REFINEMENTS OF FEJÉR'S INEQUALITY FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper, we establish some new refinements for the celebrated Fejér's and Hermite-Hadamard's integral inequalities for convex functions.

1. Introduction

One of the most important integral inequalities with various applications for generalised means, information measures, quadrature rules, etc., is the well known *Hermite-Hadamard inequality* [1]

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where $f:[a,b]\to\mathbb{R}$ is a convex function on the interval $[a,b]$.

In order to refine and generalize this classical result for weighted integrals, we define the following functions on $[0,1]$, namely

$$G(t) = \frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right) + f\left(tb+(1-t)\frac{a+b}{2}\right)\right];$$

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx;$$

$$H_g(t) = \int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)g(x)\,dx;$$

$$I(t) = \int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right) + f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right]g(x)\,dx;$$

$$F(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)\,dxdy;$$

$$\begin{aligned} K(t) = \int_a^b\int_a^b \frac{1}{4}\Bigg[& f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right) \\ & + f\left(t\frac{x+a}{2}+(1-t)\frac{y+b}{2}\right) + f\left(t\frac{x+b}{2}+(1-t)\frac{y+a}{2}\right) \\ & + f\left(t\frac{x+b}{2}+(1-t)\frac{y+b}{2}\right)\Bigg] g(x)\,g(y)\,dxdy; \end{aligned}$$

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$$L(t)=\frac{1}{2(b-a)}\int_a^b [f(ta+(1-t)x)+f(tb+(1-t)x)]\,dx;$$

$$L_g(t)=\frac{1}{2}\int_a^b [f(ta+(1-t)x)+f(tb+(1-t)x)]\,g(x)\,dx;$$

$$S_g(t)=\frac{1}{4}\int_a^b \left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(ta+(1-t)\frac{x+b}{2}\right)\right.$$
$$\left.+f\left(tb+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)\,dx$$

and

$$N(t)=\int_a^b \frac{1}{2}\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]g(x)\,dx.$$

where $f:[a,b]\to\mathbb{R}$ is convex, $g:[a,b]\to[0,\infty)$ is integrable and symmetric to $\frac{a+b}{2}$.

Remark 1. *We note that* $H=H_g=I$, $F=K$ *and* $L=L_g=S_g$ *on* $[0,1]$ *as* $g(x)=\frac{1}{b-a}\,(x\in[a,b])$.

For some results which generalize, improve, and extend the famous Hermite-Hadamard integral inequality see [2] – [20].

In [8], Fejér established the following weighted generalization of (1.1).

Theorem A. *Let* f,g *be defined as above. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\le\int_a^b f(x)\,g(x)\,dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{1.2}$$

In [11], Tseng et al. established the following Fejér-type inequalities.

Theorem B. *Let* f,g *be defined as above. Then we have*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx&\le\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}\int_a^b g(x)\,dx\\&\le\int_a^b\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\\&\le\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\int_a^b g(x)\,dx\\&\le\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx.\end{aligned} \tag{1.3}$$

In [2], Dragomir improved the first part of the Hermite-Hadamard inequality by considering the functions H,F as follows:

Theorem C. *Let* f,H *be defined as above. Then* H *is convex, increasing on* $[0,1]$, *and for all* $t\in[0,1]$, *we have*

$$f\left(\frac{a+b}{2}\right)=H(0)\le H(t)\le H(1)=\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.4}$$

Theorem D. *Let* f,F *be defined as above. Then*

(1) *F is convex on $[0,1]$, symmetric about $\frac{1}{2}$, F is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$, and for all $t\in[0,1]$,*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy.$$

(2) *We have:*

$$f\left(\frac{a+b}{2}\right) \le F\left(\frac{1}{2}\right); \qquad H(t) \le F(t), \qquad t\in[0,1]. \tag{1.5}$$

In [11], Tseng et al. established the following Fejér-type inequality related to the functions I, N, which is also the weighted generalization of Theorem C.

