SOME LANDAU TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATES ARE HÖLDER CONTINUOUS

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Abstract. Some inequalities of Landau type for functions whose derivates satisfy Hölder's condition are pointed out.

1. Introduction

Let $I = \mathbb{R}_+$ or $I = \mathbb{R}$. If $f : I \to \mathbb{R}$ is twice differentiable and $f, f^{''} \in L_p(I)$, $p \in [1, \infty]$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of the function f, so that

$$\|f'\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \cdot \|f^{''}\|_{p,I}^{\frac{1}{2}}, \tag{1.1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, i.e, we recall

$$\|h\|_{\infty,I} := ess \sup_{t \in I} |h(t)|$$

and

$$\|h\|_{p,I} := \left(\int_I |h(t)|^p dt \right)^{\frac{1}{p}},$$

if $p \in [1, \infty)$.

The investigation of such inequalities was initiated by E. Landau [1] in 1913. He considered the case $p = \infty$ and showed that

$$C_\infty(\mathbb{R}_+) = 2 \qquad \text{and} \qquad C_\infty(\mathbb{R}) = \sqrt{2}, \tag{1.2}$$

are the best constants for which (1.1) holds.

In 1932, G.H. Hardy and J.E. Littlewood [2] proved (1.1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \qquad \text{and} \qquad C_2(\mathbb{R}) = 1.$$

In 1935, G.H. Hardy, E. Landau and J.E. Littlewood [3] showed that the best constant $C_p(\mathbb{R}_+)$ in (1.1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2 \quad \text{for} \quad p \in [1, \infty), \tag{1.3}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. Actually $C_p(\mathbb{R}) \leq \sqrt{2}$ (see [4] by R.R. Kallman and G.-C. Rota and [5] by Z. Ditzian).

For other results concerning this problem, see Chapter I of [7].

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2. Some Results for f Bounded and f' Hölder Continuous

The following lemma is useful in what follows.

Lemma 1. *Let $C, D > 0$ and $r, u \in (0, 1]$. Consider the function $g_{r,u} : (0, \infty) \to \mathbb{R}$ given by*

$$g_{r,u}(\lambda) = \frac{C}{\lambda^u} + D\lambda^r. \tag{2.1}$$

Define $\lambda_0 := \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0, \infty)$. Then, for $\lambda_1 \in (0, \infty)$ we have the bound

$$\inf_{\lambda \in (0,\lambda_1]} g_{r,u}(\lambda) = \begin{cases} \frac{r+u}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}}} \cdot C^{\frac{r}{r+u}} \cdot D^{\frac{u}{r+u}} & \text{if } \lambda_1 \geq \lambda_0, \\ \\ \frac{C}{\lambda_1^u} + D\lambda_1^r & \text{if } 0 < \lambda_1 < \lambda_0. \end{cases} \tag{2.2}$$

Proof. We observe that

$$g'_{r,u}(\lambda) = \frac{rD\lambda^{r+u} - Cu}{\lambda^{u+1}}, \quad \lambda \in (0, \infty).$$

The unique solution of the equation $g'_{r,u}(\lambda) = 0$, $\lambda \in (0, \infty)$ is $\lambda_0 = \left(\frac{uC}{rD}\right)^{\frac{1}{r+u}} \in (0, \infty)$. The function $g_{r,u}$ is decreasing on $(0, \lambda_0)$ and increasing on (λ_0, ∞). The global minimum for $g_{r,u}$ on $(0, \infty)$ is

$$\begin{aligned} g_{r,u}(\lambda_0) &= \frac{C}{\left(\frac{uC}{rD}\right)^{\frac{u}{r+u}}} + D\left(\frac{uC}{rD}\right)^{\frac{r}{r+u}} = \frac{C(rD)^{\frac{u}{r+u}}}{(uC)^{\frac{u}{r+u}}} + \frac{D(uC)^{\frac{r}{r+u}}}{(rD)^{\frac{r}{r+u}}} \\ &= \frac{CrD + DuC}{(uC)^{\frac{u}{r+u}}(rD)^{\frac{r}{r+u}}} = \frac{CD(r+u)}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}} \cdot C^{\frac{u}{r+u}} \cdot D^{\frac{r}{r+u}}} \\ &= \frac{r+u}{u^{\frac{u}{r+u}} \cdot r^{\frac{r}{r+u}}} C^{\frac{r}{r+u}} \cdot D^{\frac{u}{r+u}}, \end{aligned}$$

which proves that equality (2.2) ∎

The following particular cases are useful:

