# ON THE BOMBIERI INEQUALITY IN INNER PRODUCT SPACES

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Abstract. New results related to the Bombieri generalisation of Bessel's inequality in inner product spaces are given.

## 1. Introduction

Let $(H;(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in the inner product space $H$, i.e., $(e_i,e_j)=\delta_{ij}$ for all $i,j\in\{1,\dots,n\}$ where $\delta_{ij}$ is the Kronecker delta, then the following inequality is well known in the literature as Bessel's inequality (see for example [8, p. 391]):

$$\sum_{i=1}^{n}|(x,e_i)|^2\leq\|x\|^2\quad\text{for any }x\in H. \tag{1.1}$$

For other results related to Bessel's inequality, see [4] – [6] and Chapter XV in the book [8].

In 1971, E. Bombieri [3] (see also [8, p. 394]) gave the following generalisation of Bessel's inequality.

**Theorem 1.** *If $x,y_1,\dots,y_n$ are vectors in the inner product space $(H;(\cdot,\cdot))$, then the following inequality:*

$$\sum_{i=1}^{n}|(x,y_i)|^2\leq\|x\|^2\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|(y_i,y_j)|\right\}, \tag{1.2}$$

*holds.*

It is obvious that if $(y_i)_{1\leq i\leq n}$ are supposed to be orthonormal, then from (1.2) one would deduce Bessel's inequality (1.1).

Another generalisation of Bessel's inequality was obtained by A. Selberg (see for example [8, p. 394]):

**Theorem 2.** *Let $x,y_1,\dots,y_n$ be vectors in $H$ with $y_i\neq 0$ $(i=1,\dots,n)$. Then one has the inequality:*

$$\sum_{i=1}^{n}\frac{|(x,y_i)|^2}{\sum_{j=1}^{n}|(y_i,y_j)|}\leq\|x\|^2. \tag{1.3}$$

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In this case, also, if $(y_i)_{1\leq i\leq n}$ are orthonormal, then from (1.3) one may deduce Bessel's inequality.

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [7] (see also [8, p. 395]).

**Theorem 3.** *With the assumptions in Theorem 1, one has*

$$\sum_{i=1}^{n}|(x,y_i)|\leq\|x\|\left(\sum_{i,j=1}^{n}|(y_i,y_j)|\right)^{\frac{1}{2}}. \tag{1.4}$$

If in (1.4) one chooses $y_i=e_i$ $(i=1,\dots,n)$, where $(e_i)_{1\leq i\leq n}$ are orthonormal vectors in $H$, then

$$\sum_{i=1}^{n}|(x,e_i)|\leq\sqrt{n}\,\|x\|\,,\quad\text{for any}\quad x\in H. \tag{1.5}$$

In 1992 J.E. Pečarić [9] (see also [8, p. 394]) proved the following general inequality in inner product spaces.

**Theorem 4.** *Let $x,y_1,\dots,y_n\in H$ and $c_1,\dots,c_n\in\mathbb{K}$. Then*

$$\begin{aligned}\left|\sum_{i=1}^{n}c_i\,(x,y_i)\right|^2 &\leq\|x\|^2\sum_{i=1}^{n}|c_i|^2\left(\sum_{j=1}^{n}|(y_i,y_j)|\right)\\ &\leq\|x\|^2\sum_{i=1}^{n}|c_i|^2\max_{1\leq i\leq n}\left\{\sum_{j=1}^{n}|(y_i,y_j)|\right\}.\end{aligned} \tag{1.6}$$

He showed that the Bombieri inequality (1.2) may be obtained from (1.6) for the choice $c_i=\overline{(x,y_i)}$ (using the second inequality), the Selberg inequality (1.3) may be obtained from the first part of (1.6) for the choice

$$c_i=\frac{\overline{(x,y_i)}}{\sum_{j=1}^{n}|(y_i,y_j)|},i\in\{1,\dots,n\};$$

while the Heilbronn inequality (1.4) may be obtained from the first part of (1.6) if one chooses $c_i=\frac{\overline{(x,y_i)}}{|(x,y_i)|}$, for any $i\in\{1,\dots,n\}$.

For other results connected with the above ones, see [5] and [6].

## 2. Some Preliminary Results

We start with the following lemma which is also interesting in itself.