Theorem E. *Let f, g, I, N be defined as above. Then I, N are convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &= I(0) \le I(t) \le I(1) \\ &= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &= N(0) \le N(t) \le N(1) \\ &= \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.6}$$

In [7], Dragomir et al. established the following Hermite-Hadamard-type inequality related to the functions H, G, L.

Theorem F. *Let f, H, G, L be defined as above. Then G is convex, increasing on $[0,1]$, L is convex on $[0,1]$, and for all $t\in[0,1]$, we have*

$$\begin{aligned} H(t) \le G(t) \le L(t) &\le \frac{1-t}{b-a}\int_a^b f(x)\,dx + t\cdot\frac{f(a)+f(b)}{2} \\ &\le \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.7}$$

In [12] – [13], Tseng et al. established the following theorem related to Fejér-type inequalities concerning the functions G, H_g, L_g, I, S_g and which provides a weighted generalizations of the inequality (1.7).

Theorem G ([12])**.** *Let f, g, G, H_g, L_g be defined as above. Then L_g is convex, increasing on $[0,1]$, and for all $t\in[0,1]$, we have*

$$\begin{aligned} H_g(t) &\le G(t)\int_a^b g(x)\,dx \le L_g(t) \\ &\le (1-t)\int_a^b f(x)\,g(x)\,dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.8}$$

Theorem H ([13])**.** *Let f, g, G, I, S_g be defined as above. Then S_g is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

$$\begin{aligned} I(t) &\le G(t)\int_a^b g(x)\,dx \le S_g(t) \\ &\le (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \\ &\quad + t\cdot\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \end{aligned} \tag{1.9}$$

Finally, we notice that in [5], Dragomir established the following Hermite-Hadamard-type inequalities related to the functions H, F, L.

Theorem I. *Let F, H, L be defined as above. Then we have the inequality*

$$0 \le F(t) - H(t) \le L(1-t) - F(t) \tag{1.10}$$

for all $t \in [0,1]$.

In this paper, we establish some Fejér-type and Hermite-Hadamard-type inequalities related to the functions $H, F, L, H_g, L_g, I, S_g, K$ defined above. As an important consequence we also obtain the weighted generalizations of Theorems D and I.

2. Main Results

The following lemma plays a key role in proving the new results:

Lemma 2 (*see* [9])**.** *Let $f : [a,b] \to \mathbb{R}$ be a convex function and let $a \le A \le C \le D \le B \le b$ with $A + B = C + D$. Then*

$$f(C) + f(D) \le f(A) + f(B).$$

We can state now the following result:

Theorem 3. *Let f, g, I, K be defined as above. Then:*

1. *K is convex on $[0,1]$ and symmetric about $\frac{1}{2}$.*
2. *K is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$,*

$$\begin{aligned} \sup_{t\in[0,1]} K(t) &= K(0) = K(1) \\ &= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \cdot \int_a^b g(x)\,dx \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \inf_{t\in[0,1]} K(t) &= K\left(\frac{1}{2}\right) \\ &= \int_a^b\int_a^b \frac{1}{4}\left[f\left(\frac{x+y+2a}{4}\right)+2f\left(\frac{x+y+a+b}{4}\right)\right. \\ &\quad \left. + f\left(\frac{x+y+2b}{4}\right)\right]g(x)\,g(y)\,dxdy. \end{aligned} \tag{2.2}$$

(3) *We have*

$$I(t)\int_a^b g(x)\,dx\le K(t) \tag{2.3}$$

and

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2\le K\left(\frac{1}{2}\right) \tag{2.4}$$

for all $t\in[0,1]$.

Proof. (1) It is easily observed from the convexity of f that K is convex on $[0,1]$. By changing the variable, we have that

$$K(t)=K(1-t),\qquad t\in[0,1],$$

from which we get that K is symmetric about $\frac{1}{2}$.