Corollary 1. *Let $C, D > 0$ and $r \in (0, 1]$. Consider the function $g_r : (0, \infty) \to \mathbb{R}$ given by*

$$g_r(\lambda) = \frac{C}{\lambda} + D\lambda^r.$$

Define $\overline{\lambda_0} = \left(\frac{C}{rD}\right)^{\frac{1}{r+1}} \in (0, \infty)$. Then for $\lambda_1 \in (0, \infty)$ one has

$$\inf_{\lambda \in (0,\lambda_1]} g_r(\lambda) = \begin{cases} \frac{r+1}{r^{\frac{r}{r+1}}} \cdot C^{\frac{r}{r+1}} \cdot D^{\frac{1}{r+1}} & \text{if } \lambda_1 \geq \overline{\lambda_0}, \\ \\ \frac{C}{\lambda_1} + D\lambda_1^r & \text{if } 0 < \lambda_1 < \overline{\lambda_0}. \end{cases} \tag{2.3}$$

Corollary 2. *Let $C, D > 0$ and $u \in (0, 1]$. Consider the function $g_u : (0, \infty) \to \mathbb{R}$ given by*

$$g_u(\lambda) = \frac{C}{\lambda^u} + D\lambda.$$

Define $\widetilde{\lambda_0} = \left(\frac{uC}{D}\right)^{\frac{1}{1+u}} \in (0,\infty)$. *Then for* $\lambda_1 \in (0,\infty)$ *one has*

$$\inf_{\lambda\in(0,\lambda_1]} g_u(\lambda) = \begin{cases} \frac{1+u}{u^{\frac{u}{1+u}}}\cdot C^{\frac{1}{1+u}}\cdot D^{\frac{u}{1+u}} & \text{if } \lambda_1 \ge \widetilde{\lambda_0}, \\ \frac{C}{\lambda_1^u} + D\lambda_1 & \text{if } 0<\lambda_1<\widetilde{\lambda_0}. \end{cases} \tag{2.4}$$

Remark 1. *If* $r = u = 1$ *then the following bound holds*

$$\inf_{\lambda\in(0,\lambda_1]} \left(\frac{C}{\lambda} + D\lambda\right) = \begin{cases} 2\sqrt{CD} & \text{if } \lambda_1 \ge \sqrt{\frac{C}{D}}, \\ \frac{C}{\lambda_1} + D\lambda_1 & \text{if } 0<\lambda_1<\sqrt{\frac{C}{D}}. \end{cases} \tag{2.5}$$

The following theorem holds:

Theorem 1. *Let* I *be an interval in* $\mathbb{R}$ *and* $f : I \to \mathbb{R}$ *a locally absolutely continuous function on* I. *If* $f \in L_\infty(I)$ *and the derivative* $f' : I \to \mathbb{R}$ *satisfies the Hölder condition:*

$$|f'(t) - f'(s)| \le H|t-s|^r \text{ for any } t,s \in I, \tag{2.6}$$

where $H > 0$ *and* $r \in (0,1]$ *are given, then* $f' \in L_\infty(I)$ *and one has the inequalities*

$$\|f'\|_{I,\infty} \le \begin{cases} 2^{\frac{r}{r+1}}\left(1+\frac{1}{r}\right)^{\frac{r}{r+1}}\|f\|_{I,\infty}^{\frac{r}{r+1}} H^{\frac{1}{r+1}} \\ \qquad \text{if} \quad m(I) \ge 2^{\frac{r+2}{r+1}}\left(\frac{\|f\|_{I,\infty}}{H}\right)^{\frac{1}{r+1}}\left(1+\frac{1}{r}\right)^{\frac{1}{r+1}}, \\ \frac{4\|f\|_{I,\infty}}{m(I)} + \frac{H}{2^r(r+1)}[m(I)]^r \\ \qquad \text{if} \quad 0 \le m(I) \le 2^{\frac{r+2}{r+1}}\left(\frac{\|f\|_{I,\infty}}{H}\right)^{\frac{1}{r+1}}\left(1+\frac{1}{r}\right). \end{cases} \tag{2.7}$$

Proof. We start with the following identity

$$f(t) = f(a) + (t-a)f'(a) + \int_a^t [f'(s) - f'(a)]ds \tag{2.8}$$

to get

$$|f'(a)| \le \left|\frac{f(t)-f(a)}{t-a}\right| + \frac{1}{|t-a|}\left|\int_a^t |f'(s) - f'(a)|ds\right|, \tag{2.9}$$

for any $t \in I$ and a.e. $a \in I$, $t \ne a$.