**Lemma 1.** *Let $z_1,\dots,z_n\in H$ and $\alpha_1,\dots,\alpha_n\in\mathbb{K}$. Then one has the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \tag{2.1}$$

$$\leq \begin{cases} \max\limits_{1\leq k\leq n} |\alpha_k|^2 \sum\limits_{i,j=1}^{n} |(z_i, z_j)| ; \\ \max\limits_{1\leq k\leq n} |\alpha_k| \left(\sum\limits_{i=1}^{n} |\alpha_i|^r\right)^{\frac{1}{r}} \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|\right)^s\right)^{\frac{1}{s}}, \quad r > 1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \max\limits_{1\leq k\leq n} |\alpha_k| \sum\limits_{k=1}^{n} |\alpha_k| \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|\right); \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} |\alpha_i| \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|\right)^q\right)^{\frac{1}{q}}, \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |\alpha_i|^t\right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \qquad\qquad\qquad\qquad\qquad\qquad t > 1,\ \frac{1}{t}+\frac{1}{u}=1; \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\leq i\leq n}\left\{\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}\right\}, \quad p > 1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n} |\alpha_k| \max\limits_{1\leq i\leq n} |\alpha_i| \sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n} |(z_i, z_j)|\right]; \\ \sum\limits_{k=1}^{n} |\alpha_k| \left(\sum\limits_{i=1}^{n} |\alpha_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n} |(z_i, z_j)|\right]^l\right)^{\frac{1}{l}}, \quad m > 1,\ \frac{1}{m}+\frac{1}{l}=1; \\ \left(\sum\limits_{k=1}^{n} |\alpha_k|\right)^2 \max\limits_{i,1\leq j\leq n} |(z_i, z_j)| . \end{cases}$$

*Proof.* We observe that

$$\begin{aligned} (2.2) \qquad \left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 &= \left(\sum_{i=1}^{n} \alpha_i z_i, \sum_{j=1}^{n} \alpha_j z_j\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j}\,(z_i, z_j) = \left|\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j}\,(z_i, z_j)\right| \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_i|\,|\alpha_j|\,|(z_i, z_j)| = \sum_{i=1}^{n} |\alpha_i| \left(\sum_{j=1}^{n} |\alpha_j|\,|(z_i, z_j)|\right) \\ &:= M. \end{aligned}$$

Using Hölder's inequality, we may write that

(2.3)
$$\sum_{j=1}^{n}|\alpha_j|\,|(z_i,z_j)| \leq \begin{cases} \max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{j=1}^{n}|(z_i,z_j)| \\ \left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n}|\alpha_k|\ \max\limits_{1\leq j\leq n}|(z_i,z_j)| \end{cases}$$

for any $i\in\{1,\dots,n\}$, giving

$$M \leq \begin{cases} \max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\sum\limits_{j=1}^{n}|(z_i,z_j)| =: M_1; \\ \left(\sum\limits_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}}\sum\limits_{i=1}^{n}|\alpha_i|\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|^q\right)^{\frac{1}{q}} := M_p, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\ \max\limits_{1\leq j\leq n}|(z_i,z_j)| =: M_\infty. \end{cases} \tag{2.4}$$

By Hölder's inequality we also have:

$$\sum_{i=1}^{n}|\alpha_i|\left(\sum_{j=1}^{n}|(z_i,z_j)|\right) \tag{2.5}$$
$$\leq \begin{cases} \max\limits_{1\leq i\leq n}|\alpha_i|\sum\limits_{i,j=1}^{n}|(z_i,z_j)|; \\ \left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \sum\limits_{i=1}^{n}|\alpha_i|\ \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right); \end{cases}$$

and thus

$$M_1 \leq \begin{cases} \max\limits_{1\leq k\leq n}|\alpha_k|^2\sum\limits_{i,j=1}^{n}|(z_i,z_j)|; \\ \max\limits_{1\leq k\leq n}|\alpha_k|\left(\sum\limits_{i=1}^{n}|\alpha_i|^r\right)^{\frac{1}{r}}\left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right)^s\right)^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \max\limits_{1\leq k\leq n}|\alpha_k|\sum\limits_{i=1}^{n}|\alpha_i|\ \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n}|(z_i,z_j)|\right); \end{cases}$$

and the first 3 inequalities in (2.1) are obtained.

