# A NOTE ON OSTROWSKI'S INEQUALITY

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Abstract. This paper deals with the problem of estimating the deviation of the values of a function from its mean value. We consider the following special cases: i) the case of random variables (attached to arbitrary probability fields); ii) the comparison is performed additively or multiplicatively; iii) the mean value is attached to a multiplicative averaging process.

## 1. Introduction

The inequality of Ostrowski [10] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, *if* $f : [a, b] \to \mathbb{R}$ *is a differentiable function with bounded derivative, then*

$$(O) \qquad \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[ \frac{1}{4} + \frac{(x - (a+b)/2)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

*for every* $x \in [a, b]$. *Moreover the constant* $1/4$ *is the best possible.*

The proof is an application of Lagrangian's mean value theorem:

$$\begin{aligned} \left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| &= \left| \frac{1}{b-a} \int_a^b (f(x) - f(t))\, dt \right| \\ &\leq \frac{1}{b-a} \int_a^b |f(x) - f(t)|\, dt \\ &\leq \frac{\|f'\|_\infty}{b-a} \int_a^b |x - t|\, dt \\ &= \left[ \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right] \|f'\|_\infty \\ &= \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f'\|_\infty . \end{aligned}$$

The optimality of the constant $1/4$ is also immediate, checking the inequality for the family of functions $f_\alpha(t) = |x - t|^\alpha \cdot (b-a)$ ($t \in [a, b]$, $\alpha > 1$) and then passing to the limit as $\alpha \to 1+$.

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It is worth to notice that the smoothness condition can be relaxed. In fact, the Lipschitz class suffices as well, by replacing $\|f'\|_\infty$ with the Lipschitz constant of $f$, i.e.,

$$||f||_L = \sup_{x \neq y} \left| \frac{f(x) - f(y)}{x - y} \right|.$$

The extension to the context of vector-valued functions, with values in a Banach space, is straightforward.

Since a Lipschitz function on $[a, b]$ is absolutely continuous, a natural direction of generalization of the Ostrowski inequality was its investigation within this larger class of functions (with refinements for $f' \in L^p([a, b])$, $1 \leq p < \infty$). See Fink [5]. Also, several Ostrowski type inequalities are known within the framework of Hölder functions as well as for functions of bounded variation.

The problem to estimate the deviation of a function from its mean value can be investigated from many other points of view:

- by considering random variables (attached to arbitrary probability fields);
- by changing the algebraic nature of the comparison (e.g., switching to the multiplicative framework);
- by considering other means (for example, the geometric mean);
- by estimating the deviation via other norms (the classical case refers to the sup norm, but $L^p$-norms are better motivated in other situations).

The aim of this paper is to present a number of examples giving support to this program.

## 2. Ostrowski type inequalities for random variables

In what follows $X$ will denote a locally compact metric space and $E$ a Banach space.

**Theorem 1.** *The following two assertions are equivalent for $f : X \to E$ a continuous mapping:*

*i) $f$ is Lipschitz i.e., $||f||_L = \sup_{x \neq y} \frac{\|f(x)-f(y)\|}{d(x,y)} < \infty$;*

*ii) For every $x \in X$ and every Borel probability measure $\mu$ on $X$ such that $f \in \mathcal{L}^1(\mu)$ we have*

$$\left\| f(x) - \int_X f \, d\mu \right\| \leq ||f||_L \int_X^* d(x, y) \, d\mu.$$

*Here * marks the upper integral.*

*Proof.* $i) \Rightarrow ii)$. As $d(x, \cdot)$ is continuous it is also Borel measurable, so being nonnegative its upper integral is perfectly motivated. Then we can proceed as in the classical case, described in the Introduction.

