langle f(A)x,x\rangle-f(\langle Ax,x\rangle)\\ &\le \frac{1}{2}\Phi\left[\langle A^{2}x,x\rangle-\langle Ax,x\rangle^{2}\right]\end{aligned}$$

for each $x\in H$ *with* $\|x\|=1$.

We observe that the inequality (3.5) holds for selfadjoint operators that are not necessarily positive.

The following version for sequences of operators can be stated:

Corollary 2. *Let* A_j *be positive definite operators with* $Sp(A_j)\subseteq[m,M]\subset(0,\infty)$ $j\in\{1,...,n\}$. *If* f *is a twice differentiable function on* (m,M) *and for* $p\in(-\infty,0)\cup$

$(1,\infty)$ we have the condition (3.1), then

$$\gamma\left[\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\left\langle A_j x_j,x_j\right\rangle\right)^p\right]\leq\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle-f\left(\sum_{j=1}^{n}\left\langle A_j x_j,x_j\right\rangle\right)\leq\Gamma\left[\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\left\langle A_j x_j,x_j\right\rangle\right)^p\right]\tag{3.6}$$

for each $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

If we have the condition (3.3) for $p\in(0,1)$, then

$$\delta\left[\left(\sum_{j=1}^{n}\left\langle A_j x_j,x_j\right\rangle\right)^p-\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle\right]\leq\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle-f\left(\sum_{j=1}^{n}\left\langle A_j x_j,x_j\right\rangle\right)\leq\Delta\left[\left(\sum_{j=1}^{n}\left\langle A_j x_j,x_j\right\rangle\right)^p-\sum_{j=1}^{n}\left\langle A_j^p x_j,x_j\right\rangle\right]\tag{3.7}$$

for each $x_j\in H, j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

Proof. As in [3, p. 6], if we put

$$\widetilde{A}:=\begin{pmatrix}A_1&.&.&.&0\\&.&&&\\&&.&&\\&&&.&\\0&.&.&.&A_n\end{pmatrix}\text{ and }\widetilde{x}=\begin{pmatrix}x_1\\.\\.\\.\\x_n\end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right)\subseteq[m,M]$, $\|\widetilde{x}\|=1$,

$$\left\langle f\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle,\left\langle\widetilde{A}\widetilde{x},\widetilde{x}\right\rangle=\sum_{j=1}^{n}\left\langle A_j x_j,x_j\right\rangle$$

and so on.

Applying Theorem 4 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired results (3.6) and (3.7). ∎

Corollary 3. *Let A_j be positive definite operators with $Sp(A_j)\subseteq[m,M]\subset(0,\infty)$ $j\in\{1,...,n\}$ and $p_j\geq 0$, $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}p_j=1$. If f is a twice differentiable function on (m,M) and for $p\in(-\infty,0)\cup(1,\infty)$ we have the condition*

(3.1), then

$$\gamma\left[\left\langle\sum_{j=1}^{n}p_jA_j^px,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle^p\right] \tag{3.8}$$
$$\leq\left\langle\sum_{j=1}^{n}p_jf\left(A_j\right)x,x\right\rangle-f\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)$$
$$\leq\Gamma\left[\left\langle\sum_{j=1}^{n}p_jA_j^px,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle^p\right]$$

for each $x\in H$ *with* $\|x\|=1$.

If we have the condition (3.3) for $p\in(0,1)$, *then*

$$\delta\left[\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle^p-\left\langle\sum_{j=1}^{n}p_jA_j^px,x\right\rangle\right] \tag{3.9}$$
$$\leq\left\langle\sum_{j=1}^{n}p_jf\left(A_j\right)x,x\right\rangle-f\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)$$
$$\leq\Delta\left[\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle^p-\left\langle\sum_{j=1}^{n}p_jA_j^px,x\right\rangle\right]$$

for each $x\in H$ *with* $\|x\|=1$.