By Hölder's inequality we also have:

$$M_p \leq \left(\sum_{k=1}^{n} |\alpha_k|^p\right)^{\frac{1}{p}} \times \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i| \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}; \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^t\right)^{\frac{1}{t}} \left(\sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{u}{q}}\right)^{\frac{1}{u}}, \quad t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\ \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\leq i\leq n} \left\{\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{1}{q}}\right\}; \end{cases}$$

and the next 3 inequalities in (2.1) are proved.

Finally, by the same Hölder inequality we may state that:

$$M_\infty \leq \sum_{k=1}^{n} |\alpha_k| \times \begin{cases} \max\limits_{1\leq i\leq n} |\alpha_i| \sum\limits_{i=1}^{n} \left(\max\limits_{1\leq j\leq n} |(z_i, z_j)|\right); \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n} \left(\max\limits_{1\leq j\leq n} |(z_i, z_j)|\right)^l\right)^{\frac{1}{l}}, \quad m>1,\ \frac{1}{m}+\frac{1}{l}=1; \\ \sum\limits_{i=1}^{n} |\alpha_i| \max\limits_{1\leq i,j\leq n} |(z_i, z_j)|; \end{cases}$$

and the last 3 inequalities in (2.1) are proved. □

If we would like to have some bounds for $\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2$ in terms of $\sum_{i=1}^{n} |\alpha_i|^2$, then the following corollaries may be used.

**Corollary 1.** *Let $z_1, \ldots, z_n$ and $\alpha_1, \ldots, \alpha_n$ be as in Lemma 1. If $1 < p \leq 2$, $1 < t \leq 2$, then one has the inequality*

$$\left\|\sum_{i=1}^{n} \alpha_i z_i\right\|^2 \leq n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{k=1}^{n} |\alpha_k|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(z_i, z_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}} \tag{2.6}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{t}+\frac{1}{u}=1$.

*Proof.* Observe, by the monotonicity of power means, we may write that

$$\left(\frac{\sum_{k=1}^{n} |\alpha_k|^p}{n}\right)^{\frac{1}{p}} \leq \left(\frac{\sum_{k=1}^{n} |\alpha_k|^2}{n}\right)^{\frac{1}{2}}; \quad 1 < p \leq 2,$$

$$\left(\frac{\sum_{k=1}^{n} |\alpha_k|^t}{n}\right)^{\frac{1}{t}} \leq \left(\frac{\sum_{k=1}^{n} |\alpha_k|^2}{n}\right)^{\frac{1}{2}}; \quad 1 < t \leq 2,$$

from where we get

$$\left(\sum_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$
$$\left(\sum_{k=1}^{n}|\alpha_k|^t\right)^{\frac{1}{t}} \leq n^{\frac{1}{t}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}}.$$

Using the fifth inequality in (2.1), we then deduce (2.6). □

**Remark 1.** *An interesting particular case is the one for* $p = q = t = u = 2$, *giving*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq \sum_{k=1}^{n}|\alpha_k|^2\left(\sum_{i,j=1}^{n}|(z_i, z_j)|^2\right)^{\frac{1}{2}}. \tag{2.7}$$

**Corollary 2.** *With the assumptions of Lemma 1 and if* $1 < p \leq 2$, *then*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq n^{\frac{1}{p}}\sum_{k=1}^{n}|\alpha_k|^2 \max_{1\leq i\leq n}\left[\left(\sum_{j=1}^{n}|(z_i, z_j)|^q\right)^{\frac{1}{q}}\right], \tag{2.8}$$

*where* $\frac{1}{p}+\frac{1}{q}=1$.

*Proof.* Since

$$\left(\sum_{k=1}^{n}|\alpha_k|^p\right)^{\frac{1}{p}} \leq n^{\frac{1}{p}-\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$

and

$$\sum_{k=1}^{n}|\alpha_k| \leq n^{\frac{1}{2}}\left(\sum_{k=1}^{n}|\alpha_k|^2\right)^{\frac{1}{2}},$$

then by the sixth inequality in (2.1) we deduce (2.8). □

In a similar fashion, one may prove the following two corollaries.

**Corollary 3.** *With the assumptions of Lemma 1 and if* $1 < m \leq 2$, *then*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq n^{\frac{1}{m}}\sum_{k=1}^{n}|\alpha_k|^2\left(\sum_{i=1}^{n}\left[\max_{1\leq j\leq n}|(z_i, z_j)|\right]^l\right)^{\frac{1}{l}}, \tag{2.9}$$

*where* $\frac{1}{m}+\frac{1}{l}=1$.

