SOME GRÜSS' TYPE INEQUALITIES IN 2-INNER PRODUCT SPACES AND APPLICATIONS FOR DETERMINANTAL INTEGRAL INEQUALITIES

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Abstract. Some new Grüss type inequalities in 2-inner product spaces are given. Using this framework, some determinantal integral inequalities for synchronous functions are also derived.

1. Introduction

In [3], the author has proved the following Grüss' type inequality in real or complex inner product spaces.

Theorem 1. *Let $(H, \langle ., . \rangle)$ be an inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e \in H$, $\|e\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the conditions*

$$\operatorname{Re}\langle \Phi e - x, x - \varphi e\rangle \geq 0 \text{ and } \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.1}$$

or, equivalently,

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi| \quad \text{and} \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.2}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| . \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in the sense that it can not be replaced by a smaller constant.

For integral and discrete inequalities that are particular cases of the above theorem we refer to the recent paper [4].

In [6], an extension for real 2-inner product spaces was obtained and some applications for sequences and integrals were pointed out.

It is the main aim of this paper to establish the corresponding version of Grüss' inequality for both real and complex 2-inner product spaces. A refinement and some related inequalities that complement the results from [6] are also given. Further, using the natural framework of 2-inner product spaces, some determinantal integral inequalities for synchronous functions are pointed out as well.

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2. Preliminaries

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

($2\mathrm{I}_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,

($2\mathrm{I}_2$) $(x, x|z) = (z, z|x)$,

($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,

($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,

($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product $(\cdot, \cdot|\cdot)$ can be immediately obtained as follows [2]:

(1) If $\mathbb{K} = \mathbb{R}$, then ($2\mathrm{I}_3$) reduces to

$$(y, x|z) = (x, y|z).$$

(2) From ($2\mathrm{I}_3$) and ($2\mathrm{I}_4$), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{2.1}$$

(3) Using ($2I_2$)–($2I_5$), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{2.2}$$

In the real case, (2.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{2.3}$$

and, using this formula, it is easy to see, for any $\alpha \in \mathbb{R}$, that

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{2.4}$$

In the complex case, using (2.1) and (2.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (2.2), yields

(2.5) $$(x,y|z)=\frac{1}{4}[(z,z|x+y)-(z,z|x-y)]+\frac{i}{4}[(z,z|x+iy)-(z,z|x-iy)].$$

Using the above formula and (2.1), we have, for any $\alpha\in\mathbb{C}$, that

(2.6) $$(x,y|\alpha z)=|\alpha|^2(x,y|z).$$

However, for $\alpha\in\mathbb{R}$, (2.6) reduces to (2.4).

Also, from (2.6) it follows that

$$(x,y|0)=0.$$

(4) For any three given vectors $x,y,z\in X$, consider the vector $u=(y,y|z)x-(x,y|z)y$. By $(2I_1)$, we know that $(u,u|z)\geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u,u|z)\geq 0$ can be rewritten as

(2.7) $$(y,y|z)[(x,x|z)(y,y|z)-|(x,y|z)|^2]\geq 0.$$

For $x=z$, (2.7) becomes

$$-(y,y|z)|(z,y|z)|^2\geq 0,$$

which implies that

(2.8) $$(z,y|z)=(y,z|z)=0$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (2.8) holds too. Thus (2.8) is true for any two vectors $y,z\in X$. Now, if y and z are linearly independent, then $(y,y|z)>0$ and, from (2.7), it follows

(2.9) $$|(x,y|z)|^2\leq (x,x|z)(y,y|z).$$

Using (2.8), it is easy to check that (2.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (2.9) holds for any three vectors $x,y,z\in X$ and is strict unless the vectors $u=(y,y|z)x-(x,y|z)y$ and z are linearly dependent. *In fact, we have the equality in (2.9) if and only if the three vectors x,y and z are linearly dependent* [2].

In any given 2-inner product space $(X,(\cdot,\cdot|\cdot))$, we can define a function $\|\cdot|\cdot\|$ on $X\times X$ by

(2.10) $$\|x|z\|=\sqrt{(x,x|z)}$$

for all $x,z\in X$.