Proof. Follows from Corollary 2 on choosing $x_j=\sqrt{p_j}\cdot x,\ j\in\{1,...,n\}$, where $p_j\geq 0, j\in\{1,...,n\}$, $\sum_{j=1}^{n}p_j=1$ and $x\in H$, with $\|x\|=1$. The details are omitted. ∎

Remark 2. *We observe that if* f *is a twice differentiable function on* (m,M) *with* $-\infty<m<M<\infty$, $Sp\left(A_j\right)\subset[m,M]$, $j\in\{1,...,n\}$ *and* $\varphi:=\inf_{t\in(m,M)}f''(t)$, $\Phi:=\sup_{t\in(m,M)}f''(t)$, *then*

$$\varphi\left[\sum_{j=1}^{n}\left\langle A_j^2x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\left\langle A_jx_j,x_j\right\rangle\right)^2\right] \tag{3.10}$$
$$\leq\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle-f\left(\sum_{j=1}^{n}\left\langle A_jx_j,x_j\right\rangle\right)$$
$$\leq\Phi\left[\sum_{j=1}^{n}\left\langle A_j^2x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\left\langle A_jx_j,x_j\right\rangle\right)^2\right]$$

for each $x_j\in H, j\in\{1,...,n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2=1$.

Also, if $p_j \geq 0,\ j \in \{1,...,n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$\varphi\left[\left\langle\sum_{j=1}^{n} p_j A_j^2 x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^2\right] \leq \left\langle\sum_{j=1}^{n} p_j f(A_j)\, x, x\right\rangle - f\left(\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle\right) \leq \Phi\left[\left\langle\sum_{j=1}^{n} p_j A_j^2 x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^2\right] \tag{3.11}$$

The next result provides some inequalities for the function f which replace the cases $p = 0$ and $p = 1$ that were not alowed in Theorem 4:

Theorem 5. *Let* A *be a positive definite operator on the Hilbert space* H *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* m, M *with* $0 < m < M$. *If* f *is a twice differentiable function on* (m, M) *and we have for some* $\gamma < \Gamma$ *that*

$$\gamma \leq t^2 \cdot f''(t) \leq \Gamma \text{ for any } t \in (m, M), \tag{3.12}$$

then

$$\gamma(\ln(\langle Ax, x\rangle) - \langle \ln Ax, x\rangle) \leq \langle f(A)\, x, x\rangle - f(\langle Ax, x\rangle) \leq \Gamma(\ln(\langle Ax, x\rangle) - \langle \ln Ax, x\rangle) \tag{3.13}$$

for each $x \in H$ *with* $\|x\| = 1$.

If

$$\delta \leq t \cdot f''(t) \leq \Delta \text{ for any } t \in (m, M) \tag{3.14}$$

for some $\delta < \Delta$, *then*

$$\delta(\langle A \ln Ax, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle)) \leq \langle f(A)\, x, x\rangle - f(\langle Ax, x\rangle) \leq \Delta(\langle A \ln Ax, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle)) \tag{3.15}$$

for each $x \in H$ *with* $\|x\| = 1$.

Proof. Consider the function $g_{\gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\gamma,0}(t) = f(t) + \gamma \ln t$. The function $g_{\gamma,0}$ is twice differentiable,

$$g''_{\gamma,p}(t) = f''(t) - \gamma t^{-2}$$

for any $t \in (m, M)$ and by (3.12) we deduce that $g_{\gamma,0}$ is convex on (m, M). Now, applying the Mond & Pečarić inequality for $g_{\gamma,0}$ we have

$$\begin{aligned} 0 &\leq \langle(f(A) + \gamma \ln A)\, x, x\rangle - [f(\langle Ax, x\rangle) + \gamma \ln(\langle Ax, x\rangle)] \\ &= \langle f(A)\, x, x\rangle - f(\langle Ax, x\rangle) - \gamma[\ln(\langle Ax, x\rangle) - \langle \ln Ax, x\rangle] \end{aligned}$$

which is equivalent with the first inequality in (3.13).

By defining the function $g_{\Gamma,0} : (m, M) \to \mathbb{R}$ given by $g_{\Gamma,0}(t) = -\Gamma \ln t - f(t)$ and applying the same argument we deduce the second part of (3.13).

The rest goes likewise for the functions

$$g_{\delta,1}(t) = f(t) - \delta t \ln t \text{ and } g_{\Delta,0}(t) = \Delta t \ln t - f(t)$$

and the details are omitted. ∎

Corollary 4. *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, ..., n\}$. If f is a twice differentiable function on (m, M) and we have the condition (3.12), then*

$$\gamma \left(\ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right) \leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \leq \Gamma \left(\ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle \right) \tag{3.16}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If we have the condition (3.14), then

$$\delta \left(\sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right) \leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \leq \Delta \left(\sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right) \tag{3.17}$$

for each $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

The following particular case also holds:

Corollary 5. *Let A_j be positive definite operators with $Sp(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, ..., n\}$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$. If f is a twice differentiable function on (m, M) and we have the condition (3.12), then*

$$\gamma \left(\ln \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) - \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle \right) \leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \leq \Gamma \left(\ln \left(\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle \right) - \left\langle \sum_{j=1}^{n} p_j \ln A_j x, x \right\rangle \right) \tag{3.18}$$

for each $x \in H$ with $\|x\| = 1$.