$ii) \Rightarrow i)$. Consider the particular case of Dirac measure $\delta_y$ (concentrated at $y$). Then

$$\|f(x) - f(y)\| \leq ||f||_L \, d(x, y)$$

which shows that $f$ must be Lipschitz. ∎

If $X$ is a bounded metric space, the above theorem works for all continuous mappings. In fact, if $||f||_L < \infty$ then $f$ is necessarily bounded (and thus it belongs to $\mathcal{L}^\infty(\mu) \subset \mathcal{L}^1(\mu)$). Also, the mappings $d(x, \cdot)$ are $\mu$-integrable (which makes * unnecessary).

The condition $f \in \mathcal{L}^1(\mu)$ is automatically fulfilled by all continuous bounded functions regardless what Borel probability measure $\mu$ we consider on $X$; in fact, they are in $\mathcal{L}^\infty(\mu) \subset \mathcal{L}^1(\mu)$. In general, not every continuous function $f$ is $\mu$-integrable. For, think at the case where $X = \mathbb{R}$, $f(x) = x$, and $\mu = \frac{1}{\pi(1+x^2)}\, dx$.

We shall illustrate Theorem 1 in a number of particular situations. The first one, concerns the case of classical probability fields:

**Corollary 1.** *Let $E$ be a normed vector space and let $x_1, ..., x_n$ be $n$ vectors in $E$. Then*

$$\left\| x_i - \frac{1}{n}\sum_{k=1}^{n} x_k \right\| \leq \frac{1}{n}\left[\left(i - \frac{n+1}{2}\right)^2 + \frac{n^2-1}{4}\right] \cdot \sup_{1\leq k\leq n-1} ||x_{k+1} - x_k|| \,.$$

*Proof.* We consider the measure space $(X\,, \Sigma, \mu)$, where $X = \{1, ..., n\}$, $\Sigma = \mathcal{P}(X)$ and $\mu(A) = |A|$ for every $A \subset X$.

$X$ has a natural structure of metric subspace of $\mathbb{R}$. The function

$$f : X \to E, \quad f(i) = x_i,$$

is Lipschitz, with Lipschitz constant

$$(L) \qquad L = \sup_{1\leq k\leq n-1} ||x_{k+1} - x_k|| \,.$$

In fact, if $i < j$,

$$\begin{aligned} \|f(i) - f(j)\| &= ||x_i - x_j|| \\ &\leq ||x_i - x_{i+1}|| + ... + ||x_{j-1} - x_j|| \\ &\leq (j - i) \cdot \sup_{1\leq k\leq n-1} ||x_{k+1} - x_k|| \,, \end{aligned}$$

which proves the inequality $\leq$ in $(L)$. The other inequality is clear.

According to Theorem 1,

$$\left\| f(i) - \frac{1}{\mu(X)}\int_X f(k)\; d\mu(k) \right\| \leq \frac{L}{\mu(X)} \int_X |i-k|\; d\mu(k),$$

which can be easily shown to be equivalent to the inequality in the statement of Corollary 1 because

$$\int_X |i-k|\; d\mu(k) = \sum_{k=1}^{n} |i-k| = \left(i - \frac{n+1}{2}\right)^2 + \frac{n^2-1}{4}. \quad \blacksquare$$

Notice that the right hand side of the inequality in Corollary 1 is $\geq \sqrt{\mathrm{var}\,(f)}$, where

$$\mathrm{var}\,(f) = \frac{1}{\mu(X)}\int_X \left| f(x) - \frac{1}{\mu(X)}\int_X f(t) d\mu(t)\right|^2 d\mu(x)$$

represents the variance of $f$. According to the classical Chebyshev inequality,

$$\mu\left\{ \left| f - \frac{1}{\mu(X)}\int_X f\; d\mu \right| \leq \varepsilon \right\} \geq 1 - \frac{\mathrm{var}\; f}{\varepsilon^2}$$

and the discussion above shows that the range of interest in this inequality is precisely

$$\mathrm{var}\; f < \varepsilon \leq \frac{1}{n}\left[\left(i - \frac{n+1}{2}\right)^2 + \frac{n^2-1}{4}\right] \cdot \sup_{1\leq k\leq n-1} ||x_{k+1} - x_k|| \,.$$

