REPRESENTATION OF MULTIVARIATE FUNCTIONS VIA THE POTENTIAL THEORY

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Abstract. In this paper, by the use of Potential Theory, some representation results for multivariate functions from the Sobolev spaces $W^{1,p}(\Omega)$, in terms of the double layer potential and the fundamental solution of Laplace's equation are pointed out. Applications for multivariate inequalities of Ostrowski type are also provided.

1. Introduction

The following representation for an absolutely continuous function $f:[a,b]\to\mathbb{R}$ in terms of the integral mean is known in the literature as Montgomery identity

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b p(t,x)\,f'(t)\,dt,\ \ x\in[a,b];$$

where $p:[a,b]^2\to\mathbb{R}$, is given by

$$p(t,x)=\begin{cases} t-a & \text{if } a\le t\le x\\ t-b & \text{if } x<t\le b\end{cases}. \tag{1.1}$$

In the last decade, many authors (see for example [2] and the references therein) have extended the above result for different classes of functions defined on a compact interval, including: functions of bounded variation, monotonic functions, convex functions, n-time differentiable functions whose derivatives are absolutely continuous or satisfy different convexity properties etc...and pointed out sharp inequalities for the absolute value of the difference

$$D(f;x):=f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt,\ \ x\in[a,b].$$

The obtained results have been applied in Approximation Theory, Numerical Integration, Information Theory and other related domains.

We have, see for instance [2, p. 2], the following *Ostrowski type inequalities*

$$|D(f;x)|$$
$$\le\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_\infty & \text{if } f'\in L_\infty[a,b];\\ \frac{1}{(p+1)^{1/p}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right]^{1/p}(b-a)^{1/p}\left\|f'\right\|_q & \begin{array}{c}\text{if } f'\in L_q[a,b]\\ q>1,\frac{1}{p}+\frac{1}{q}=1\end{array}\\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left\|f'\right\|_1;\end{cases};$$

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provided f is absolutely continuous and $L_r[a,b]$ $(1 \le r \le \infty)$ are the usual Lebesgue spaces. The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{1/p}}$ and $\frac{1}{2}$ are best possible in the sense that they cannot be replaced by smaller constants.

If the functions $f : [a,b] \times [c,d] \to \mathbb{R}$ has the partial derivatives $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$, and $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ continuous on $[a,b] \times [c,d]$, then one has the representation [2, p. 307]

$$\begin{aligned} f(x,y) &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, dtds \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(t,x) \frac{\partial f(t,s)}{\partial t} dtds \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d q(s,y) \frac{\partial f(t,s)}{\partial s} dtds \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(t,x)\, q(s,y) \frac{\partial^2 f(t,s)}{\partial t \partial s} dtds, \end{aligned}$$

for each $(x,y) \in [a,b] \times [c,d]$, where p is defined by (1.1) and q is the corresponding kernel for the interval $[c,d]$.

Another representation for $f : [a,b] \times [c,d] \to \mathbb{R}$ is [2, p. 294]

$$\begin{aligned} f(x,y) &= \frac{1}{b-a} \int_a^b f(t,y)\, dt + \frac{1}{d-c} \int_c^d f(x,s)\, ds \\ &- \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)\, dtds \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d p(t,x)\, q(s,y) \frac{\partial^2 f(t,s)}{\partial t \partial s} dtds, \end{aligned}$$

for each $(x,y) \in [a,b] \times [c,d]$, provided $\frac{\partial^2 f(t,s)}{\partial t \partial s}$ is continuous in $[a,b] \times [c,d]$.

Different Ostrowski type inequalities for multivariate functions may be stated, see Chapters 5 & 6 of [2].

In this paper, by the use of Potential Theory, some representation results for multivariate functions from the Sobolev spaces $W^{1,p}(\Omega)$, where Ω is an open bounded set with smooth boundary in $\mathbb{R}^N, N \ge 2, p \in (N, \infty]$, in terms of the double layer potential and the fundamental solution of Laplace's equation are pointed out. Applications for multivariate inequalities of Ostrowski type are also provided.

2. Preliminaries

For $\Omega \subset \mathbb{R}^N$, we denote by $\overline{\Omega}$ its closure and by $\partial\Omega$ the boundary of Ω.

By a *vector field* we understand an $\mathbb{R}^N$-valued function on a subset of $\mathbb{R}^N$. If $Z = (z_1, z_2, \ldots, z_N)$ is a differentiable vector field on an open set $\Omega \subset \mathbb{R}^N$, the *divergence* of Z on Ω is defined by

$$\mathrm{div} Z = \sum_{i=1}^N \frac{\partial z_i}{\partial x_i}.$$

Proposition 1 (The Divergence Theorem)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with C^1 boundary and let Z be a vector field of class $C^1(\Omega) \cap C(\overline{\Omega})$. Then,*

$$\int_\Omega \mathrm{div} Z(y)\, dy = \int_{\partial\Omega} \langle Z(x), \nu(x) \rangle\, d\sigma(x).$$

Here, $\nu(x)$ is the unit outward normal to $\partial\Omega$ at x and $d\sigma$ denotes the Euclidian measure on $\partial\Omega$. We denote by $\langle \cdot, \cdot \rangle$ the canonical inner product on $\mathbb{R}^N \times \mathbb{R}^N$.

If u is a differentiable function defined near $\partial\Omega$, we can define the *normal derivative* of u on $\partial\Omega$ by

$$\frac{\partial u}{\partial \nu} = \langle \nabla u, \nu \rangle, \quad \text{where } \nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N} \right).$$

If Ω is a domain for which the divergence theorem applies, then we have

Proposition 2 (Green's first identity)**.** *Assume that $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. The following holds*

$$\int_\Omega v(x)\Delta u(x)\,dx + \int_\Omega \langle \nabla u(x), \nabla v(x) \rangle\,dx = \int_{\partial\Omega} v(x)\frac{\partial u}{\partial \nu}(x)\,d\sigma(x).$$

Let $\|\cdot\|_{L^m(\Omega)}$ denote the usual norm on $L^m(\Omega)$, i.e.,

$$\|u\|_{L^m(\Omega)} = \left(\int_\Omega |u(x)|^m\,dx \right)^{1/m}, \quad \text{if } u \in L^m(\Omega) \text{ with } 1 \le m < \infty$$

respectively

$$\|u\|_{L^\infty(\Omega)} = \inf\{C > 0 :\ |u(x)| \le C \text{ a.e. on } \Omega\}, \quad \text{if } u \in L^\infty(\Omega).$$

By $W^{1,m}(\Omega)$, $1 \le m \le \infty$, we understand the Sobolev space defined by

$$W^{1,m}(\Omega) = \left\{ u \in L^m(\Omega) \,\middle|\, \begin{array}{l} \exists g_1, g_2, \ldots g_N \in L^m(\Omega) \text{ such that} \\ \displaystyle\int_\Omega u\frac{\partial \phi}{\partial x_i} = -\int_\Omega g_i\phi, \quad \forall \phi \in C_c^\infty(\Omega), \quad \forall i = \overline{1, N} \end{array} \right\}.$$

For $u \in W^{1,m}(\Omega)$ we define $g_i = \frac{\partial u}{\partial x_i}$ and we write

$$\nabla u = \operatorname{grad} u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N} \right).$$

The Sobolev space $W^{1,m}(\Omega)$ is endowed with the norm

$$\|u\|_{W^{1,m}(\Omega)} = \|u\|_{L^m(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^m(\Omega)}.$$

For $x \in \mathbb{R}^N$ and $r > 0$, set $B_r(x) = \{y \in \mathbb{R}^N :\ |x-y| < r\}$, where $|x| = \langle x, x \rangle^{1/2}$.