If we have the condition (3.14), then

$$\delta\left(\left\langle\sum_{j=1}^{n}p_jA_j\ln A_jx,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\ln\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)\right)$$
$$\leq\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle-f\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)$$
$$\leq\Delta\left(\left\langle\sum_{j=1}^{n}p_jA_j\ln A_jx,x\right\rangle-\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\ln\left(\left\langle\sum_{j=1}^{n}p_jA_jx,x\right\rangle\right)\right) \tag{3.19}$$

for each $x\in H$ with $\|x\|=1$.

4. Applications

It is clear that the results from the previous section can be applied for various particular functions which are twice differentiable and the second derivatives satisfy the boundedness conditions from the statements of the Theorems 4, 5 and the Remark 1.

We point out here only some simple examples that are, in our opinion, of large interest.

1. For a given $\alpha>0$, consider the function $f(t)=\exp(\alpha t),t\in\mathbb{R}$. We have $f''(t)=\alpha^2\exp(\alpha t)$ and for a selfadjoint operator A with $Sp(A)\subset[m,M]$ (for some real numbers $m<M$) we also have

$$\varphi:=\inf_{t\in(m,M)}f''(t)=\alpha^2\exp(\alpha m)\ \text{ and }\ \Phi:=\sup_{t\in(m,M)}f''(t)=\alpha^2\exp(\alpha M).$$

Utilising the inequality (3.5) we get

$$\frac{1}{2}\alpha^2\exp(\alpha m)\left[\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right]\leq\langle\exp(\alpha A)x,x\rangle-\exp(\langle\alpha Ax,x\rangle)\leq\frac{1}{2}\alpha^2\exp(\alpha M)\left[\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right], \tag{4.1}$$

for each $x\in H$ with $\|x\|=1$.

Now, if $\beta>0$, then we also have

$$\frac{1}{2}\beta^2\exp(-\beta M)\left[\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right]\leq\langle\exp(-\beta A)x,x\rangle-\exp(-\langle\beta Ax,x\rangle)\leq\frac{1}{2}\beta^2\exp(-\beta m)\left[\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right], \tag{4.2}$$

for each $x\in H$ with $\|x\|=1$.

2. Now, assume that $0<m<M$ and the operator A satisfies the condition $m\cdot 1_H\leq A\leq M\cdot 1_H$. If we consider the function $f:(0,\infty)\to(0,\infty)$ defined by $f(t)=t^p$ with $p\in(-\infty,0)\cup(0,1)\cup(1,\infty)$. Then $f''(t)=p(p-1)t^{p-2}$ and if we consider $\varphi:=\inf_{t\in(m,M)}f''(t)$ and $\Phi:=\sup_{t\in(m,M)}f''(t)$, then we have

$$\varphi=p(p-1)m^{p-2},\Phi=p(p-1)M^{p-2}\text{ for }p\in[2,\infty),$$

$$\varphi = p\,(p-1)\,M^{p-2}, \Phi = p\,(p-1)\,m^{p-2} \text{ for } p \in (1,2)\,,$$

$$\varphi = p\,(p-1)\,m^{p-2}, \Phi = p\,(p-1)\,M^{p-2} \text{ for } p \in (0,1)\,,$$

and

$$\varphi = p\,(p-1)\,M^{p-2}, \Phi = p\,(p-1)\,m^{p-2} \text{ for } p \in (-\infty,0)\,.$$