As well known, convolution by smooth kernels leads to good approximation schemes. Theorem 1 allows us to estimate the speed of convergence. Here is an example:

**Corollary 2.** *Let $f : \mathbb{R} \to E$ be a Lipschitz mapping. Then*

$$\left\| f(x) - \frac{n}{(2\pi)^{1/2}} \int_{\mathbb{R}} f(t) e^{-n^2(x-t)^2/2} dt \right\| \leq \frac{2 \|f\|_L}{n(2\pi)^{1/2}}$$

*for every $x \in \mathbb{R}$. Particularly, $f$ is the uniform limit of a sequence $(f_n)_n$ of Lipschitz functions of class $C^\infty$, with $\|f_n\|_L \leq \|f\|_L$ for every $n$.*

We end this section with the case of functions of several variables:

**Corollary 3.** *Let $f = f(x, y)$ be a differentiable function defined on a compact $2-$dimensional interval $X = [a, b] \times [c, d]$ such that $|\partial f/\partial x| \leq L$ and $|\partial f/\partial y| \leq M$. Then*

$$\left\| f(x,y) - \frac{1}{Area\, X} \iint_X f(u,v) du dv \right\| \leq L \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] Area\, X +$$

$$+ M \left[ \frac{1}{4} + \left( \frac{y - \frac{c+d}{2}}{d-c} \right)^2 \right] Area\, X.$$

*Proof.* Clearly, we may assume that $L, M > 0$. Then $f$ is a Lipschitz function (with Lipschitz constant 1) provided that $X$ is endowed with the metric

$$d\left((x,y),(u,v)\right) = L\left|x-u\right| + M\left|y-v\right|.$$

Now the conclusion follows from Theorem 1. ■

## 3. Comparison of arithmetic means

Suppose that $X$ is a locally compact bounded metric space on which there are given two Borel probability measures $\mu$ and $\nu$. We are interested to estimate the difference

$$\int_X f d\mu - \int_X f d\nu$$

for $f$ a Lipschitz function on $X$, with values in a Banach space $E$. For $\nu = \delta_x$, this reduces to the classical Ostrowski inequality.

Following the ideas in the preceding section we are led to

$$\left\| \int_X f d\mu - \int_X f d\nu \right\| \leq \|f\|_L \int_X \int_X d(x,y) d\mu(x) d\nu(y)$$

but this is not always the best result.

For example, for $\mu = dx/(b-a)$ and $\nu = (\delta_a + \delta_b)/2$ (on $X = [a, b]$) it yields the trapezoid inequality

$$\left\| \frac{1}{b-a} \int_a^b f(x)\ dx - \frac{f(a)+f(b)}{2} \right\| \leq C \|f\|_L (b-a),$$

with $C = 1/2$. This can be improved up to $C = 1/4$ within Ostrowski theory (by applying $(O)$ to $f|[a,(a+b)/2]$ and $f|[(a+b)/2, b]$, for $x = a$ and $x = b$ respectively).

However, the Iyengar inequality gives us a better upper bound:

$$(Iy)\qquad \left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{\|f\|_L}{4}(b-a) - \frac{(f(b)-f(a))^2}{4(b-a)\,\|f\|_L}.$$

See [6], [7], or [9] for details. We can combine $(O)$ and $(Iy)$ to get a more general result:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \left[\lambda f\left(\frac{a+b}{2}\right) + (1-\lambda)\frac{f(a)+f(b)}{2}\right]\right|$$
$$\le \frac{\|f\|_L}{4}(b-a) - (1-\lambda)\frac{(f(b)-f(a))^2}{4(b-a)\,\|f\|_L},$$

for every Lipschitz function $f$ and every $\lambda \in [0,1]$. This is optimal in the Lipschitz class, but better results are known for smooth functions. See [7].

There is a large activity (motivated by the problems in numerical integration) concerning the approximation of probability measures by convex combinations of Dirac measures. However, sharp general formulas remain to be found.