(2) Let $t_1<t_2$ in $\left[0,\frac{1}{2}\right]$. Using the symmetry of K, we have

$$K(t_1)=\frac{1}{2}\left[K(t_1)+K(1-t_1)\right], \tag{2.5}$$

$$K(t_2)=\frac{1}{2}\left[K(t_2)+K(1-t_2)\right] \tag{2.6}$$

and, by Lemma 2, we have

$$\frac{1}{2}\left[K(t_2)+K(1-t_2)\right]\le\frac{1}{2}\left[K(t_1)+K(1-t_1)\right]. \tag{2.7}$$

From (2.5) – (2.7), we obtain that K is decreasing on $\left[0,\frac{1}{2}\right]$. Since K is symmetric about $\frac{1}{2}$ and K is decreasing on $\left[0,\frac{1}{2}\right]$, we get that K is increasing on $\left[\frac{1}{2},1\right]$. Using the symmetry and monotonicity of K, we derive (2.1) and (2.2).

(3) Using substitution rules for integration and the hypothesis of g, we have the following identity

$$\begin{aligned}K(t)=\int_a^b\int_a^b\frac{1}{4}\Bigg[&f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)\\&+f\left(t\frac{x+a}{2}+(1-t)\frac{a+2b-y}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{y+a}{2}\right)\\&+f\left(t\frac{x+b}{2}+(1-t)\frac{a+2b-y}{2}\right)\Bigg]g(x)\,g(y)\,dydx\end{aligned} \tag{2.8}$$

for all $t\in[0,1]$.

By Lemma 2, the following inequalities hold for all $t\in[0,1]$, $x\in[a,b]$ and $y\in[a,b]$. The inequality

$$\begin{aligned}&\frac{1}{2}f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\\&\le\frac{1}{4}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)+f\left(t\frac{x+a}{2}+(1-t)\frac{a+2b-y}{2}\right)\right]\end{aligned} \tag{2.9}$$

holds when

$$A = t\frac{x+a}{2} + (1-t)\frac{y+a}{2},$$
$$C = D = t\frac{x+a}{2} + (1-t)\frac{a+b}{2} \quad \text{and}$$
$$B = t\frac{x+a}{2} + (1-t)\frac{a+2b-y}{2}$$

in Lemma 2. The inequality

$$\frac{1}{2}f\left(t\frac{x+b}{2} + (1-t)\frac{a+b}{2}\right) \leq \frac{1}{4}\left[f\left(t\frac{x+b}{2} + (1-t)\frac{y+a}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{a+2b-y}{2}\right)\right] \tag{2.10}$$

holds when

$$A = t\frac{x+b}{2} + (1-t)\frac{y+a}{2},$$
$$C = D = t\frac{x+b}{2} + (1-t)\frac{a+b}{2} \quad \text{and}$$
$$B = t\frac{x+b}{2} + (1-t)\frac{a+2b-y}{2}$$

in Lemma 2.

Multiplying the inequalities (2.9) and (2.10) by $g(x)g(y)$, integrating them over x on $[a,b]$, over y on $[a,b]$ and using identities (2.8), we derive the inequality (2.3).

From the inequality (2.3) and the monotonicity of I, we have

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 = I(0)\int_a^b g(x)\,dx \leq I\left(\frac{1}{2}\right)\int_a^b g(x)\,dx \leq K\left(\frac{1}{2}\right)$$

from which we derive the inequality (2.4).

This completes the proof. □

Remark 4. *Let* $g(x) = \frac{1}{b-a}$ $(x \in [a,b])$ *in Theorem 3. Then* $I(t) = H(t)$, $K(t) = F(t)$ $(t \in [0,1])$ *and Theorem 3 reduces to Theorem D.*

Remark 5. *From Theorem E and Theorem 3, we obtain the following Fejér-type inequality*

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 \leq I(t)\int_a^b g(x)\,dx \leq K(t) \leq \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right) + f\left(\frac{x+b}{2}\right)\right]g(x)\,dx \cdot \int_a^b g(x)\,dx.$$

Theorem 6. *Let* f, g, I, K, S_g *be defined as above. Then we have the inequality*

$$0 \leq K(t) - I(t)\int_a^b g(x)\,dx \leq S_g(1-t)\int_a^b g(x)\,dx - K(t), \tag{2.11}$$

for all $t \in [0,1]$.