**Corollary 4.** *With the assumptions of Lemma 1, we have:*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq n\sum_{k=1}^{n}|\alpha_k|^2 \max_{1\leq i,j\leq n}|(z_i, z_j)|. \tag{2.10}$$

The following lemma may be of interest as well.

**Lemma 2.** *With the assumptions of Lemma 1, one has the inequalities*

$$\left\|\sum_{i=1}^{n}\alpha_i z_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\sum_{j=1}^{n}|(z_i, z_j)| \tag{2.11}$$

$$\leq \begin{cases} \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{1\leq i\leq n} \left[\sum\limits_{j=1}^{n} |(z_i, z_j)|\right]; \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2p}\right)^{\frac{1}{p}} \left(\left(\sum\limits_{j=1}^{n} |(z_i, z_j)|\right)^{q}\right)^{\frac{1}{q}}, \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \max\limits_{1\leq i\leq n} |\alpha_i|^2 \sum\limits_{i,j=1}^{n} |(z_i, z_j)|. \end{cases}$$

*Proof.* As in Lemma 1, we know that

$$\left\| \sum_{i=1}^{n} \alpha_i z_i \right\|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)|. \tag{2.12}$$

Using the simple observation that (see also [8, p. 394])

$$|\alpha_i| |\alpha_j| \leq \frac{1}{2} \left(|\alpha_i|^2 + |\alpha_j|^2\right), \quad i, j \in \{1, \dots, n\}$$

we have

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_i| |\alpha_j| |(z_i, z_j)| &\leq \frac{1}{2} \sum_{i,j=1}^{n} \left(|\alpha_i|^2 + |\alpha_j|^2\right) |(z_i, z_j)| \\ &= \frac{1}{2} \left[\sum_{i,j=1}^{n} |\alpha_i|^2 |(z_i, z_j)| + \sum_{i,j=1}^{n} |\alpha_j|^2 |(z_i, z_j)|\right] \\ &= \sum_{i,j=1}^{n} |\alpha_i|^2 |(z_i, z_j)|, \end{aligned}$$

which proves the first inequality in (2.11).

The second part follows by Hölder's inequality and we omit the details. □

**Remark 2.** *The first part in (2.11) is the inequality obtained by Pečarić in* [9].

## 3. Some Pečarić Type Inequalities

We are now able to point out the following result which complements the inequality (1.6) due to J.E. Pečarić [9] (see also [8, p. 394]).

**Theorem 5.** *Let $x, y_1, \dots, y_n$ be vectors of an inner product space $(H; (\cdot, \cdot))$ and $c_1, \dots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left| \sum_{i=1}^{n} c_i (x, y_i) \right|^2 \tag{3.1}$$

$$\leq \|x\|^2 \times \begin{cases} \max\limits_{1\leq k\leq n} |c_k|^2 \sum\limits_{i,j=1}^{n} |(y_i, y_j)| ; \\ \max\limits_{1\leq k\leq n} |c_k| \left(\sum\limits_{i=1}^{n} |c_i|^r\right)^{\frac{1}{r}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^s\right]^{\frac{1}{s}}, \quad r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \max\limits_{1\leq k\leq n} |c_k| \sum\limits_{k=1}^{n} |c_k| \max\limits_{1\leq i\leq n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right); \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \max\limits_{1\leq i\leq n} |c_i| \left(\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|\right)^q\right)^{\frac{1}{q}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} |c_i|^t\right)^{\frac{1}{t}} \left[\sum\limits_{i=1}^{n}\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \qquad\qquad t>1,\ \frac{1}{t}+\frac{1}{u}=1; \\ \left(\sum\limits_{k=1}^{n} |c_k|^p\right)^{\frac{1}{p}} \sum\limits_{i=1}^{n} |c_i| \max\limits_{1\leq i\leq n}\left\{\left(\sum\limits_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right\}, \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \sum\limits_{k=1}^{n} |c_k| \max\limits_{1\leq i\leq n} |c_i| \sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n} |(y_i, y_j)|\right]; \\ \sum\limits_{k=1}^{n} |c_k| \left(\sum\limits_{i=1}^{n} |c_i|^m\right)^{\frac{1}{m}} \left(\sum\limits_{i=1}^{n}\left[\max\limits_{1\leq j\leq n} |(y_i, y_j)|\right]^l\right)^{\frac{1}{l}}, \quad m>1,\ \frac{1}{m}+\frac{1}{l}=1; \\ \left(\sum\limits_{k=1}^{n} |c_k|\right)^2 \max\limits_{i,1\leq j\leq n} |(y_i, y_j)| . \end{cases}$$