Let $E(x)$ define the fundamental solution of Laplace's equation $\Delta E(x) = 0$ in $\mathbb{R}^N$ $(N \ge 2)$, i.e.,

$$E(x) = \begin{cases} \dfrac{1}{2\pi}\ln|x|, & x \ne 0 \text{ (if } N = 2) \\ \dfrac{1}{(2-N)\omega_N|x|^{N-2}}, & x \ne 0 \text{ (if } N \ge 3) \end{cases}$$

where ω_N stands for the area of the unit sphere in $\mathbb{R}^N$. By [4, Proposition 0.7], we know that the value of ω_N is

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

where $\Gamma(s)$ represents the Gamma function defined for $\operatorname{Re} s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt.$$

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded subset with C^2 boundary. For a continuous function h on $\partial\Omega$, the *double layer potential with moment* h is defined as

$$\bar{\bar{u}}_h(y) = \int_{\partial\Omega} h(x) \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x). \tag{2.1}$$

For details about the next results, we refer to [4].

Proposition 3. *If h is a continuous function on $\partial\Omega$, then*
(a) *$\bar{\bar{u}}_h(y)$ is well defined for all $y \in \mathbb{R}^N$.*
(b) *$\Delta\bar{\bar{u}}_h(y) = 0$ for all $y \notin \partial\Omega$.*

Lemma 1 (Gauss' Lemma)**.** *Let $\bar{\bar{v}}$ be the double layer potential with moment $h \equiv 1$, i.e.,*

$$\bar{\bar{v}}(y) = \int_{\partial\Omega} \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x).$$

Then, we have

$$\bar{\bar{v}}(y) = \begin{cases} 1 & \text{if } y \in \Omega, \\ 1/2 & \text{if } y \in \partial\Omega, \\ 0 & \text{if } y \in \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

The next result states the limits of the $\bar{\bar{u}}_h(y)$ (defined by (2.1)) as we approach $\partial\Omega$ from the interior or exterior of Ω.

Proposition 4. *Let h be continuous on $\partial\Omega$ and $y_0 \in \partial\Omega$. Then,*

$$\lim_{\Omega \ni y \to y_0} \bar{\bar{u}}_h(y) = \frac{1}{2}h(y_0) + \bar{\bar{u}}_h(y_0) \ \text{ and } \lim_{\mathbb{R}^N \setminus \overline{\Omega} \ni y \to y_0} \bar{\bar{u}}_h(y) = -\frac{1}{2}h(y_0) + \bar{\bar{u}}_h(y_0). \tag{2.2}$$

Remark 1. If $h \in C(\partial\Omega)$ then $\bar{\bar{u}}_h \in C(\partial\Omega) \cap L^m(\Omega)$, for each $1 \leq m \leq \infty$.

Indeed, by Propositions 3 and 4, the function $\phi : \overline{\Omega} \to \mathbb{R}$ defined by $\phi(y) = \bar{\bar{u}}_h(y)$, $\forall y \in \Omega$ and $\phi(y_0) = \frac{1}{2}h(y_0) + \bar{\bar{u}}_h(y_0)$, $\forall y_0 \in \partial\Omega$ is continuous on $\overline{\Omega}$. It follows that $\bar{\bar{u}}_h \in C(\partial\Omega)$ and $\phi \in L^\infty(\Omega)$. But $\phi \equiv \bar{\bar{u}}_h$ on Ω so that $\bar{\bar{u}}_h \in L^\infty(\Omega)$. Thus, for each $1 \leq m < \infty$, we have

$$\int_\Omega |\bar{\bar{u}}_h|^m\, dx \leq \|\bar{\bar{u}}_h\|^m_{L^\infty(\Omega)} \operatorname{meas}(\Omega) < \infty,$$

which shows that $\bar{\bar{u}}_h \in L^m(\Omega)$.

3. Main results

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary and $A = (a_i)_{i \in I}$ be a finite family of points in Ω.

We assume throughout that $f \in C(\overline{\Omega}) \cap C^1(\Omega \setminus A)$ and, for some $\alpha \in (0,1)$,

$$\limsup_{x \to a_i} \frac{|f(x) - f(a_i)|}{|x - a_i|^\alpha} < \infty, \quad \forall i \in I. \tag{H}$$

We adopt the following notations

$$⨏_\Omega f\, dx = \frac{1}{\operatorname{meas}(\Omega)} \int_\Omega f(x)\, dx \ \text{ and } ⨏_{\partial\Omega} f\, d\sigma(x) = \frac{1}{\operatorname{meas}(\partial\Omega)} \int_{\partial\Omega} f(x)\, d\sigma(x).$$

Theorem 1. *Suppose $f \in W^{1,p}(\Omega)$ for some $p \in (N, \infty]$. Then*

$$f(y) = \bar{\bar{u}}_f(y) - \int_\Omega \langle \nabla E(x-y), \nabla f(x) \rangle \, dx, \qquad \forall y \in \Omega \tag{3.1}$$

resp.,

$$\int_\Omega f(x)\, dx = \frac{1}{N} \int_{\partial\Omega} f(x) \langle x-y, \nu \rangle \, d\sigma(x) - \frac{1}{N} \int_\Omega \langle \nabla f(x), x-y \rangle \, dx, \quad \forall y \in \mathbb{R}^N. \tag{3.2}$$

Proof. Let $y \in \Omega$ be fixed. We first recall that, for each $\gamma \in (0, N)$, the mapping $x \longmapsto |x-y|^{-\gamma} \in L^1(\Omega)$. Indeed, for $r > 0$ fixed so that $B_r(y) \subset\subset \Omega$, we have

$$\begin{aligned} \int_\Omega \frac{dx}{|x-y|^\gamma} &= \int_{\Omega \setminus B_r(y)} \frac{dx}{|x-y|^\gamma} + \int_{B_r(y)} \frac{dx}{|x-y|^\gamma} \\ &\le \frac{\mathrm{meas}\,(\Omega)}{r^\gamma} + \lim_{\epsilon \to 0} \int_\epsilon^r \left(\int_{\partial B_\rho(y)} \frac{d\sigma(x)}{|x-y|^\gamma} \right) d\rho \\ &= \frac{\mathrm{meas}\,(\Omega)}{r^\gamma} + \lim_{\epsilon \to 0} \int_\epsilon^r \frac{\mathrm{meas}\,(\partial B_\rho(y))}{\rho^\gamma}\, d\rho \\ &= \frac{\mathrm{meas}\,(\Omega)}{r^\gamma} + \frac{\omega_N r^{N-\gamma}}{N-\gamma} < \infty. \end{aligned}$$