It is easy to see that this function satisfies the following conditions:

($2N_1$) $\|x|z\|\geq 0$ and $\|x|z\|=0$ if and only if x and z are linearly dependent,

($2N_2$) $\|z|x\|=\|x|z\|$,

($2N_3$) $\|\alpha x|z\|=|\alpha|\|x|z\|$ for any scalar $\alpha\in\mathbb{K}$,

($2N_4$) $\|x+x'|z\|\leq\|x|z\|+\|x'|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X\times X$ and satisfying the conditions $(2N_1)$–$(2N_4)$ is called a 2-*norm* on X and $(X,\|\cdot|\cdot\|)$ is called a *linear* 2-*normed space* [5].

Whenever a 2-inner product space $(X,(\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X,\|\cdot|\cdot\|)$ with the 2-norm defined by (2.10).

3. A Grüss' Type Inequality in 2-Inner Product Spaces

The following lemma holds.

Lemma 1. *Let a,x,z,A be vectors in the 2-inner product space $(X,(.,.|))$ over $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ with $a\neq A$. Then*

$$\operatorname{Re}(A-x,x-a|z)\geq 0$$

if and only if

$$\left\|x-\frac{a+A}{2}|z\right\|\leq\frac{1}{2}\|A-a|z\|.$$

Proof. Define

$$I_1:=\operatorname{Re}(A-x,x-a|z)\,,I_2:=\frac{1}{4}\|A-a|z\|^2-\left\|x-\frac{a+A}{2}|z\right\|^2.$$

A simple calculation shows that

$$I_1=I_2=\operatorname{Re}[(x,a|z)+(A,x|z)]-\operatorname{Re}(A,a|z)-\|x|z\|^2$$

and thus, obviously, $I_1\geq 0$ iff $I_2\geq 0$ showing the required equivalence. ∎

The following corollary is obvious

Corollary 1. *Let $x,z,e\in X$ with $\|e,z\|=1$ and $\delta,\Delta\in\mathbb{K}$ with $\delta\neq\Delta$. Then*

$$\operatorname{Re}(\Delta e-x,x-\delta e|z)\geq 0$$

if and only if

$$\left\|x-\frac{\delta+\Delta}{2}\cdot e|z\right\|\leq\frac{1}{2}|\Delta-\delta|.$$

The following lemma also holds.

Lemma 2. *Let $x,z,e\in X$ with $\|e,z\|=1$. Then one has the following representation*

$$0\leq\|x|z\|^2-|(x,e|z)|^2=\inf_{\lambda\in\mathbb{K}}\|x-\lambda e|z\|^2. \tag{3.1}$$

Proof. Observe, for any $\lambda\in\mathbb{K}$, that

$$\begin{aligned}(x-\lambda e,x-(x,e|z)\,e|z)&=\|x|z\|^2-|(x,e|z)|^2-\lambda\left[(e,x|z)-(e,x|z)\,\|e|z\|^2\right]\\&=\|x|z\|^2-|(x,e|z)|^2.\end{aligned}$$

Using Schwarz's inequality for 2-inner products, we have

$$\begin{aligned}\left[\|x|z\|^2-|(x,e|z)|^2\right]^2&=|(x-\lambda e,x-(x,e|z)\,e|z)|^2\\&\leq\|x-\lambda e|z\|^2\,\|x-(x,e|z)\,e|z\|^2\\&=\|x-\lambda e|z\|^2\left[\|x|z\|^2-|(x,e|z)|^2\right],\end{aligned}$$

which gives the bound

$$\|x|z\|^2-|(x,e|z)|^2\leq\|x-\lambda e|z\|^2,\ \lambda\in\mathbb{K}. \tag{3.2}$$

Taking the infimum in (3.2) over $\lambda \in \mathbb{K}$, we deduce

$$\|x|z\|^2 - |(x,e|z)|^2 \leq \inf_{\lambda \in \mathbb{K}} \|x - \lambda e|z\|^2 .$$

Since, for $\lambda_0 = (x,e|z)$, we get $\|x - \lambda_0 e|z\|^2 = \|x|z\|^2 - |(x,e|z)|^2$, then the representation (3.1) is proved. ∎

We are able now to provide a different proof for the Grüss' type inequality in 2-inner product spaces, than the one from paper [6]. The sharpness of the constant is also proved.