## 4. The multiplicative setting

According to [8], the *multiplicative mean value* of a continuous function $f : [a,b] \to (0,\infty)$ (where $0 < a < b$) is defined by the formula

$$M_\star(f) = \exp\left(\frac{1}{\log b - \log a}\int_{\log a}^{\log b} \log f(e^t)dt\right)$$
$$= \exp\left(\frac{1}{\log b - \log a}\int_a^b \log f(t)\,\frac{dt}{t}\right)$$

The main properties of the multiplicative mean are listed below:

$$M_*(1) = 1$$
$$m \le f \le M \ \Rightarrow m \le M_*(f) \le M$$
$$M_*(fg) = M_*(f)\,M_*(g).$$

Given a function $f : I \to (0,\infty)$ (with $I \subset (0,\infty)$) we shall say that $f$ is *multiplicatively Lipschitzian* provided there exists a constant $L > 0$ such that

$$\max\left\{\frac{f(x)}{f(y)}, \frac{f(y)}{f(x)}\right\} \le \left(\frac{y}{x}\right)^L$$

for all $x < y$ in $I$; the smallest $L$ for which the above inequality holds constitutes the *multiplicative Lipschitz* constant of $f$ and it will be denoted by $||f||_{\star Lip}$.

**Remark 1.** *Though the family of multiplicatively Lipschitz functions is large enough (to deserve attention in its own), we know the exact value of the multiplicative Lipschitz constant only in few cases:*

*i) If* $f$ *is of the form* $f(x) = x^\alpha$, *then* $||f||_{\star Lip} = \alpha$.

*ii) If* $f = \exp|[a,b]$ *(where* $0 < a < b$*), then* $||f||_{\star Lip} = b$.

*iii) Clearly,* $||f||_{\star Lip} \le 1$ *for every non-decreasing functions* $f$ *such that* $f(x)/x$ *is non-increasing. For example, this is the case of the functions* sin *and* sec *on* $(0, \pi/2)$.

*iv) If $f$ and $g$ are two multiplicatively Lipschitzian functions (defined on the same interval) and $\alpha, \beta \in \mathbb{R}$, then $f^\alpha g^\beta$ is multiplicatively Lipschitzian too. Moreover,*

$$||f^\alpha g^\beta||_{\star Lip} \leq |\alpha| \cdot ||f||_{\star Lip} + |\beta| \cdot ||g||_{\star Lip}.$$

The following result represents the multiplicative counterpart of the classical Ostrowski inequality:

**Theorem 2.** *Let $f : [a, b] \to (0, \infty)$ be a multiplicatively Lipschitz function with $||f||_{\star Lip} = L$. Then*

$$\max\left\{\frac{f(x)}{M_*(f)}, \frac{M_*(f)}{f(x)}\right\} \leq \left(\frac{b}{a}\right)^{L\left(1/4+\log^2(x/\sqrt{ab})/\log^2(b/a)\right)}.$$

*Proof.* In fact,

$$\begin{aligned}
\frac{M_*(f)}{f(x)} &= \exp\left(\frac{1}{\log b - \log a}\int_a^b \log\frac{f(t)}{f(x)}\,\frac{dt}{t}\right)\\
&\leq \exp\left(\frac{L}{\log(b/a)}\left(\int_a^x \log\frac{x}{t}\,\frac{dt}{t} + \int_x^b \log\frac{t}{x}\,\frac{dt}{t}\right)\right)\\
&= \exp\left(\frac{L}{\log(b/a)}\left(\log x(2\log x - \log a - \log b) - \log^2 x + \frac{\log^2 a}{2} + \log^2 b\right)\right)\\
&= \exp\left(\frac{L}{\log b/a}\cdot\frac{(\log x - \log a)^2 + (\log b - \log x)^2}{2}\right)\\
&= \left(\frac{b}{a}\right)^{L\left(1/4+\log^2(x/\sqrt{ab})/\log^2(b/a)\right)}
\end{aligned}$$