We now define $F : \overline{\Omega} \setminus \{y\} \to \mathbb{R}^N$ as follows

$$F(x) = (f(x) - f(y)) \nabla E(x-y) = \frac{f(x) - f(y)}{\omega_N |x-y|^N} (x-y).$$

Note that $F(x)$ is not smooth for all $x \in \Omega$. We overcome this problem by choosing $\epsilon > 0$ small enough such that $B_\epsilon(y)$ resp., $B_\epsilon(a_i)$ ($a_i \in A \setminus \{y\}$) is contained within Ω and each two such balls are disjoint. Therefore, $F \in C^1(D_\epsilon) \cap C(\overline{D}_\epsilon)$ where $D_\epsilon = \Omega \setminus \left(\cup_{i \in I} \overline{B}_\epsilon(a_i) \cup \overline{B}_\epsilon(y) \right)$. Using the Divergence Theorem, we arrive at

$$\begin{aligned} \int_{D_\epsilon} \mathrm{div}\, F(x)\, dx &= \int_{\partial\Omega} (f(x) - f(y)) \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x) \\ &\quad - \frac{1}{\omega_N \epsilon^{N-1-\alpha}} \int_{\partial B_\epsilon(y)} \frac{f(x) - f(y)}{|x-y|^\alpha}\, d\sigma(x) \\ &\quad - \frac{1}{\omega_N} \sum_{i \in I, a_i \neq y} \int_{\partial B_\epsilon(a_i)} \frac{f(x) - f(y)}{\epsilon |x-y|^N} \langle x-y, x-a_i \rangle \, d\sigma(x). \end{aligned} \tag{3.3}$$

We see that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{N-1-\alpha}} \int_{\partial B_\epsilon(y)} \frac{f(x) - f(y)}{|x-y|^\alpha}\, d\sigma(x) = 0. \tag{3.4}$$

Indeed, in view of (H), for some constant $L > 0$ and $\epsilon > 0$ small enough, we have

$$\begin{aligned} 0 &\le \frac{1}{\epsilon^{N-1-\alpha}} \left| \int_{\partial B_\epsilon(y)} \frac{f(x) - f(y)}{|x-y|^\alpha}\, d\sigma(x) \right| \\ &\le \frac{L}{\epsilon^{N-1-\alpha}} \int_{\partial B_\epsilon(y)} d\sigma(x) = L \omega_N \epsilon^\alpha \to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

Notice that, for each $i \in I$ with $a_i \neq y$, there exists a constant $C_i > 0$ such that

$$|f(x) - f(y)| \le C_i |x-y|^{N-1}, \ \forall x \in \overline{B}_\epsilon(a_i)$$

(since $y \notin \overline{B}_\epsilon(a_i)$). Hence

$$\left|\int_{\partial B_\epsilon(a_i)} \frac{f(x)-f(y)}{\epsilon|x-y|^N}\langle x-y, x-a_i\rangle\, d\sigma(x)\right| \le \int_{\partial B_\epsilon(a_i)} \frac{|f(x)-f(y)|}{|x-y|^{N-1}}\, d\sigma(x) \tag{3.5}$$
$$\le C_i\omega_N\epsilon^{N-1} \to 0 \quad \text{as } \epsilon \to 0,$$

provided $i \in I$ such that $a_i \neq y$. By (3.3)–(3.5), it follows that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{D_\epsilon} \operatorname{div} F(x)\, dx &= \int_{\partial\Omega} (f(x)-f(y))\frac{\partial E}{\partial\nu}(x-y)\, d\sigma(x)\\ &= \int_{\partial\Omega} f(x)\frac{\partial E}{\partial\nu}(x-y)\, d\sigma(x) - f(y)\end{aligned} \tag{3.6}$$

by using Gauss' Lemma. On the other hand, for each $x \in D_\epsilon$,

$$\begin{aligned}\operatorname{div} F(x) &= \langle \nabla f(x), \nabla E(x-y)\rangle + (f(x)-f(y))\Delta_x E(x-y)\\ &= \langle \nabla f(x), \nabla E(x-y)\rangle\end{aligned}$$

since $x \longmapsto E(x-y)$ is harmonic on $\mathbb{R}^N \setminus \{y\}$. By Hölder's inequality, we obtain

$$\int_\Omega |\langle \nabla f(x), \nabla E(x-y)\rangle|\, dx \le \frac{\|\nabla f\|_{L^p(\Omega)}}{\omega_N}\left(\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}}\right)^{\frac{1}{p'}} < \infty$$

which is due to $|\nabla f| \in L^p(\Omega)$ and $(N-1)p' < N$. Hence, the mapping $x \longmapsto \langle \nabla f(x), \nabla E(x-y)\rangle$ is integrable on Ω. Thus, using (3.6) we deduce that

$$\begin{aligned}\int_\Omega \langle \nabla f(x), \nabla E(x-y)\rangle\, dx &= \lim_{\epsilon\to 0}\int_{D_\epsilon} \langle \nabla f(x), \nabla E(x-y)\rangle\, dx\\ &= \int_{\partial\Omega} f(x)\frac{\partial E}{\partial\nu}(x-y)\, d\sigma(x) - f(y)\end{aligned}$$

which concludes our first assertion.

Let $y \in \mathbb{R}^N$ be arbitrary. We define $G : \overline{\Omega} \to \mathbb{R}^N$ by $G(x) = f(x)(x-y)$. Let $\epsilon > 0$ be small such that $\overline{B}_\epsilon(a_i) \subset \Omega$, $\forall i \in I$ and $\overline{B}_\epsilon(a_i) \cap \overline{B}_\epsilon(a_j) = \emptyset$, $\forall i, j \in I$ with $i \neq j$. Set $U_\epsilon = \Omega \setminus \left(\cup_{i\in I}\overline{B}_\epsilon(a_i)\right)$. We have $G \in C^1(U_\epsilon) \cap C(\overline{U}_\epsilon)$. By Proposition 1, we find that

$$\begin{aligned}\int_{U_\epsilon} \operatorname{div} G(x)\, dx &= \int_{\partial\Omega} f(x)\langle x-y, \nu\rangle\, d\sigma(x)\\ &\quad - \sum_{i\in I}\int_{\partial B_\epsilon(a_i)} \frac{f(x)}{\epsilon}\langle x-y, x-a_i\rangle\, d\sigma(x).\end{aligned} \tag{3.7}$$