Utilising the inequality (3.5) we then get the following refinements an reverses of Hölder-McCarthy's inequalities from Theorem 2:

$$\begin{aligned}(4.3)\quad & \frac{1}{2}p\,(p-1)\,m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \\ & \le \langle A^px,x\rangle - \langle Ax,x\rangle^p \\ & \le \frac{1}{2}p\,(p-1)\,M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \text{ for } p \in [2,\infty),\end{aligned}$$

$$\begin{aligned}(4.4)\quad & \frac{1}{2}p\,(p-1)\,M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \\ & \le \langle A^px,x\rangle - \langle Ax,x\rangle^p \\ & \le \frac{1}{2}p\,(p-1)\,m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \text{ for } p \in (1,2)\,,\end{aligned}$$

$$\begin{aligned}(4.5)\quad & \frac{1}{2}p\,(1-p)\,M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \\ & \le \langle Ax,x\rangle^p - \langle A^px,x\rangle \\ & \le \frac{1}{2}p\,(1-p)\,m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \text{ for } p \in (0,1)\end{aligned}$$

and

$$\begin{aligned}(4.6)\quad & \frac{1}{2}p\,(p-1)\,M^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \\ & \le \langle A^px,x\rangle - \langle Ax,x\rangle^p \\ & \le \frac{1}{2}p\,(p-1)\,m^{p-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \text{ for } p \in (-\infty,0)\,,\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

3. Now, if we consider the function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$, then $f''(t) = t^{-2}$ which gives that $\varphi = M^{-2}$ and $\Phi = m^{-2}$. Utilising the inequality (3.5) we then deduce the bounds

$$\begin{aligned}(4.7)\quad & \frac{1}{2}M^{-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \\ & \le \ln\left(\langle Ax,x\rangle\right) - \langle \ln Ax,x\rangle \\ & \le \frac{1}{2}m^{-2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\end{aligned}$$

for each $x \in H$ with $\|x\| = 1$.

Moreover, if we consider the function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t\ln t$, then $f''(t) = t^{-1}$ which gives that $\varphi = M^{-1}$ and $\Phi = m^{-1}$. Utilising the inequality (3.5) we then deduce the bounds

$$\frac{1}{2}M^{-1}\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right]\leq\langle A\ln Ax,x\rangle-\langle Ax,x\rangle\ln\left(\langle Ax,x\rangle\right)\leq\frac{1}{2}m^{-1}\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right]\tag{4.8}$$

for each $x\in H$ with $\|x\|=1$.

Remark 3. *Utilising Theorem 4 for the particular value of $p=-1$ we can state the inequality*

$$\frac{1}{2}\psi\left(\left\langle A^{-1}x,x\right\rangle-\langle Ax,x\rangle^{-1}\right)\leq\langle f\left(A\right)x,x\rangle-f\left(\langle Ax,x\rangle\right)\leq\frac{1}{2}\Psi\left(\left\langle A^{-1}x,x\right\rangle-\langle Ax,x\rangle^{-1}\right)\tag{4.9}$$

for each $x\in H$ with $\|x\|=1$, provided that f is twice differentiable on $(m,M)\subset(0,\infty)$ and

$$\psi=\inf_{t\in(m,M)}t^{3}f''\left(t\right)\ \ \textit{while}\ \ \Psi=\sup_{t\in(m,M)}t^{3}f''\left(t\right)$$

are assumed to be finite.

We observe that, by utilising the inequality (4.9) instead of the inequality (3.5) we may obtain similar results in terms of the quantity $\left\langle A^{-1}x,x\right\rangle-\langle Ax,x\rangle^{-1}$, $x\in H$ with $\|x\|=1$. However the details are left to the interested reader.

References

[1] S.S. Dragomir, Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint, *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 11. [ONLINE: `http://www.staff.vu.edu.au/RGMIA/v11(E).asp`]

[2] S.S. Dragomir, Some new Grüss' type Inequalities for functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 12. [ONLINE: `http://www.staff.vu.edu.au/RGMIA/v11(E).asp`]

[3] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[4] A. Matković, J. Pečarić and I. Perić, A variant of Jensen's inequality of Mercer's type for operators with applications. *Linear Algebra Appl.* **418** (2006), no. 2-3, 551–564.

[5] C.A. McCarthy, c_p, *Israel J. Math.*, **5**(1967), 249-271.

[6] J. Mićić, Y.Seo, S.-E. Takahasi and M. Tominaga, Inequalities of Furuta and Mond-Pečarić, *Math. Ineq. Appl.*, **2**(1999), 83-111.

[7] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19**(1993), 405-420.

[8] B. Mond and J. Pečarić, On some operator inequalities, *Indian J. Math.*, **35**(1993), 221-232.

[9] B. Mond and J. Pečarić, Classical inequalities for matrix functions, *Utilitas Math.*, **46**(1994), 155-166.

Research Group in Mathematical Inequalities & Applications, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://www.staff.vu.edu.au/rgmia/dragomir/`