and a similar estimate is valid for $f(x)/M_*(f)$. ■

For $f = \exp|[a, b]$ (where $0 < a < b$), we have $M_*(f) = \exp\left(\frac{b-a}{\log b - \log a}\right)$ and $||f||_{\star Lip} = b$. By Theorem 3.4, we infer the inequalities

$$\exp\left|\frac{b-a}{\log b - \log a} - x\right| \leq \left(\frac{b}{a}\right)^{b\left(1/4+\log^2(x/\sqrt{ab})/\log^2(b/a)\right)}$$

i.e.,

$$\left|\frac{b-a}{\log b - \log a} - x\right| \leq b\left(1/4 + \frac{\log^2(x/\sqrt{ab})}{(\log b - \log a)^2}\right)(\log b - \log a)$$

for every $x \in [a, b]$. Particularly,

$$\left|\frac{b-a}{\log b - \log a} - \sqrt{ab}\right| \leq \frac{b}{4}(\log b - \log a).$$

## 5. An open problem

The deviation of the values of a function from its mean value can be estimated via a variety of norms. For example, the Ostrowski inequality yields

$$(O') \qquad \left\|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right\|_\infty \leq \frac{b-a}{2}\|f'\|_\infty$$

for every $f \in C^1([a, b])$.

In some instances, the $L^p$-norms are more suitable. An old result in this direction is the following inequality due to W. Stekloff [12], [13], [7],

$$(S) \qquad \left(\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right|^2 dx\right)^{1/2} \leq \frac{b-a}{\pi}\left(\int_a^b |f'(x)|^2\, dx\right)^{1/2}$$

that works also for every $f \in C^1([a,b])$. In terms of variance, $(S)$ may be read as

$$\operatorname{var}(f) \leq \frac{b-a}{\pi^2}\int_a^b |f'(t)|^2\, dt,$$

so that, combined with the Schwarz inequality, it yields the following estimate for the covariance of two random variables (of class $C^1$) :

$$\begin{aligned} \operatorname{cov}(f,g) &\leq \operatorname{var}^{1/2}(f)\cdot \operatorname{var}^{1/2}(g) \\ &\leq \frac{b-a}{\pi^2}\left(\int_a^b |f'(t)|^2\, dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\, dt\right)^{1/2}. \end{aligned}$$

There is a large literature in this area, including deep results in the higher dimensional case. See [7].

A natural way to pack together $(O')$ and $(S)$ is as follows.

Consider a separable Banach lattice $E$. Then $E$ contains quasi-interior points $u > 0$ and admits strictly positive functionals $x' \in E'$. This means that

$$\lim_{n\to\infty} \|x - x \wedge nu\| = 0 \text{ for every } x \in E,\ x \geq 0,$$

and

$$x \geq 0,\ x'(x) = 0 \text{ implies } x = 0.$$

See H. H. Schaefer [11], for details.

To each such a pair $(u, x')$ with $x'(u) = 1$, one can associate a positive linear projection,

$$M : E \to E, \quad M(x) = x'(x)\cdot u,$$

whose image is $\mathbb{R}\cdot u$. Clearly, this projection provides an analogue of the integral mean.

**Open Problem.** *Characterize all the pairs $(u, x')$ as above, for which there exist a densely defined linear operator $D : \operatorname{dom} D \subset E \to E$ and a positive constant $C = C(x', u)$ such that*

$$\|x - M(x)\| \leq C\,\|D(x)\| \text{ for every } x \in \operatorname{dom} D.$$

In the examples at the beginning of this section, $E$ is one of the spaces $C([a,b])$ or $L^2([a,b])$, $u$ is the function identically 1, $x'$ is the normalized Lebesgue measure and $D$ is the differential. The same picture follows from Corollary 3 above. The problem is how general could be the existence of a differential like operator in the context of separable Banach lattices.

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