For each $i \in I$, we have

$$\begin{aligned}\left|\int_{\partial B_\epsilon(a_i)} \frac{f(x)}{\epsilon}\langle x-y, x-a_i\rangle\, d\sigma(x)\right| &\le \int_{\partial B_\epsilon(a_i)} \frac{|f(x)|}{\epsilon}|\langle x-y, x-a_i\rangle|\, d\sigma(x)\\ &\le \int_{\partial B_\epsilon(a_i)} |f(x)||x-y|\, d\sigma(x)\\ &\le C_i\|f\|_{L^\infty(\Omega)}\operatorname{meas}(\partial B_\epsilon(a_i))\\ &= C_i\|f\|_{L^\infty(\Omega)}\omega_N\epsilon^{N-1} \to 0 \text{ as } \epsilon \to 0\end{aligned}$$

for some constant $C_i > 0$ that satisfies $|x-y| \le C_i$, $\forall x \in \partial B_k(a_i)$, $\forall k \in (0, \epsilon]$.

It follows that

$$\lim_{n\to\infty}\int_{\partial B_\epsilon(a_i)}\frac{f(x)}{\epsilon}\langle x-y,x-a_i\rangle\,d\sigma(x)=0,\quad \forall i\in I. \tag{3.8}$$

We see that

$$\operatorname{div}G(x)=\langle\nabla f(x),x-y\rangle+Nf(x),\quad \forall x\in U_\epsilon.$$

By $f\in C(\overline{\Omega})\cap W^{1,p}(\Omega)$ and Hölder's inequality, we deduce $f\in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega|\langle\nabla f(x),x-y\rangle|\,dx&\le\int_\Omega|\nabla f(x)||x-y|\,dx\\&\le\left(\int_\Omega|\nabla f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_\Omega|x-y|^{p'}\,dx\right)^{\frac{1}{p'}}\\&=\|\nabla f\|_{L^p(\Omega)}\left(\int_\Omega|x-y|^{p'}\,dx\right)^{\frac{1}{p'}}<\infty.\end{aligned}$$

Therefore,

$$\lim_{\epsilon\to0}\int_{U_\epsilon}\operatorname{div}G(x)\,dx=\int_\Omega\langle\nabla f(x),x-y\rangle+N\int_\Omega f(x)\,dx. \tag{3.9}$$

Passing to the limit $\epsilon\to0$ in (3.7) and using (3.8) resp., (3.9), we conclude that

$$\int_\Omega\langle\nabla f(x),x-y\rangle+N\int_\Omega f(x)\,dx=\int_{\partial\Omega}f(x)\langle x-y,\nu\,d\sigma(x)$$

which proves (3.2). □

To our next aim, we recall the following results.

Lemma 2. *Let $\Omega\subset\mathbb{R}^N$ be an open set. Let (h_n) be a sequence in $L^p(\Omega)$, $1\le p\le\infty$, and let $h\in L^p(\Omega)$ be such that $\|h_n-h\|_{L^p(\Omega)}\to0$.*

Then, there exists a subsequence (h_{n_k}) and a function $\phi\in L^p(\Omega)$ such that

(a) *$h_{n_k}(x)\to h(x)$ a.e. in Ω,*
(b) *$|h_{n_k}(x)|\le\phi(x)$ $\forall k$, a.e. in Ω.*

The interested reader may find the proof of Lemma 2 in [1, Theorem IV.9].

Lemma 3. *Suppose that Ω is of class C^1 and let $u\in W^{1,p}(\Omega)$ with $1\le p<\infty$.*

Then, there exists a sequence (u_n) in $C_c^\infty(\mathbb{R}^N)$ such that $u_n|_\Omega\to u$ in $W^{1,p}(\Omega)$. In other words, the restrictions to Ω of functions belonging to $C_c^\infty(\mathbb{R}^N)$ form a subspace which is dense in $W^{1,p}(\Omega)$.

For the proof of Lemma 3 we refer to [1, Corollary IX.8].

We are now ready to give a representation theorem of functions in any Sobolev space $W^{1,p}(\Omega)$, $p\in(N,\infty)$. More precisely, we prove

Theorem 2. *Let Ω be an open bounded C^1 set in $\mathbb{R}^N$, $N\ge2$. Then, for any $g\in W^{1,p}(\Omega)$ with $p\in(N,\infty)$, there exists a sequence $(g_n)\subset C_c^\infty(\mathbb{R}^N)$ so that*

$$\begin{aligned}g(y)=&\lim_{n\to\infty}\int_{\partial\Omega}g_n(x)\frac{\partial E}{\partial\nu}(x-y)\,d\sigma(x)\\&-\int_\Omega\langle\nabla E(x-y),\nabla g(x)\rangle\,dx\quad a.e.\ y\in\Omega.\end{aligned} \tag{3.10}$$

Proof. By Lemma 3, we know that there exists a sequence $g_n \in C_c^\infty(\mathbb{R}^N)$ such that $g_n|_\Omega \to g$ in $W^{1,p}(\Omega)$. Hence,

$$\lim_{n\to\infty} \|g_n|_\Omega - g\|_{L^p(\Omega)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \left\| \frac{\partial g_n}{\partial x_i} - \frac{\partial g}{\partial x_i} \right\|_{L^p(\Omega)} = 0, \ \forall i = \overline{1, N}.$$

Applying Lemma 2 we have that, up to a subsequence (relabelled (g_n)),

$$g_n|_\Omega \to g \quad \text{a.e. in } \Omega. \tag{3.11}$$

Using Theorem 1, we obtain

$$g_n(y) = \int_{\partial\Omega} g_n(x) \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x) - \int_\Omega \langle \nabla E(x-y), \nabla g_n(x) \rangle\, dx, \quad \forall y \in \Omega. \tag{3.12}$$

We now show that

$$\lim_{n\to\infty} \int_\Omega \langle \nabla E(x-y), \nabla g_n(x) \rangle\, dx = \int_\Omega \langle \nabla E(x-y), \nabla g(x) \rangle\, dx, \quad \forall y \in \Omega. \tag{3.13}$$

Indeed, by Hölder's inequality, we deduce

$$\begin{aligned}
0 &\le \int_\Omega |\langle E(x-y), \nabla g_n(x) - \nabla g(x) \rangle|\, dx \\
&= \int_\Omega \left| \sum_{i=1}^N \frac{\partial E}{\partial x_i}(x-y)\, \frac{\partial (g_n - g)}{\partial x_i} \right| \le \sum_{i=1}^N \int_\Omega \left| \frac{\partial E}{\partial x_i}(x-y)\, \frac{\partial (g_n - g)}{\partial x_i} \right| dx \\
&\le \sum_{i=1}^N \left(\int_\Omega \left| \frac{\partial E}{\partial x_i}(x-y) \right|^{p'} dx \right)^{1/p'} \cdot \left(\int_\Omega \left| \frac{\partial (g_n - g)}{\partial x_i} \right|^p dx \right)^{1/p} \\
&\le \left(\int_\Omega |\nabla E(x-y)|^{p'}\, dx \right)^{1/p'} \sum_{i=1}^N \left\| \frac{\partial (g_n - g)}{\partial x_i} \right\|_{L^p(\Omega)} \\
&\le \frac{1}{\omega_N} \left(\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}} \right)^{1/p'} \cdot \sum_{i=1}^N \left\| \frac{\partial (g_n - g)}{\partial x_i} \right\|_{L^p(\Omega)} \to 0 \text{ as } n \to \infty
\end{aligned}$$

By (3.11)–(3.13) we conclude the proof. □

4. Special cases

A function $u \in C^2(\Omega)$ is called *harmonic* in Ω if it satisfies $\Delta u = 0$ in Ω.