Theorem 2. *Let $(X, (.,.|.))$ be a 2-inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R},\mathbb{C})$ and $e, z \in X, \|e|z\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in X such that the conditions*

$$\text{(3.3)} \qquad \operatorname{Re}(\Phi e - x, x - \varphi e|z) \geq 0, \quad \operatorname{Re}(\Gamma e - y, y - \gamma e|z) \geq 0$$

hold or, equivalently, the following assumptions

$$\text{(3.4)} \qquad \left\| x - \frac{\varphi + \Phi}{2} \cdot e|z \right\| \leq \frac{1}{2} |\Phi - \varphi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} \cdot e|z \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

are valid, then one has the inequality

$$\text{(3.5)} \qquad |(x,y|z) - (x,e|z)(e,y|z)| \leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| .$$

The constant $\frac{1}{4}$ is best possible.

Proof. If we apply Schwarz's inequality in 2-inner product spaces for the vectors $x - (x,e|z)\,e, y - (y,e|z)\,e$, then it can be easily shown (see, for example, the proof of Theorem 1 from [6]), that

$$\text{(3.6)} \quad |(x,y|z) - (x,e|z)(e,y|z)| \leq \left[\|x|z\|^2 - |(x,e|z)|^2 \right]^{\frac{1}{2}} \left[\|y|z\|^2 - |(y,e|z)|^2 \right]^{\frac{1}{2}},$$

for any $x, y, z \in X$ and $e \in X, \|e,z\| = 1$. Using Lemma 2 and the conditions (3.4) we obviously have that

$$\left[\|x|z\|^2 - |(x,e|z)|^2 \right]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e|z\| \leq \left\| x - \frac{\varphi + \Phi}{2} \cdot e|z \right\| \leq \frac{1}{2} |\Phi - \varphi|$$

and

$$\left[\|y|z\|^2 - |(y,e|z)|^2 \right]^{\frac{1}{2}} = \inf_{\lambda \in \mathbb{K}} \|y - \lambda e|z\| \leq \left\| y - \frac{\gamma + \Gamma}{2} \cdot e|z \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

and so, by (3.6), the desired inequality (3.5) is obtained.

To prove the sharpness of the constant $\frac{1}{4}$, assume that (3.5) holds with $x = y$ and a constant $C > 0$, *i.e.*,

$$\text{(3.7)} \qquad \|x|z\|^2 - |(x,e|z)|^2 \leq C\,|\Phi - \varphi|^2$$

provided x, e, z, φ and Φ satisfy the hypothesis of the theorem.

For $e, z \in X, \|e,z\| = 1$ and $\varphi \neq \Phi$, consider $m \in X$ with $\|e,z\| = 1$ and $(m,e|z) = 0$. If we define

$$x = \frac{\varphi + \Phi}{2} e + \frac{\Phi - \varphi}{2} m,$$

then we have

$$(\Phi e - x, x - \varphi e|z) = \frac{|\Phi - \varphi|^2}{4}(e - m, m + e|z) = 0$$

and thus the condition (3.3) is fulfilled. From (3.7) we deduce

$$\left\| \frac{\varphi + \Phi}{2} e + \frac{\Phi - \varphi}{2} m|z \right\|^2 - \left| \left(\frac{\varphi + \Phi}{2} e + \frac{\Phi - \varphi}{2} m, e|z \right) \right|^2 \leq C |\Phi - \varphi|^2 \tag{3.8}$$

and, since

$$\left\| \frac{\varphi + \Phi}{2} e + \frac{\Phi - \varphi}{2} m|z \right\|^2 = \frac{|\Phi + \varphi|^2}{4} + \frac{|\Phi - \varphi|^2}{4}$$

and

$$\left| \left(\frac{\varphi + \Phi}{2} e + \frac{\Phi - \varphi}{2} m, e|z \right) \right|^2 = \frac{|\Phi + \varphi|^2}{4},$$

then, by (3.8), we get

$$\frac{|\Phi - \varphi|^2}{4} \leq C |\Phi - \varphi|^2,$$

for $\Phi \neq \varphi$, which implies that $C \geq \frac{1}{4}$, and the proof is completed. ∎

4. A Refinement of Grüss Inequality in 2-Inner Product Spaces

The following result improving (3.5) holds.