*Proof.* We note that

$$\sum_{i=1}^{n} c_i (x, y_i) = \left(x, \sum_{i=1}^{n} \overline{c_i} y_i\right).$$

Using Schwarz's inequality in inner product spaces, we have

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \left\|\sum_{i=1}^{n} \overline{c_i} y_i\right\|^2. \tag{3.2}$$

Finally, using Lemma 1 with $\alpha_i = \overline{c_i}$, $z_i = y_i$ $(i = 1, \ldots, n)$, we deduce the desired inequality (3.1). We omit the details. □

The following corollaries may be useful if one needs bounds in terms of $\sum_{i=1}^{n} |c_i|^2$.

**Corollary 5.** *With the assumptions in Theorem 5 and if* $1 < p \leq 2$, $1 < t \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$, *one has the inequality:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n^{\frac{1}{p}+\frac{1}{t}-1} \sum_{i=1}^{n} |c_i|^2 \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{u}}, \tag{3.3}$$

*and, in particular, for* $p = q = t = u = 2$,

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \left(\sum_{i,j=1}^{n} |(y_i, y_j)|^2\right)^{\frac{1}{2}}. \tag{3.4}$$

The proof is similar to the one in Corollary 1.

**Corollary 6.** *With the assumptions in Theorem 5 and if* $1 < p \leq 2$, *then*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n^{\frac{1}{p}} \sum_{k=1}^{n} |c_k|^2 \max_{1\leq i\leq n} \left[\sum_{j=1}^{n} |(y_i, y_j)|^q\right]^{\frac{1}{q}}, \tag{3.5}$$

*where* $\frac{1}{p} + \frac{1}{q} = 1$.

The proof is similar to the one in Corollary 2.

The following two inequalities also hold.

**Corollary 7.** *With the above assumptions for* $X, y_i, c_i$ *and if* $1 < m \leq 2$, *then*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n^{\frac{1}{m}} \sum_{k=1}^{n} |c_k|^2 \left(\sum_{i=1}^{n} \left[\max_{1\leq j\leq n} |(y_i, y_j)|\right]^l\right)^{\frac{1}{l}}, \tag{3.6}$$

*where* $\frac{1}{m} + \frac{1}{l} = 1$.

**Corollary 8.** *With the above assumptions for* $X, y_i, c_i$, *one has*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 n \sum_{k=1}^{n} |c_k|^2 \max_{i,1\leq j\leq n} |(y_i, y_j)|. \tag{3.7}$$

Using Lemma 2, we may state the following result as well.

**Remark 3.** *With the assumptions of Theorem 5, one has the inequalities:*

$$\left|\sum_{i=1}^{n} c_i (x, y_i)\right|^2 \leq \|x\|^2 \sum_{i=1}^{n} |c_i|^2 \sum_{j=1}^{n} |(y_i, y_j)| \tag{3.8}$$

$$\leq \|x\|^2 \times \begin{cases} \sum_{i=1}^{n} |c_i|^2 \max_{1\leq i\leq n} \left[\sum_{j=1}^{n} |(y_i, y_j)|\right]; \\[2ex] \left(\sum_{i=1}^{n} |c_i|^{2p}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^q\right)^{\frac{1}{q}}, \quad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex] \max_{1\leq i\leq n} |c_i|^2 \sum_{i,j=1}^{n} |(y_i, y_j)|; \end{cases}$$

*that provide some alternatives to Pečarić's result (1.6).*

## 4. Some Inequalities of Bombieri Type

In this section we point out some inequalities of Bombieri type that may be obtained from (3.1) on choosing $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$.

If the above choice was made in the first inequality in (3.1), then one would obtain:

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \le \|x\|^2 \max_{1\le i\le n} |(x, y_i)|^2 \sum_{i,j=1}^{n} |(y_i, y_j)|$$

giving, by taking the square root,

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \max_{1\le i\le n} |(x, y_i)| \left(\sum_{i,j=1}^{n} |(y_i, y_j)|\right)^{\frac{1}{2}}, \quad x \in H. \tag{4.1}$$

If the same choice for $c_i$ is made in the second inequality in (3.1), then one would get

$$\left(\sum_{i=1}^{n} |(x, y_i)|^2\right)^2 \le \|x\|^2 \max_{1\le i\le n} |(x, y_i)| \left(\sum_{i=1}^{n} |(x, y_i)|^r\right)^{\frac{1}{r}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^s\right]^{\frac{1}{s}},$$

implying

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^r\right)^{\frac{1}{2r}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right)^s\right]^{\frac{1}{2s}}, \tag{4.2}$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$.