The mean value theorem for harmonic functions says that the function value at the center of the ball $B_R(a) \subset \Omega$ is equal to the integral mean values over both the surface $\partial B_R(a)$ and $B_R(a)$ itself. More precisely,

Proposition 5 (Theorem 2.1 in [5]). *Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $\Delta u = 0$ in Ω. Then for any ball $B_R(a) \subset \Omega$, we have*

$$u(a) = \fint_{\partial B_R(a)} u(x)\, d\sigma(x), \tag{4.1}$$

$$u(a) = \fint_{B_R(a)} u(x)\, dx. \tag{4.2}$$

The *Poisson integral formula*, together with an approximation argument, gives the representation form for harmonic functions $u \in C^2(B_R(a)) \cap C(\overline{B}_R(a))$, that is (see [5], pp. 20)

$$u(y) = \frac{R^2 - |y-a|^2}{R\omega_N} \int_{\partial B_R(a)} \frac{u(x)}{|x-y|^N}\, d\sigma(x), \quad \forall y \in B_R(a). \tag{4.3}$$

Moreover, we have

Proposition 6 (Theorem 2.6 in [5])**.** *Let φ be a continuous function on ∂B. Then the function u defined by*

$$u(y) = \begin{cases} \dfrac{R^2 - |y-a|^2}{R\omega_N} \displaystyle\int_{\partial B_R(a)} \frac{u(x)}{|x-y|^N}\, d\sigma(x), & \forall y \in B_R(a), \\ \varphi(y), \quad \forall y \in \partial B_R(a) \end{cases} \tag{4.4}$$

belongs to $C^2(B_R(a)) \cap C(\overline{B}_R(a))$ and satisfies $\Delta u = 0$ in $B_R(a)$.

It is now natural to ask what are the corresponding representation formulas for functions satisfying weaker regularity assumptions and not necessarily harmonic.

To this aim, we state some consequences of Theorem 1, whose preliminary assumptions are self-understood. As a common hypothesis for Corollaries 1–7, we have $f \in W^{1,p}(\Omega)$ for some $p \in (N, \infty]$.

Corollary 1. *For any ball $B_R(a) \subset \Omega$, we have*

$$f(y) = \int_{\partial B_R(a)} \frac{f(x)\langle x-y, x-a\rangle}{R\omega_N |x-y|^N}\, d\sigma(x) - \int_{B_R(a)} \frac{\langle \nabla f(x), x-y\rangle}{\omega_N |x-y|^N}\, dx, \tag{4.5}$$

where $y \in B_R(a)$ is arbitrary.

Using Proposition 6 and Corollary 1, we arrive at

Corollary 2. *For any $a \in \Omega$ and $R > 0$ such that $B_R(a) \subset \Omega$, we find*

$$\begin{aligned} f(y) =& \chi(y) + \int_{\partial B_R(a)} \frac{\langle y-a, y-x\rangle}{R\omega_N |x-y|^N} f(x)\, d\sigma(x) \\ &- \int_{B_R(a)} \frac{\langle \nabla f(x), x-y\rangle}{\omega_N |x-y|^N}\, dx, \quad \forall y \in B_R(a) \end{aligned} \tag{4.6}$$

where χ is the unique classical solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } B_R(a) \\ u = f, & \text{on } \partial B_R(a). \end{cases}$$

Corollary 3. *The following representation formula holds*

$$\begin{aligned} f(y) = & \fint_\Omega f(x)\, dx + \int_{\partial\Omega} \left(\frac{\langle x-y, \nu\rangle}{\omega_N |x-y|^N} - \frac{\langle x-z, \nu\rangle}{N \operatorname{meas}(\Omega)} \right) f(x)\, d\sigma(x) \\ & - \int_\Omega \left(\frac{\langle \nabla f(x), x-y\rangle}{\omega_N |x-y|^N} - \frac{\langle \nabla f(x), x-z\rangle}{N \operatorname{meas}(\Omega)} \right) dx, \quad \forall y \in \Omega, \quad \forall z \in \mathbb{R}^N. \end{aligned} \tag{4.7}$$

In particular, for $z = y$ we obtain

$$\begin{aligned} f(y) = & \fint_\Omega f(x)\, dx + \int_{\partial\Omega} \left(\frac{1}{\omega_N |x-y|^N} - \frac{1}{N \operatorname{meas}(\Omega)} \right) f(x) \langle x-y, \nu\rangle\, d\sigma(x) \\ & - \int_\Omega \left(\frac{1}{\omega_N |x-y|^N} - \frac{1}{N \operatorname{meas}(\Omega)} \right) \langle \nabla f(x), x-y\rangle\, dx, \quad \forall y \in \Omega. \end{aligned} \tag{4.8}$$

Corollary 4. *For each $a \in \Omega$ and $R > 0$ such that $B_R(a) \subset \Omega$, we obtain*

$$f(y) = \oint_{B_R(a)} f(x)\,dx - \oint_{\partial B_R(a)} f(x)\,d\sigma(x) + \int_{\partial B_R(a)} \frac{f(x)\langle x-y, x-a\rangle}{R\omega_N |x-y|^N}\,d\sigma(x)$$
$$- \frac{1}{\omega_N}\int_{B_R(a)} \left(\frac{\langle \nabla f(x), x-y\rangle}{|x-y|^N} - \frac{\langle \nabla f(x), x-a\rangle}{R^N} \right) dx, \qquad \forall y \in B_R(a).$$

The particular case $y = a$ leads to

$$f(a) = \oint_{B_R(a)} f(x)\,dx - \frac{1}{\omega_N}\int_{B_R(a)} \left(\frac{1}{|x-a|^N} - \frac{1}{R^N} \right) \langle \nabla f(x), x-a\rangle\,dx. \tag{4.9}$$

resp.,

$$f(a) = \oint_{\partial B_R(a)} f(x)\,d\sigma(x) - \frac{1}{\omega_N}\int_{B_R(a)} \frac{\langle \nabla f(x), x-a\rangle}{|x-a|^N}\,dx. \tag{4.10}$$

Corollary 5. *An arbitrary value of f is below compared with the double layer potential with moment f*

$$|f(y) - \bar{\bar{u}}_f(y)| \le \frac{\|\nabla f\|_{L^p(\Omega)}}{\omega_N} \left(\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}} \right)^{\frac{1}{p'}}, \quad \forall y \in \Omega \tag{4.11}$$

where p' denotes the conjugate coefficient of p (i.e., $1/p + 1/p' = 1$). Moreover, for $y \in \Omega$ fixed, the equality in (4.11) is established for the nontrivial function $f(x) = \pm|x-y|$ if $p = \infty$ resp., $f(x) = \pm|x-y|^\beta$ with $\beta = (p-N)/(p-1)$ if $p \in (N, \infty)$.