Theorem 3. *Let $(X, (.,.|.))$ be an inner product space over $\mathbb{K}\,(\mathbb{K} = \mathbb{R}, \mathbb{C})$ and $e, z \in X$, $\|e|z\| = 1$. If $\varphi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in X such that the conditions* (3.3) *or, equivalently,* (3.4) *hold, then we have the inequality*

$$\begin{aligned} &|(x, y|z) - (x, e|z)(e, y|z)| \\ &\leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| - [\operatorname{Re}(\Phi e - x, x - \varphi e|z)]^{\frac{1}{2}} [\operatorname{Re}(\Gamma e - y, y - \gamma e|z)]^{\frac{1}{2}} \\ &\left(\leq \frac{1}{4} |\Phi - \varphi| \cdot |\Gamma - \gamma| \right). \end{aligned} \tag{4.1}$$

The constant $\frac{1}{4}$ is best possible.

Proof. As above, we have

$$|(x, y|z) - (x, e|z)(e, y|z)|^2 \leq \left[\|x|z\|^2 - |(x, e|z)|^2 \right] \left[\|y|z\|^2 - |(y, e|z)|^2 \right]. \tag{4.2}$$

By simple computation, we also observe that the following identities are valid:

$$\begin{aligned} &\|x|z\|^2 - |(x, e|z)|^2 \\ &= \operatorname{Re}\left[(\Phi - (x, e|z)) \left(\overline{(x, e|z)} - \overline{\varphi} \right) \right] - \operatorname{Re}(\Phi e - x, x - \varphi e|z) \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\|y|z\|^2 - |(y, e|z)|^2 \\ &= \operatorname{Re}\left[(\Gamma - (y, e|z)) \left(\overline{(y, e|z)} - \overline{\gamma} \right) \right] - \operatorname{Re}(\Gamma e - x, x - \gamma e|z). \end{aligned} \tag{4.4}$$

Using the elementary inequality for complex numbers

$$4\operatorname{Re}\left(a\overline{b}\right) \le |a+b|^2\,;a,b\in\mathbb{K}\,(\mathbb{K}=\mathbb{R},\,\mathbb{C})$$

we may state that

$$\operatorname{Re}\left[(\Phi-(x,e|z))\left(\overline{(x,e|z)}-\overline{\varphi}\right)\right] \le \frac{1}{4}|\Phi-\varphi|^2 \tag{4.5}$$

and

$$\operatorname{Re}\left[(\Gamma-(y,e|z))\left(\overline{(y,e|z)}-\overline{\gamma}\right)\right] \le \frac{1}{4}|\Gamma-\gamma|^2. \tag{4.6}$$

Consequently, by $(4.2)-(4.6)$, we may state that

$$\begin{aligned}&|(x,y|z)-(x,e|z)\,(e,y|z)|^2\\ &\le \left[\frac{1}{4}|\Phi-\varphi|^2-\left([\operatorname{Re}(\Phi e-x,x-\varphi e|z)]^{\frac{1}{2}}\right)^2\right]\\ &\quad\times\left[\frac{1}{4}|\Gamma-\gamma|^2-\left([\operatorname{Re}(\Gamma e-y,y-\gamma e|z)]^{\frac{1}{2}}\right)^2\right].\end{aligned}$$

Finally, using the elementary inequality for positive real numbers

$$\left(m^2-n^2\right)\left(p^2-q^2\right)\le(mp-nq)^2,$$

we have

$$\begin{aligned}&\left[\frac{1}{4}|\Phi-\varphi|^2-\left([\operatorname{Re}(\Phi e-x,x-\varphi e|z)]^{\frac{1}{2}}\right)^2\right]\\ &\times\left[\frac{1}{4}|\Gamma-\gamma|^2-\left([\operatorname{Re}(\Gamma e-y,y-\gamma e|z)]^{\frac{1}{2}}\right)^2\right]\\ &\le\left(\frac{1}{4}|\Phi-\varphi|\cdot|\Gamma-\gamma|-[\operatorname{Re}(\Phi e-x,x-\varphi e|z)]^{\frac{1}{2}}\,[\operatorname{Re}(\Gamma e-y,y-\gamma e|z)]^{\frac{1}{2}}\right)^2,\end{aligned}$$

which gives the desired inequality (4.1).