Since f' is of $r - H$- Hölder type, then

$$\left|\int_a^t |f'(s) - f'(a)ds\right| \le H\left|\int_a^t |s-a|^r ds\right| = \frac{H}{r+1}|t-a|^{r+1}. \tag{2.10}$$

So then by (2.9) and (2.10) we deduce

$$|f'(a)| \le \frac{|f(t)-f(a)|}{|t-a|} + \frac{H}{r+1}|t-a|^r, \tag{2.11}$$

for any $t \in I$ and a.e. $a \in I$, $t \ne a$.

Since $f \in L_\infty(I)$, then by (2.11) we obviously get that

$$|f'(a)| \le \frac{2\|f\|_{I,\infty}}{|t-a|} + \frac{H}{r+1}|t-a|^r \tag{2.12}$$

for any $t \in I$ and a.e. $a \in I$, $t \ne a$.

Now observe that for any $a \in I$ and any $s \in \left(0, \frac{m(I)}{2}\right)$ there exists a $t \in I$ so that $s = |t-a|$ and then, by (2.12), we deduce

$$|f'(a)| \leq \frac{2\|f\|_{I,\infty}}{s} + \frac{H}{r+1} s^r \tag{2.13}$$

for a.e. $a \in I$ and every $s \in \left(0, \frac{m(I)}{2}\right)$. By taking the inequality (2.13) to the infimum over s on $\left(0, \frac{m(I)}{2}\right)$, we get that

$$|f'(a)| \leq \inf_{s \in \left(0, \frac{m(I)}{2}\right)} \left[\frac{2\|f\|_{I,\infty}}{s} + \frac{H}{r+1} s^r\right] = K \tag{2.14}$$

for a.e. $a \in I$.

If we take the essential supremum over $a \in I$ in (2.14), we conclude that

$$\|f'\|_{I,\infty} \leq K. \tag{2.15}$$

Making use of Corollary 1, we get

$$K = \begin{cases} \frac{r+1}{r^{\frac{r}{r+1}}} (2\|f\|_{I,\infty})^{\frac{r}{r+1}} \left(\frac{H}{r+1}\right)^{\frac{1}{r+1}} & \text{if } \frac{m(I)}{2} \geq \left(\frac{2\|f\|_{I,\infty}(r+1)}{rH}\right)^{\frac{1}{r+1}}, \\ \frac{2\|f\|_{I,\infty}}{\frac{m(I)}{2}} + \frac{H}{r+1} \cdot \left(\frac{m(I)}{2}\right)^r & \text{if } 0 < \frac{m(I)}{2} < \left(\frac{2\|f\|_{I,\infty}(r+1)}{rH}\right)^{\frac{1}{r+1}}. \end{cases}$$

giving the desired result (2.7). ∎

The following result also holds

Corollary 3. *With the assumption in Theorem 1 and if f' is L-Lipschitz then*

$$\|f'\|_{I,\infty} \leq \begin{cases} 2\sqrt{\|f\|_{I,\infty} \cdot L} & \text{if } m(I) \geq \sqrt{\frac{\|f\|_{I,\infty}}{L}}; \\ \frac{4\|f\|_{I,\infty}}{m(I)} + \frac{H}{4} m(I) & \text{if } 0 < m(I) \leq \sqrt{\frac{\|f\|_{I,\infty}}{L}}. \end{cases} \tag{2.16}$$

Remark 2. *This result was obtained by Niculescu and Buşe in* [6]*, see Theorem 3.*

3. Some Bounds for f and f' Hölder Continuous

The following result also holds:

Theorem 2. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If f is $l - K$-Hölder type, i.e. it satisfies the condition*

$$|f(t) - f(s)| \leq K|t-s|^l \quad \text{for any } t, s, \in \overset{\circ}{I}, \tag{3.1}$$

where $K > 0$ and $l \in (0,1)$ are given, and the derivative $f' : I \to \mathbb{R}$ satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality

$$\|f'\|_{I,\infty} \leq \begin{cases} \frac{1-l+r}{(1-l)^{\frac{1-l}{1-l+r}} \cdot r^{\frac{r}{1-l+r}} \cdot (r+1)^{\frac{1-l}{r+1-l}}} K^{\frac{r}{r+1-l}} \cdot H^{\frac{1-l}{r+1-l}} \\ \qquad \text{if } m(I) \geq 2\left[\frac{(1-l)K}{H}\right]^{\frac{1}{1-l+r}} \left(1+\frac{1}{r}\right)^{\frac{1}{1-l+r}}; \\ \frac{2(1-l)K}{[m(I)]^{1-l}} + \frac{H}{2^r(r+1)} [m(I)]^r \\ \qquad \text{if } 0 < m(I) < 2\left[\frac{(1-l)K}{H}\right]^{\frac{1}{1-l+r}} \left(1+\frac{1}{r}\right)^{\frac{1}{1-l+r}}. \end{cases} \tag{3.2}$$

Proof. We know (see the proof of Theorem 1) that

$$|f'(a)| \le \frac{|f(t)-f(a)|}{|t-a|} + \frac{H}{r+1}|t-a|^r \tag{3.3}$$

for any $t \in I$ and a.e. $a \in I$ with $a \neq t$.