Proof. By (3.1) and Hölder's inequality, we have

$$\begin{aligned}
|f(y) - \bar{\bar{u}}_f(y)| &= \left| \int_\Omega \langle \nabla E(x-y), \nabla f(x)\rangle\,dx \right| = \left| \int_\Omega \frac{\langle x-y, \nabla f(x)\rangle}{\omega_N |x-y|^N}\,dx \right| \\
&\le \frac{1}{\omega_N}\int_\Omega \frac{|\langle x-y, \nabla f(x)\rangle|}{|x-y|^N}\,dx \le \frac{1}{\omega_N}\int_\Omega \frac{|\nabla f(x)|}{|x-y|^{N-1}}\,dx \\
&\le \frac{1}{\omega_N}\left(\int_\Omega |\nabla f(x)|^p\,dx \right)^{1/p} \left(\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}} \right)^{1/p'} \\
&= \frac{\|\nabla f\|_{L^p(\Omega)}}{\omega_N} \left(\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}} \right)^{1/p'}.
\end{aligned}$$

Let $y \in \Omega$ be fixed. We define $f^\pm_{p,y} : \overline{\Omega} \to \mathbb{R}$ by

$$f^\pm_{p,y}(x) = \begin{cases} \pm\,|x-y|, & \text{if } p = \infty \\ \pm\,|x-y|^{\frac{p-N}{p-1}}, & \text{if } p \in (N, \infty). \end{cases}$$

Clearly, we have $f^\pm_{p,y} \in C(\overline{\Omega})$. Moreover, $f^\pm_{p,y} \in C^1(\Omega \setminus \{y\})$ and

$$\nabla f^\pm_{p,y}(x) = \begin{cases} \pm\,\dfrac{x-y}{|x-y|}, & \forall x \in \Omega \setminus \{y\}, \quad \text{if } p = \infty \\[2ex] \pm\,\dfrac{p-N}{p-1}\,\dfrac{x-y}{|x-y|^{\frac{p+N-2}{p-1}}}, & \forall x \in \Omega \setminus \{y\}, \quad \text{if } p \in (N, \infty). \end{cases} \tag{4.12}$$

Since $C(\overline{\Omega}) \subset L^p(\Omega)$, we infer that $f^{\pm}_{p,y} \in W^{1,p}(\Omega)$ and

$$\left\|\nabla f^{\pm}_{p,y}(x)\right\|_{L^p(\Omega)} = \begin{cases} 1, & \text{if } p = \infty \\ \dfrac{p-N}{p-1}\left(\displaystyle\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}}\right)^{1/p}, & \text{if } p \in (N,\infty). \end{cases}$$

It follows that the right hand side (RHS) of (4.11) for $f^{\pm}_{p,y}$ is

$$\text{RHS} = \begin{cases} \dfrac{1}{\omega_N}\left(\displaystyle\int_\Omega \frac{dx}{|x-y|^{N-1}}\right), & \text{if } p = \infty \\ \dfrac{p-N}{\omega_N(p-1)}\displaystyle\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}}, & \text{if } p \in (N,\infty). \end{cases} \tag{4.13}$$

By (3.1) and (4.12), we have that the left hand side (LHS) of (4.11) for $f^{\pm}_{p,y}$ is

$$\begin{aligned} \text{LHS} &= \left|\int_\Omega \langle \nabla E(x-y), \nabla f^{\pm}_{p,y}(x)\rangle\, dx\right| = \left|\int_\Omega \frac{\langle x-y, \nabla f^{\pm}_{p,y}(x)\rangle}{\omega_N|x-y|^N}\, dx\right| \\ &= \begin{cases} \dfrac{1}{\omega_N}\left(\displaystyle\int_\Omega \frac{dx}{|x-y|^{N-1}}\right), & \text{if } p = \infty \\ \dfrac{p-N}{\omega_N(p-1)}\displaystyle\int_\Omega \frac{dx}{|x-y|^{(N-1)p'}}, & \text{if } p \in (N,\infty). \end{cases} \end{aligned} \tag{4.14}$$

Using (4.13) and (4.14) we obtain equality in (4.11) for $f(x) = f^{\pm}_{p,y}(x)$. $\square$

Corollary 6. *For $a \in \Omega$ and $R > 0$ such that $B = B_R(a) \subset \overline{B}_R(a) \subset \Omega$, we have*

$$\left|f(a) - \fint_{\partial B} f(x)\, d\sigma(x)\right| \leq \omega_N^{\frac{1}{p'}-1}\left(\frac{R^{N-(N-1)p'}}{N-(N-1)p'}\right)^{\frac{1}{p'}} \|\nabla f\|_{L^p(B)}\,. \tag{4.15}$$

Moreover, the constant is sharp and the function $f(x) = \pm|x-a|$ if $p = \infty$ resp., $f(x) = \pm|x-a|^{(p-N)/(p-1)}$ if $p \in (N,\infty)$ achieves the equality.

Proof. Note that $f \in C(\overline{B}) \cap C^1(B \setminus A_i)$ resp., $f \in W^{1,p}(B)$ with $p \in (N,\infty]$. Therefore, we can apply Corollary 5 with $y = a$ and $\Omega = B$. More precisely,

$$\left|f(a) - \int_{\partial B} f(x)\frac{\partial E}{\partial\nu}(x-a)\, d\sigma(x)\right| \leq \frac{\|\nabla f\|_{L^p(B)}}{\omega_N}\left(\int_B \frac{dx}{|x-a|^{(N-1)p'}}\right)^{1/p'} \tag{4.16}$$

where the equality holds for $f(x) = \pm|x-y|$ if $p = \infty$ and $f(x) = \pm|x-y|^{(p-N)/(p-1)}$ if $p \in (N,\infty)$.