The sharpness of the constant $\frac{1}{4}$ may be proven in a similar way to the one incorporated in Theorem 2 and we omit the details. ∎

5. Some Companion Inequalities

The following companion of Grüss inequality in inner product spaces holds.

Theorem 4. *Let* $(X,(.,.|.))$ *be a 2-inner product space over* $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\,\mathbb{C})$ *and* $e,z\in X$, $\|e|z\|=1$. *If* $\gamma,\Gamma\in\mathbb{K}$ *and* $x,y\in X$ *are so that*

$$\operatorname{Re}\left(\Gamma e-\frac{x+y}{2},\frac{x+y}{2}-\gamma e|z\right)\ge 0 \tag{5.1}$$

or, equivalently,

$$\left\|\frac{x+y}{2}-\frac{\gamma+\Gamma}{2}\cdot e|z\right\|\le\frac{1}{2}|\Gamma-\gamma| \tag{5.2}$$

holds, then we have the inequality

$$\operatorname{Re}\left[(x,y|z)-(x,e|z)\,(e,y|z)\right]\le\frac{1}{4}|\Gamma-\gamma|^2. \tag{5.3}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Start with the well known inequality

$$\operatorname{Re}(w,u|z) \leq \frac{1}{4}\|w+u|z\|^2, \ w,u \in X. \tag{5.4}$$

Since

$$(x,y|z)-(x,e|z)(e,y|z) = (x-(x,e|z)e, y-(y,e|z)e|z),$$

then, using (5.4), we may write

$$\begin{aligned} \operatorname{Re}[(x,y|z)-(x,e|z)(e,y|z)] &= \operatorname{Re}[(x-(x,e|z)e, y-(y,e|z)e|z)] \\ &\leq \frac{1}{4}\|x-\langle x,e\rangle e+y-\langle y,e\rangle e|z\|^2 \\ &= \left\|\frac{x+y}{2}-\left(\frac{x+y}{2},e|z\right)\cdot e|z\right\|^2 \\ &= \left\|\frac{x+y}{2}|z\right\|^2-\left|\left(\frac{x+y}{2},e|z\right)\right|^2. \end{aligned} \tag{5.5}$$

If we apply the Grüss inequality (3.5) for, say, $a=b=\frac{x+y}{2}$, then we get

$$\left\|\frac{x+y}{2}|z\right\|^2-\left|\left(\frac{x+y}{2},e|z\right)\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2. \tag{5.6}$$

Making use of (5.5) and (5.6) we deduce (5.3).

The fact that $\frac{1}{4}$ is the best possible constant in (5.3) follows by the fact that if in (5.1) we choose $x=y$, then it becomes $\operatorname{Re}(\Gamma e-x, x-\gamma e|z) \geq 0$, implying $0 \leq \|x|z\|^2-|(x,e|z)|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2$, for which, by Grüss' inequality in 2-inner product spaces, we know that the constant $\frac{1}{4}$ is best possible. ∎

The following corollary might be of interest if one wanted to evaluate the absolute value of

$$\operatorname{Re}[(x,y|z)-(x,e|z)(e,y|z)].$$

Corollary 2. *Let $(X,(.,.|.))$ be a 2-inner product space over $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ and $e,z \in X$, $\|e,z\|=1$. If $\gamma,\Gamma \in \mathbb{K}$ and $x,y \in X$ are so that*

$$\operatorname{Re}\left(\Gamma e-\frac{x\pm y}{2}, \frac{x\pm y}{2}-\gamma e|z\right) \geq 0 \tag{5.7}$$

or, equivalently,

$$\left\|\frac{x\pm y}{2}-\frac{\gamma+\Gamma}{2}\cdot e|z\right\| \leq \frac{1}{2}|\Gamma-\gamma|, \tag{5.8}$$

then we have the inequality

$$|\operatorname{Re}[(x,y|z)-(x,e|z)(e,y|z)]| \leq \frac{1}{4}|\Gamma-\gamma|^2. \tag{5.9}$$