Proof. Using substitution rules for integration and the hypothesis of g, we have the following identity

$$K(t) = \int_a^b \int_a^b \frac{1}{4}\left[f\left(t\frac{x+a}{2} + (1-t)\frac{y+a}{2}\right)\right.$$
$$+ f\left(t\frac{x+a}{2} + (1-t)\frac{a+2b-y}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{y+a}{2}\right)$$
$$\left.+ f\left(t\frac{x+b}{2} + (1-t)\frac{a+2b-y}{2}\right)\right] g(x)\, g(y)\, dydx$$
$$= \int_a^b \int_a^{\frac{a+b}{2}} \frac{1}{2}\left[f\left(t\frac{x+a}{2} + (1-t)\,y\right)\right.$$
$$+ f\left(t\frac{x+a}{2} + (1-t)(a+b-y)\right) + f\left(t\frac{x+b}{2} + (1-t)\,y\right)$$
$$\left.+ f\left(t\frac{x+b}{2} + (1-t)(a+b-y)\right)\right] g(x)\, g(2y-a)\, dydx$$
$$= \frac{1}{2}\int_a^b \int_a^{\frac{3a+b}{4}} \left[f\left(t\frac{x+a}{2} + (1-t)\,y\right)\right.$$
$$+ f\left(t\frac{x+a}{2} + (1-t)\left(\frac{3a+b}{2} - y\right)\right)$$
$$+ f\left(t\frac{x+a}{2} + (1-t)(a+b-y)\right)$$
$$+ f\left(t\frac{x+a}{2} + (1-t)\left(\frac{b-a}{2} + y\right)\right)$$
$$+ f\left(t\frac{x+b}{2} + (1-t)\,y\right) + f\left(t\frac{x+b}{2} + (1-t)\left(\frac{3a+b}{2} - y\right)\right)$$
$$+ f\left(t\frac{x+b}{2} + (1-t)(a+b-y)\right)$$
$$\left.+ f\left(t\frac{x+b}{2} + (1-t)\left(\frac{b-a}{2} + y\right)\right)\right] g(x)\, g(2y-a)\, dydx \tag{2.12}$$

for all $t \in [0,1]$.

By Lemma 2, the following inequalities hold for all $t \in [0,1]$, $x \in [a,b]$ and $y \in \left[a, \frac{3a+b}{4}\right]$. The inequality

$$f\left(t\frac{x+a}{2} + (1-t)\,y\right) + f\left(t\frac{x+a}{2} + (1-t)\left(\frac{3a+b}{2} - y\right)\right)$$
$$\leq f\left(t\frac{x+a}{2} + (1-t)\,a\right) + f\left(t\frac{x+a}{2} + (1-t)\frac{a+b}{2}\right) \tag{2.13}$$

holds when

$$A = t\frac{x+a}{2} + (1-t)\,a, \qquad C = t\frac{x+a}{2} + (1-t)\,y,$$
$$D = t\frac{x+a}{2} + (1-t)\left(\frac{3a+b}{2} - y\right) \quad \text{and} \quad B = t\frac{x+a}{2} + (1-t)\frac{a+b}{2}$$

in Lemma 2. The inequality

$$f\left(t\frac{x+a}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right)+f\left(t\frac{x+a}{2}+(1-t)(a+b-y)\right) \le f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+a}{2}+(1-t)b\right) \tag{2.14}$$

holds when

$$A=t\frac{x+a}{2}+(1-t)\frac{a+b}{2}, \qquad C=t\frac{x+a}{2}+(1-t)\left(\frac{b-a}{2}+y\right),$$
$$D=t\frac{x+a}{2}+(1-t)(a+b-y) \quad \text{and} \quad B=t\frac{x+a}{2}+(1-t)b$$

in Lemma 2. The inequality

$$f\left(t\frac{x+b}{2}+(1-t)y\right)+f\left(t\frac{x+b}{2}+(1-t)\left(\frac{3a+b}{2}-y\right)\right) \le f\left(t\frac{x+b}{2}+(1-t)a\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right) \tag{2.15}$$