Notice that, for each $x \in \partial B$, we have

$$\begin{aligned} \frac{\partial E}{\partial\nu}(x-a) &= \langle \nabla E(x-a), \nu(x)\rangle = \langle \frac{x-a}{\omega_N|x-a|^N}, \frac{x-a}{|x-a|}\rangle \\ &= \frac{1}{\omega_N|x-a|^{N-1}} = \frac{1}{\omega_N R^{N-1}} = \text{meas}\,(\partial B). \end{aligned}$$

It follows that

$$\int_{\partial B} f(x)\frac{\partial E}{\partial\nu}\,(x-a)\, d\sigma(x) = \frac{1}{\text{meas}\,(\partial B)}\int_{\partial B} f(x)\, d\sigma(x) = \fint_{\partial B} f(x)\, d\sigma(x). \tag{4.17}$$

On the other hand,

$$
\begin{aligned}
\int_B \frac{dx}{|x-a|^{(N-1)p'}} &= \int_0^R \left(\int_{\partial B_\rho(a)} \frac{d\sigma(x)}{|x-a|^{(N-1)p'}} \right) d\rho \\
&= \int_0^R \left(\frac{1}{\rho^{(N-1)p'}} \int_{\partial B_\rho(a)} d\sigma(x) \right) d\rho = \int_0^R \frac{\omega_N \rho^{N-1}}{\rho^{(N-1)p'}}\, d\rho \\
&= \frac{\omega_N R^{N-(N-1)p'}}{N-(N-1)p'}.
\end{aligned} \tag{4.18}
$$

Replacing (4.17) and (4.18) in (4.16) we obtain (4.15). □

Corollary 7. *The following identities hold*

$$
\begin{aligned}
\int_\Omega \bar{\bar{u}}_f(y)\, dy =& \frac{1}{N} \int_{\partial\Omega} f(x) \langle x - z, \nu \rangle\, d\sigma(x) - \frac{1}{N} \int_\Omega \langle \nabla f(x), x - z \rangle\, dx \\
&+ \int_\Omega \left(\int_\Omega \langle \nabla E(x-y), \nabla f(x) \rangle\, dx \right) dy, \quad \forall z \in \mathbb{R}^N
\end{aligned} \tag{4.19}
$$

resp.,

$$
\int_{\partial\Omega} \bar{\bar{u}}_f(z)\, d\sigma(z) = \frac{1}{2} \int_{\partial\Omega} f(z)\, d\sigma(z) + \int_{\partial\Omega} \zeta(z)\, d\sigma(z), \tag{4.20}
$$

where we define

$$
\zeta(z) = \lim_{\Omega \ni t \to z} \int_\Omega \langle \nabla E(x-t), \nabla f(x) \rangle\, dx, \quad \textit{for each } z \in \partial\Omega.
$$

Remark 2. Note that ζ is well defined because of (2.2) and (3.1).

Proof. By virtue of Remark 1, $\bar{\bar{u}}_f \in L^1(\Omega)$. Obviously, $f \in L^1(\Omega)$ since $f \in C(\overline{\Omega})$ and Ω is bounded. Therefore, we can integrate (3.1) over Ω to obtain

$$
\int_\Omega f(y)\, dy = \int_\Omega \bar{\bar{u}}_f(y)\, dy - \int_\Omega \left(\int_\Omega \langle \nabla E(x-y), \nabla f(x) \rangle\, dx \right) dy.
$$

Using now (3.2), we arrive at (4.19).

Let $z \in \partial\Omega$ be arbitrary. By the continuity of f on $\overline{\Omega}$ and Proposition 3, we find

$$
\lim_{\Omega \ni y \to z} [f(y) - \bar{\bar{u}}_f(y)] = \frac{f(z)}{2} - \bar{\bar{u}}_f(z).
$$

Combining this with (3.1), we derive that

$$
f(z) = 2\bar{\bar{u}}_f(z) - 2\zeta(z), \quad \forall z \in \partial\Omega. \tag{4.21}
$$

By Remark 1, $\bar{\bar{u}}_f(z) \in C(\partial\Omega)$. Hence integrating (4.21) over $\partial\Omega$ we find (4.20). □

Corollary 8 (Gauss' Lemma extension)**.** *Assume $f \in W^{1,p}(\Omega)$, for some $p \in [1, \infty]$. Then the following representation holds*

$$
\bar{\bar{u}}_f(y) = \begin{cases} f(y) + \displaystyle\int_\Omega \langle \nabla E(x-y), \nabla f(x) \rangle\, dx, & \forall y \in \Omega, \ \textit{if } p \in (N, \infty], \\ \zeta(y) + f(y)/2, & \forall y \in \partial\Omega, \ \textit{if } p \in (N, \infty], \\ \displaystyle\int_\Omega \langle \nabla E(x-y), \nabla f(x) \rangle\, dx, & \forall y \in \mathbb{R}^N \setminus \overline{\Omega}, \ \forall p \in [1, \infty]. \end{cases} \tag{4.22}
$$

Proof. In view of (3.1) and (4.21), we need only to show that

$$\bar{\bar{u}}_f(y) = \int_\Omega \langle \nabla E(x-y), \nabla f(x)\rangle\, dx, \quad \forall y \in \mathbb{R}^N \setminus \overline{\Omega}, \quad \forall p \in [1,\infty]. \tag{4.23}$$

For $y \in \mathbb{R}^N \setminus \overline{\Omega}$ fixed, we define the vector field $Z : \overline{\Omega} \to \mathbb{R}^N$ by

$$Z(x) = f(x)\nabla E(x-y) = \frac{f(x)}{\omega_N |x-y|^N}\,(x-y), \quad \forall x \in \overline{\Omega}.$$

Clearly, $Z \in C^1(\Omega \setminus A) \cap C(\overline{\Omega})$. Let $\epsilon > 0$ be fixed such that $\overline{B}_\epsilon(a_i) \subset \Omega$, $\forall i \in I$ and $\overline{B}_\epsilon(a_i) \cap \overline{B}_\epsilon(a_j) = \emptyset$, $\forall i, j \in I$ with $i \neq j$. We denote $\Omega_\epsilon := \Omega \setminus \left(\cup_{i\in I} \overline{B}_\epsilon(a_i)\right)$. By applying Proposition 1 for $Z : \Omega_\epsilon \to \mathbb{R}^N$, we obtain

$$\begin{aligned} \int_{\Omega_\epsilon} \operatorname{div} Z(x)\, dx = \int_{\partial\Omega} & f(x) \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x) \\ & - \frac{1}{\omega_N} \sum_{i\in I} \int_{\partial B_\epsilon(a_i)} \frac{f(x)}{\epsilon |x-y|^N} \langle x-y, x-a_i\rangle\, d\sigma(x). \end{aligned} \tag{4.24}$$

Since $y \notin \overline{\Omega}$, for each $i \in I$, there exists a constant $M_i > 0$ such that

$$|x-y| > M_i, \quad \forall x \in \partial B_j(a_i), \quad \forall j \in (0, \epsilon].$$