If the inner product space X is real, then (for $m,M \in \mathbb{R}$, $M>m$)

$$\left(Me-\frac{x\pm y}{2}, \frac{x\pm y}{2}-me|z\right) \geq 0 \tag{5.10}$$

or, equivalently,

$$\left\|\frac{x\pm y}{2}-\frac{m+M}{2}\cdot e|z\right\| \leq \frac{1}{2}(M-m), \tag{5.11}$$

imply

$$|(x,y|z)-(x,e|z)(e,y|z)|\le\frac{1}{4}(M-m)^2. \tag{5.12}$$

In both inequalities (5.9) *and* (5.12), *the constant* $\frac{1}{4}$ *is best possible.*

Proof. We only remark that, if

$$\operatorname{Re}\left(\Gamma e-\frac{x-y}{2},\frac{x-y}{2}-\gamma e|z\right)\ge 0$$

holds, then by Theorem 4, we get

$$\operatorname{Re}\left[(-x,y|z)+(x,e|z)(e,y|z)\right]\le\frac{1}{4}|\Gamma-\gamma|^2,$$

which shows that

$$\operatorname{Re}\left[(x,y|z)-(x,e|z)(e,y|z)\right]\ge-\frac{1}{4}|\Gamma-\gamma|^2. \tag{5.13}$$

Making use of (5.3) and (5.13) we deduce the desired result (5.9). ∎

Finally, we may state and prove the following dual result as well.

Proposition 1. *Let* $(X,(.,.|.))$ *be an inner product space over* $\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ *and* $e,z\in X,\|e,z\|=1$. *If* $\varphi,\Phi\in\mathbb{K}$ *and* $x,y\in X$ *are so that*

$$\operatorname{Re}\left[(\Phi-(x,e|z))\left(\overline{(x,e|z)}-\overline{\varphi}\right)\right]\le 0, \tag{5.14}$$

then we have the inequalities

$$\begin{aligned}\|x-(x,e|z)\,e|z\| &\le\left[\operatorname{Re}\langle x-\Phi e,x-\varphi e|z\rangle\right]^{\frac{1}{2}}\\ &\le\frac{\sqrt{2}}{2}\left[\|x-\Phi e|z\|^2+\|x-\varphi e|z\|^2\right]^{\frac{1}{2}}.\end{aligned} \tag{5.15}$$

Proof. We know that the following identity holds true (see (4.3))

$$\begin{aligned}&\|x|z\|^2-|(x,e|z)|^2\\ =\ &\operatorname{Re}\left[(\Phi-(x,e|z))\left(\overline{(x,e|z)}-\overline{\varphi}\right)\right]+\operatorname{Re}(x-\Phi e,x-\varphi e|z).\end{aligned} \tag{5.16}$$

Using the assumption (5.14) and the fact that

$$\|x|z\|^2-|(x,e|z)|^2=\|x-(x,e|z)\,e|z\|^2,$$

then by (5.16) we deduce the first inequality in (5.15).

The second inequality in (5.15) follows by the fact that for any $v,w\in X$ one has

$$\operatorname{Re}\langle w,v|z\rangle\le\frac{1}{2}\left(\|w|z\|^2+\|v|z\|^2\right).$$

The proposition is thus proved. ∎

6. Determinantal Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σa $\sigma-$algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(s)|f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$(f, g|h)_\rho := \frac{1}{2}\int_\Omega \int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s)\, d\mu(t), \tag{6.1}$$

where by

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}$$

we denote the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

generating the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2}\int_\Omega \int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}^2 d\mu(s)\, d\mu(t)\right)^{1/2}. \tag{6.2}$$

A simple calculation with integrals reveals that

$$(f, g|h)_\rho = \begin{vmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{6.3}$$

and

$$\|f|h\|_\rho = \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^{1/2}, \tag{6.4}$$

where, for simplicity, instead of $\int_\Omega \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_\Omega \rho f g d\mu$.