The other inequalities in (3.1) will produce the following results, respectively

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|\right)^{\frac{1}{2}} \left[\max_{1\le i\le n}\left(\sum_{j=1}^{n} |(y_i, y_j)|\right)\right]; \tag{4.3}$$

$$\sum_{i=1}^{n} |(x, y_i)|^2 \le \|x\| \max_{1\le i\le n} |(x, y_i)|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |(x, y_i)|^p\right)^{\frac{1}{2p}} \left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n} |(y_i, y_j)|^q\right)^{\frac{1}{q}}\right]^{\frac{1}{2}}, \tag{4.4}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$;

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2 \le \|x\|\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^t\right)^{\frac{1}{2t}}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|^q\right)^{\frac{u}{q}}\right]^{\frac{1}{2u}}, \tag{4.5}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ t>1,\ \frac{1}{t}+\frac{1}{u}=1$;

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2 \le \|x\|\left(\sum_{i=1}^{n}\left|(x,y_i)\right|^p\right)^{\frac{1}{2p}}\left(\sum_{i=1}^{n}\left|(x,y_i)\right|\right)^{\frac{1}{2}}\max_{1\le i\le n}\left\{\left(\sum_{j=1}^{n}\left|(y_i,y_j)\right|^q\right)^{\frac{1}{2q}}\right\}, \tag{4.6}$$

where $p>1,\ \frac{1}{p}+\frac{1}{q}=1$;

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2 \le \|x\|\left[\sum_{i=1}^{n}\left|(x,y_i)\right|\right]^{\frac{1}{2}}\max_{1\le i\le n}\left|(x,y_i)\right|^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left[\max_{1\le j\le n}\left|(y_i,y_j)\right|\right]\right)^{\frac{1}{2}}; \tag{4.7}$$

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2 \le \|x\|\left[\sum_{i=1}^{n}\left|(x,y_i)\right|^m\right]^{\frac{1}{2m}}\left[\sum_{i=1}^{n}\left[\max_{1\le j\le n}\left|(y_i,y_j)\right|^l\right]\right]^{\frac{1}{2l}}, \tag{4.8}$$

where $m>1,\ \frac{1}{m}+\frac{1}{l}=1$; and

$$\sum_{i=1}^{n}\left|(x,y_i)\right|^2 \le \|x\|\sum_{i=1}^{n}\left|(x,y_i)\right|\max_{i,1\le j\le n}\left|(y_i,y_j)\right|^{\frac{1}{2}}. \tag{4.9}$$

If in the above inequalities we assume that $(y_i)_{1\le i\le n}=(e_i)_{1\le i\le n}$, where $(e_i)_{1\le i\le n}$ are orthonormal vectors in the inner product space $(H,(\cdot,\cdot))$, then from (4.1) – (4.9) we may deduce the following inequalities similar in a sense with Bessel's inequality:

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2 \le \sqrt{n}\,\|x\|\max_{1\le i\le n}\left\{\left|(x,e_i)\right|\right\}; \tag{4.10}$$

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2 \le n^{\frac{1}{2s}}\,\|x\|\max_{1\le i\le n}\left\{\left|(x,e_i)\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{n}\left|(x,e_i)\right|^r\right)^{\frac{1}{2r}}, \tag{4.11}$$

where $r>1,\ \frac{1}{r}+\frac{1}{s}=1$;

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2 \le \|x\|\max_{1\le i\le n}\left\{\left|(x,e_i)\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{n}\left|(x,e_i)\right|\right)^{\frac{1}{2}}; \tag{4.12}$$

$$\sum_{i=1}^{n}\left|(x,e_i)\right|^2 \le \sqrt{n}\,\|x\|\max_{1\le i\le n}\left\{\left|(x,e_i)\right|^{\frac{1}{2}}\right\}\left(\sum_{i=1}^{n}\left|(x,e_i)\right|^p\right)^{\frac{1}{2p}}, \tag{4.13}$$

where $p > 1$;