holds when

$$A=t\frac{x+b}{2}+(1-t)a, \quad C=t\frac{x+b}{2}+(1-t)y,$$
$$D=t\frac{x+b}{2}+(1-t)\left(\frac{3a+b}{2}-y\right) \qquad \text{and} \qquad B=t\frac{x+b}{2}+(1-t)\frac{a+b}{2}$$

in Lemma 2. The inequality

$$f\left(t\frac{x+b}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right)+f\left(t\frac{x+b}{2}+(1-t)(a+b-y)\right) \le f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)b\right) \tag{2.16}$$

holds when

$$A=t\frac{x+b}{2}+(1-t)\frac{a+b}{2}, \qquad C=t\frac{x+b}{2}+(1-t)\left(\frac{b-a}{2}+y\right),$$
$$D=t\frac{x+b}{2}+(1-t)(a+b-y) \qquad \text{and} \qquad B=t\frac{x+b}{2}+(1-t)b$$

in Lemma 2.

Multiplying the inequalities (2.13) and (2.16) by $g(x)g(2y-a)$, integrating them over x on $[a,b]$, over y on $\left[a,\frac{3a+b}{4}\right]$ and using identity (2.12), we have the inequality

$$2K(t) \le [I(t)+S_g(1-t)]\int_a^b g(x)\,dx, \tag{2.17}$$

for all $t\in[0,1]$. Using (2.3) and (2.17), we derive (2.11). This completes the proof. □

Remark 7. *Let $g(x)=\frac{1}{b-a}$ $(x\in[a,b])$ in Theorem 6. Then $K(t)=F(t)$, $I(t)=H(t)$, $S_g(1-t)=L(1-t)$ $(t\in[0,1])$ and Theorem 6 reduces to Theorem I.*

The following two Fejér-type inequalities are natural consequences of Theorems 3, 6, E, G, H and we omit their proofs.

Theorem 8. *Let f,g,G,I,K,L_g,S_g be defined as above. Then, for all $t\in[0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 &\le I(t)\int_a^b g(x)\,dx \le K(t)\\ &\le \frac{1}{2}\left[I(t)+S_g(1-t)\right]\int_a^b g(x)\,dx\\ &\le \frac{1}{2}\left[G(t)\int_a^b g(x)\,dx+S_g(1-t)\right]\int_a^b g(x)\,dx\\ &\le \frac{1}{2}\left[L_g(t)+S_g(1-t)\right]\int_a^b g(x)\,dx\\ &\le \frac{1}{2}\left((1-t)\int_a^b f(x)\,g(x)\,dx\right.\\ &\qquad +t\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\\ &\qquad \left.+\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\right)\int_a^b g(x)\,dx\\ &\le \frac{f(a)+f(b)}{2}\left(\int_a^b g(x)\,dx\right)^2 \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left(\int_a^b g(x)\,dx\right)^2 &\le I(t)\int_a^b g(x)\,dx \le K(t)\\ &\le \frac{1}{2}\left[I(t)+S_g(1-t)\right]\int_a^b g(x)\,dx\\ &\le \frac{1}{2}\left[G(t)\int_a^b g(x)\,dx+S_g(1-t)\right]\int_a^b g(x)\,dx\\ &\le \frac{1}{2}\left[S_g(t)+S_g(1-t)\right]\int_a^b g(x)\,dx\\ &\le \frac{1}{2}\left(\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]g(x)\,dx\right.\\ &\qquad \left.+\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx\right)\int_a^b g(x)\,dx\\ &\le \frac{f(a)+f(b)}{2}\left(\int_a^b g(x)\,dx\right)^2. \end{aligned} \tag{2.19}$$

Let $g(x)=\frac{1}{b-a}\,(x\in[a,b])$. Then we have the following Hermite-Hadamard-type inequality which is a natural consequence of Theorem 8.

Corollary 9. *Let f, g, G, H, F, L be defined as above. Then, for all $t \in [0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq H(t) \leq F(t) \leq \frac{1}{2}\left[H(t) + L(1-t)\right] \\ &\leq \frac{1}{2}\left[G(t) + L(1-t)\right] \leq \frac{1}{2}\left[L(t) + L(1-t)\right] \\ &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.20}$$

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