Hence, for each $i \in I$, we have

$$\begin{aligned} \left| \int_{\partial B_\epsilon(a_i)} \frac{f(x)}{\epsilon|x-y|^N} \langle x-y, x-a_i\rangle\, d\sigma(x) \right| & \leq \int_{\partial B_\epsilon(a_i)} \frac{|f(x)|}{|x-y|^{N-1}}\, d\sigma(x) \\ \leq \frac{\|f\|_{L^\infty(\Omega)}}{M_i^{N_1}} \operatorname{meas}(\partial B_\epsilon(a_i)) & = \frac{\omega_N \|f\|_{L^\infty(\Omega)}}{M_i^{N-1}}\, \epsilon^{N-1} \to 0 \ \text{ as } \epsilon \to 0. \end{aligned} \tag{4.25}$$

By (4.24) and (4.25), it follows that

$$\lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \operatorname{div} Z(x)\, dx = \int_{\partial\Omega} f(x) \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x). \tag{4.26}$$

Since $x \longmapsto E(x-y)$ is harmonic on $\mathbb{R}^N \setminus \{y\}$, we find that

$$\begin{aligned} \operatorname{div} Z(x) &= \langle \nabla f(x), \nabla E(x-y)\rangle + f(x)\Delta_x E(x-y) \\ &= \langle \nabla f(x), \nabla E(x-y)\rangle, \quad \forall x \in \Omega_\epsilon. \end{aligned} \tag{4.27}$$

We define $\Psi(x) = |x-y|^{1-N}$, for each $x \in \Omega$. Since $y \notin \overline{\Omega}$, we have $\Psi \in C(\overline{\Omega})$ so that $\Psi \in L^m(\Omega)$, $\forall m \in [1, \infty]$. By Hölder's inequality, we infer that

$$\int_\Omega |\langle \nabla f(x), \nabla E(x-y)\rangle|\, dx \leq \frac{1}{\omega_N} \|\nabla f\|_{L^p(\Omega)} \|\Psi\|_{L^{p'}(\Omega)} < \infty, \quad \forall p \in [1,\infty]. \tag{4.28}$$

From (4.26)–(4.28), we conclude (4.23). □

Proposition 7. *If Ω is an open bounded set with C^1 boundary and $f \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $\Delta f \in C(\overline{\Omega})$, then*

$$\begin{aligned} \int_\Omega \langle \nabla E(x-y), \nabla f(x)\rangle\, dx = & \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) \\ & - \int_\Omega \Delta f(x) E(x-y)\, dx, \quad \forall y \in \mathbb{R}^N \setminus \partial\Omega. \end{aligned} \tag{4.29}$$

Proof. If $y \in \mathbb{R}^N \setminus \overline{\Omega}$, then (4.29) follows by Proposition 2 (since $x \longmapsto E(x-y)$ belongs to $C^2(\Omega) \cap C^1(\overline{\Omega})$).

For $y \in \Omega$ fixed, we choose $\epsilon > 0$ such that $\overline{B}_\epsilon(y) \subset \Omega$. By Proposition 2 (applied on $\Omega \setminus B_\epsilon(y)$), we find

$$\begin{aligned}\int_{\Omega\setminus B_\epsilon(y)} \Delta f(x) E(x-y)\, dx &= \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) \\ &- \int_{\partial B_\epsilon(y)} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) - \int_{\Omega\setminus B_\epsilon(y)} \langle \nabla f(x), \nabla E(x-y)\rangle\, dx.\end{aligned} \tag{4.30}$$

Since $x \longmapsto \Delta f(x) E(x-y)$ is integrable on Ω, we have

$$\int_\Omega \Delta f(x) E(x-y)\, dx = \lim_{\epsilon\to 0} \int_{\Omega\setminus B_\epsilon(y)} \Delta f(x) E(x-y)\, dx. \tag{4.31}$$

On the other hand, using $f \in C^1(\overline{\Omega})$, we deduce (as in the proof of Theorem 1) that $x \longmapsto \langle \nabla f(x), \nabla E(x-y)\rangle$ is integrable on Ω. It follows that

$$\int_\Omega \langle \nabla f(x), \nabla E(x-y)\rangle\, dx = \lim_{\epsilon\to 0} \int_{\Omega\setminus B_\epsilon(y)} \langle \nabla f(x), \nabla E(x-y)\rangle\, dx. \tag{4.32}$$

Our next step is to prove that

$$\lim_{\epsilon\to 0} \int_{\partial B_\epsilon(y)} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) = 0. \tag{4.33}$$

Indeed, if $N = 2$, then we have

$$\begin{aligned}\left| \int_{\partial B_\epsilon(y)} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) \right| &\le \int_{\partial B_\epsilon(y)} \left| \frac{\partial f}{\partial \nu}(x) \right| \frac{1}{2\pi} \left| \ln |x-y| \right| \, d\sigma(x) \\ &\le -C\epsilon \log \epsilon \to 0 \quad \text{as } \epsilon \to 0\end{aligned}$$

resp., if $N > 2$ then

$$\begin{aligned}\left| \int_{\partial B_\epsilon(y)} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) \right| &\le \int_{\partial B_\epsilon(y)} \left| \frac{\partial f}{\partial \nu}(x) \right| \frac{1}{\omega_N (N-2) |x-y|^{N-2}}\, d\sigma(x) \\ &\le C \frac{\operatorname{meas}(\partial B_\epsilon(y))}{\epsilon^{N-2}} = C\omega_N \epsilon \to 0 \quad \text{as } \epsilon \to 0\end{aligned}$$

where, in both cases, C denotes a positive constant.

Passing to the limit $\epsilon \to 0$ in (4.30) and using (4.31)–(4.33), we obtain (4.29). □

Remark 3. Under the assumptions of Proposition 7, Corollary 8 leads to the Green–Riemann representation formula (see [4, §2.4])

$$\begin{aligned}f(y) = \int_{\partial\Omega} f(x) \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x) &- \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) \\ &+ \int_\Omega \Delta f(x) E(x-y)\, dx, \qquad \forall y \in \Omega\end{aligned}$$

and

$$\begin{aligned}0 = \int_{\partial\Omega} f(x) \frac{\partial E}{\partial \nu}(x-y)\, d\sigma(x) &- \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(x) E(x-y)\, d\sigma(x) \\ &+ \int_\Omega \Delta f(x) E(x-y)\, dx, \qquad \forall y \in \mathbb{R}^N \setminus \overline{\Omega}.\end{aligned}$$

Moreover, if $\partial\Omega$ is smooth enough (at least C^2), then

$$f(y) = 2\int_{\partial\Omega} f(x)\frac{\partial E}{\partial \nu}(x-y)\,d\sigma(x) - 2\lim_{\Omega\ni t\to y}\int_{\Omega}\langle \nabla E(x-t), \nabla f(x)\rangle\,dx, \quad \forall y \in \partial\Omega.$$

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