We recall that the pair of functions $(q, p) \in L^2_\rho(\Omega) \times L^2_\rho(\Omega)$ is called *synchronous* if

$$(q(x) - q(y))(p(x) - p(y)) \geq 0$$

for *a.e.* $x, y \in \Omega$.

We note that, if $\Omega = [a, b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h \in L^2_\rho(\Omega)$ is such that $h(x) \neq 0$ for *a.e.* $x \in \Omega$. Then, by the definition of 2-inner product $(f, g|h)_\rho$, we have

$$\begin{aligned} &(f, g|h)_\rho \\ = \; &\frac{1}{2}\int_\Omega \int_\Omega \rho(s)\rho(t) h^2(s) h^2(t) \left(\frac{f(s)}{h(s)} - \frac{f(t)}{h(t)}\right)\left(\frac{g(s)}{h(s)} - \frac{g(t)}{h(t)}\right) d\mu(s)\, d\mu(t) \end{aligned} \tag{6.5}$$

and thus a *sufficient condition* for the inequality

$$(f, g|h)_\rho \geq 0 \tag{6.6}$$

to hold, is that, the pair of functions $\left(\frac{f}{h}, \frac{g}{h}\right)$ are synchronous. It is obvious that, this condition is not necessary.

Using the representations (6.3), (6.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, one may state some interesting determinantal integral inequalities as follows.

Proposition 2. *Let $f, g, h, u \in L^2_\rho(\Omega)$ with $h \neq 0$ a.e. and*

$$\int_\Omega \rho u^2 d\mu \int_\Omega \rho h^2 d\mu - \left(\int_\Omega \rho u h d\mu\right)^2 = 1. \tag{6.7}$$

If $M > m$ and $N > n$ are real numbers with the property that

$$\left(M \cdot \frac{u}{h} - \frac{f}{h}, \frac{f}{h} - m \cdot \frac{u}{h}\right) \text{ and } \left(N \cdot \frac{u}{h} - \frac{g}{h}, \frac{g}{h} - n \cdot \frac{u}{h}\right) \tag{6.8}$$

are synchronous on Ω, then we have the following determinantal integral Grüss type inequality

$$\begin{aligned} &\left|\det\begin{bmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right. \\ &\left.- \det\begin{bmatrix} \int_\Omega \rho f u d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix} \cdot \det\begin{bmatrix} \int_\Omega \rho g u d\mu & \int_\Omega \rho g h d\mu \\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right| \\ \leq\ & \frac{1}{4}(M-m)(N-n). \end{aligned}$$

The proof follows by Theorem 2 applied for the 2-inner product $(.,.|.)_\rho$ defined in (6.1).

If one applies Theorem 3 for the same 2-inner product, then one can state the following refinement of Grüss inequality for determinants.

Proposition 3. *With the assumptions of Proposition 2, we have the following determinantal integral inequality*

$$\begin{aligned} &\left|\det\begin{bmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right. \\ &\left.- \det\begin{bmatrix} \int_\Omega \rho f u d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix} \cdot \det\begin{bmatrix} \int_\Omega \rho g u d\mu & \int_\Omega \rho g h d\mu \\ \int_\Omega \rho u h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right| \\ \leq\ & \frac{1}{4}(M-m)(N-n) \\ &- \left(\det\begin{bmatrix} \int_\Omega \rho (Mu-f)(f-mu) d\mu & \int_\Omega \rho (Mu-f) h d\mu \\ \int_\Omega \rho (f-mu) h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right)^{1/2} \\ &\times \left(\det\begin{bmatrix} \int_\Omega \rho (Nu-g)(g-nu) d\mu & \int_\Omega \rho (Nu-g) h d\mu \\ \int_\Omega \rho (g-nu) h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix}\right)^{1/2} \end{aligned}$$

Similar integral inequalities may be stated if one uses the other results for 2-inner products established in Section 5. For the sake of brevity, we do not state them here.

Remark 1. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

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