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq n^{\frac{1}{2u}} \|x\| \left( \sum_{i=1}^{n} |(x, e_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^{n} |(x, e_i)|^t \right)^{\frac{1}{2t}}, \tag{4.14}$$

where $p > 1$, $t > 1$, $\frac{1}{t} + \frac{1}{u} = 1$;

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\| \left( \sum_{i=1}^{n} |(x, e_i)|^p \right)^{\frac{1}{2p}} \left( \sum_{i=1}^{n} |(x, e_i)| \right)^{\frac{1}{2}}, \quad p > 1; \tag{4.15}$$

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \sqrt{n} \|x\| \left( \sum_{i=1}^{n} |(x, e_i)| \right)^{\frac{1}{2}} \max_{1 \leq i \leq n} \left\{ |(x, e_i)|^{\frac{1}{2}} \right\}; \tag{4.16}$$

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq n^{\frac{1}{2l}} \|x\| \left[ \sum_{i=1}^{n} |(x, e_i)|^m \right]^{\frac{1}{m}}, \quad m > 1, \ \frac{1}{m} + \frac{1}{l} = 1; \tag{4.17}$$

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\| \sum_{i=1}^{n} |(x, e_i)|. \tag{4.18}$$

Corollaries 5 − 8 will produce the following results which do not contain the Fourier coefficients in the right side of the inequality.

Indeed, if one chooses $c_i = \overline{(x, y_i)}$ in (3.3), then

$$\left( \sum_{i=1}^{n} |(x, y_i)|^2 \right)^2 \leq \|x\|^2 n^{\frac{1}{p} + \frac{1}{t} - 1} \sum_{i=1}^{n} |(x, y_i)|^2 \left[ \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}},$$

giving the following Bombieri type inequality:

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq n^{\frac{1}{p} + \frac{1}{t} - 1} \|x\|^2 \left[ \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |(y_i, y_j)|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, \tag{4.19}$$

where $1 < p \leq 2$, $1 < t \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{t} + \frac{1}{u} = 1$.

If in this inequality we consider $p = q = t = u = 2$, then

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \left( \sum_{i,j=1}^{n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}}. \tag{4.20}$$

For a different proof of this result see also [6].

In a similar way, if $c_i = \overline{(x, y_i)}$ in (3.6), then

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq n^{\frac{1}{m}} \|x\|^2 \left( \sum_{i=1}^{n} \left[ \max_{1 \leq j \leq n} |(y_i, y_j)| \right]^l \right)^{\frac{1}{l}}, \tag{4.21}$$

where $m > 1$, $\frac{1}{m} + \frac{1}{l} = 1$.

Finally, if $c_i = \overline{(x, y_i)}$ $(i = 1, \dots, n)$, is taken in (3.7), then

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq n \|x\|^2 \max_{1 \leq i,j \leq n} |(y_i, y_j)| . \tag{4.22}$$

**Remark 4.** *Let us compare Bombieri's result*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |(y_i, y_j)| \right\} \tag{4.23}$$

*with our result*

$$\sum_{i=1}^{n} |(x, y_i)|^2 \leq \|x\|^2 \left\{ \sum_{i,j=1}^{n} |(y_i, y_j)|^2 \right\}^{\frac{1}{2}} . \tag{4.24}$$

*Denote*

$$M_1 := \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |(y_i, y_j)| \right\}$$

*and*

$$M_2 := \left[ \sum_{i,j=1}^{n} |(y_i, y_j)|^2 \right]^{\frac{1}{2}} .$$

*If we choose the inner product space $H = \mathbb{R}$, $(x, y) := xy$ and $n = 2$, then for $y_1 = a$, $y_2 = b$, $a, b > 0$, we have*

$$M_1 = \max \left\{ a^2 + ab, ab + b^2 \right\} = (a + b) \max (a, b) ,$$

$$M_2 = \left( a^4 + a^2 b^2 + a^2 b^2 + b^4 \right)^{\frac{1}{2}} = a^2 + b^2.$$

*Assume that $a \geq b$. Then $M_1 = a^2 + ab \geq a^2 + b^2 = M_2$, showing that, in this case, the bound provided by (4.24) is better than the bound provided by (4.23). If $(y_i)_{1 \leq i \leq n}$ are orthonormal vectors, then $M_1 = 1$, $M_2 = \sqrt{n}$, showing that in this case the Bombieri inequality (which becomes Bessel's inequality) provides a better bound than (4.24).*

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