MONOTONICITY AND INEQUALITIES FOR RATIO OF THE GENRALIZED LOGARITHMIC MEANS

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ABSTRACT. Let $c > b > a > 0$ be real numbers. Then the function $f(r) = \frac{L_r(a,b)}{L_r(a,c)}$ is strictly decreasing on $(-\infty, \infty)$, where $L_r(a,b)$ denotes the generalized (extended) logarithmic mean of two positive numbers a and b.

1. INTRODUCTION

If $-\infty < p < \infty$ and a, b are two positive numbers, the generalized (extended) logarithmic mean $L_p(a,b)$ of a and b is defined for $a = b$ by $L_p(a,b) = a$ and for $a \neq b$ by

$$L_p(a,b) = \begin{cases} \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{1/p}, & p \neq -1, 0; \\ \dfrac{b-a}{\ln b - \ln a}, & p = -1; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & p = 0. \end{cases} \tag{1}$$

the case $p = -1$ is called the logarithmic mean of a and b, and will be written $L(a,b)$; while the case $p = 0$ is the identric mean of a and b, written $I(a,b)$.

This definition of the generalized logarithmic mean can be found in [2, p. 6] and [33, 34].

It is well known that if $r > 0$ is a real number, then for all natural numbers n

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{2}$$

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The first inequality in (2) is called H. Alzer's inequality [1], and the second one in (2) J. S. Martins' inequality [11]. The inequality between two ends of (2) is called Minc-Sathre's inequality [12].

There exists a very rich literature on inequality (2). Alzer's inequality has been generalized and extended, for example, in [4, 5, 6, 7, 10, 14, 15, 16, 17, 22, 24, 25, 30, 31, 32, 35, 36, 37]. So does Martins's inequality in [3, 5, 17, 21, 23, 25, 26, 27, 29, 35, 37, 38] and Minc-Sathre's inequality in [1, 5, 9, 18, 19, 20, 25, 27, 28], respectively.

Recently, F. Qi and B.-N. Guo proved in [15, 23] the following double inequality: Let $b > a > 0$ and $\delta > 0$, then for any positive real number r,

$$\frac{b}{b+\delta} < \left(\frac{\frac{1}{b-a}\int_a^b x^r \,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \,\mathrm{d}x}\right)^{1/r} < \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{3}$$

The upper and lower bounds in (3) are the best possible, or more accurately say,

$$\lim_{r\to\infty}\left(\frac{\frac{1}{b-a}\int_a^b x^r \,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \,\mathrm{d}x}\right)^{1/r} = \frac{b}{b+\delta}, \tag{4}$$

$$\lim_{r\to 0}\left(\frac{\frac{1}{b-a}\int_a^b x^r \,\mathrm{d}x}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r \,\mathrm{d}x}\right)^{1/r} = \frac{[b^b/a^a]^{1/(b-a)}}{[(b+\delta)^{b+\delta}/a^a]^{1/(b+\delta-a)}}. \tag{5}$$

Inequality (3) can be taken for an integral form of (2).

It is easy to see that inequality (3) can be written for $r > 0$ as

$$\frac{b}{b+\delta} < \frac{L_r(a,b)}{L_r(a,b+\delta)} < \frac{I(a,b)}{I(a,b+\delta)}. \tag{6}$$

In this short note, we are about to extend the result presented by (3) to (5) which are established in [15, 23] by F. Qi and B.-N. Guo, and obtain the following

Theorem 1. *Let $c > b > a > 0$ be real numbers. Then the function*

$$f(r) = \frac{L_r(a,b)}{L_r(a,c)} \tag{7}$$

is strictly decreasing with $r \in (-\infty, \infty)$.

The following corollary is straightforward.

Corollary 1. *Let $c > b > a > 0$ be real numbers.*

(1) *For any real number $r \in \mathbb{R}$,*

$$\frac{b}{c} < \frac{L_r(a,b)}{L_r(a,c)} < 1. \tag{8}$$

The both bounds in (8) are the best possible.

(2) *For any positive real number $r > 0$,*

$$\frac{b}{c} < \frac{L_r(a,b)}{L_r(a,c)} < \frac{I(a,b)}{I(a,c)}. \tag{9}$$

The both bounds in (9) are also the best possible.

Remark 1. It is worthwhile pointing out that inequalities (3) and (9) are equivalent each other.

In [29] it was conjectured that the function

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r}\right)^{1/r} \tag{10}$$

is decreasing with r. Now it is still keep open. We can regard Theorem 1 as a solution to an integral form of the conjecture above.

2. Proof of Theorem 1

In order to verify Theorem 1, we shall make use of the following elementary lemma which can be found in [8, p. 395].

Lemma 1 ([8, p. 395])**.** *Let the second derivative of $\phi(x)$ be continuous with $x \in (-\infty, \infty)$ and $\phi(0) = 0$. Define*

$$g(x) = \begin{cases} \dfrac{\phi(x)}{x}, & x \neq 0; \\ \phi'(0), & x = 0. \end{cases} \tag{11}$$

Then $\phi(x)$ is (strictly) convex if and only if $g(x)$ is (strictly) increasing with $x \in (-\infty, \infty)$.

Remark 2. In [13, p. 18] a general conclusion was given: A function f is convex on $[a, b]$ if and only if $\frac{f(x)-f(x_0)}{x-x_0}$ is nondecreasing on $[a, b]$ for every point $x_0 \in [a, b]$.

Proof of Theorem 1. Define for $r \in (-\infty, \infty)$

$$\varphi(r) = \begin{cases} \ln\left(\dfrac{c-a}{b-a}\cdot\dfrac{b^{r+1}-a^{r+1}}{c^{r+1}-a^{r+1}}\right), & r \neq -1; \\ \ln\left(\dfrac{c-a}{b-a}\cdot\dfrac{\ln b-\ln a}{\ln c-\ln a}\right), & r = -1. \end{cases} \tag{12}$$

Then we have

$$\ln f(r) = \begin{cases} \dfrac{\varphi(r)}{r}, & r \neq 0, \\ \varphi'(0), & r = 0. \end{cases} \tag{13}$$

In order to prove that $\ln f(r)$ is strictly decreasing it suffices to show that φ is strictly concave in $(-\infty, \infty)$. Easy computation reveals that

$$\varphi(-1-r) = \varphi(r-1) + r\ln\frac{c}{b}, \tag{14}$$

which implies that $\varphi''(-r-1) = \varphi''(r-1)$, and then $\varphi(r)$ has the same concavity on both $(-\infty,-1)$ and $(-1,\infty)$. Hence, it is sufficient to prove that φ is strictly concave on $(-1,\infty)$.

A simple computation yields

$$\varphi''(r) = \frac{(a/c)^{r+1}[\ln(a/c)]^2}{[1-(a/c)^{r+1}]^2} - \frac{(a/b)^{r+1}[\ln(a/b)]^2}{[1-(a/b)^{r+1}]^2}. \tag{15}$$

Define for $0 < t < 1$

$$\omega(t) = \frac{t(\ln t)^2}{(1-t)^2}. \tag{16}$$

Differentiation yields

$$(1-t)t\ln t\frac{\omega'(t)}{\omega(t)} = (1+t)\ln t + 2(1-t) = -\sum_{n=2}^{\infty}\frac{n-1}{n(n+1)}t^{n+1} < 0, \tag{17}$$

which means that $\omega'(t) > 0$ for $0 < t < 1$. As a result of applying this conclusion in (15), we obtain $\varphi''(r) < 0$ for $r > -1$. Thus $\varphi(r)$ is strictly concave in $(-1,\infty)$. The proof is complete. $\square$

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