Using the assumption that (3.1) holds, then, by (3.3) we may write that

$$|f'(a)| \le \frac{K}{|t-a|^{1-l}} + \frac{H}{r+1}|t-a|^r \tag{3.4}$$

for any $t \in I$ and a.e. $a \in I$ with $t \neq a$.

Using a similar argument to the one in Theorem 1, we may conclude that $\|f'\|_{I,\infty} \le S$, where

$$\begin{aligned} S &= \inf_{\lambda \in (0,\frac{m(I)}{2})} \left[\frac{K}{\lambda^{1-l}} + \frac{H}{r+1}\lambda^r\right] \\ &= \begin{cases} \frac{1-l+r}{(1-l)^{\frac{1-l}{1-l+r}} \cdot r^{\frac{r}{1-l+r}}} K^{\frac{r}{r+1-l}} \cdot \left(\frac{H}{r+1}\right)^{\frac{1-l}{r+1-l}} & \text{if } \frac{m(I)}{2} \ge \left[\frac{(1-l)K}{r\frac{H}{r+1}}\right]^{\frac{1}{1-l+r}} \\ \frac{K}{(\frac{m(I)}{2})^{1-l}} + \frac{H}{r+1}\left(\frac{m(I)}{2}\right)^r & \text{if } 0 < \frac{m(I)}{2} \le \left[\frac{(1-l)K}{r\cdot\frac{H}{r+1}}\right]^{\frac{1}{1-l+r}} \end{cases} \end{aligned}$$

from where we deduce the desired inequality (3.2). ∎

The following corollary is useful.

Corollary 4. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If $f' \in L_p(I)$, $p > 1$ and the derivative f' satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality:*

$$\|f'\|_{I,\infty} \le \begin{cases} \frac{pr+1}{p^{\frac{pr}{pr+1}}} \cdot \frac{1}{r^{\frac{pr}{pr+1}} \cdot (r+1)^{\frac{1}{pr+1}}} \|f'\|_{I,p}^{\frac{pr}{pr+1}} H^{\frac{1}{pr+1}} \\ \qquad \text{if } m(I) \ge 2\left[\frac{\|f'\|_{I,p}}{pH}\right]^{\frac{p}{pr+1}} \cdot \left(1+\frac{1}{r}\right)^{\frac{p}{pr+1}}; \\ \frac{\|f'\|_{I,p} \cdot 2^{\frac{1}{p}}}{[m(I)]^{\frac{1}{p}}} + \frac{H}{2^r(r+1)}[m(I)]^r \\ \qquad \text{if } 0 < m(I) < 2\left[\frac{\|f'\|_{I,p}}{pH}\right]^{\frac{p}{pr+1}} \cdot \left(1+\frac{1}{r}\right)^{\frac{p}{pr+1}}. \end{cases} \tag{3.5}$$

Proof. If $f' \in L_p(I)$, then we have

$$\begin{aligned} |f(b)-f(a)| &= \left|\int_a^b f'(s)ds\right| \le \left|\int_a^b |f'(s)|ds\right| \\ &\le |b-a|^{\frac{1}{q}} \left|\int_a^b |f'(s)|^p ds\right|^{\frac{1}{p}} \\ &\le |b-a|^{1-\frac{1}{p}} \cdot \|f'\|_{I,p}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, for a.e. $a, b \in I$.

Using Theorem 2 for $l = 1 - \frac{1}{p}$ and $K = \|f'\|_{I,p}$ we deduce the desired result (3.5). ∎

Finally we may state the following corollary as well.

Corollary 5. *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ a locally absolutely continuous function on I. If $f' \in L_1(I)$ and the derivative f' satisfies the Hölder condition (2.6), then $f' \in L_\infty(I)$ and one has the inequality*

$$\|f'\|_{I,\infty} \leq \begin{cases} \left(1+\frac{1}{r}\right)^{\frac{r}{r+1}} \cdot \|f'\|_{I,1}^{\frac{r}{r+1}} H^{\frac{1}{r+1}} \\ \qquad \text{if } m(I) \geq 2\left(\frac{\|f'\|_{I,1}}{H}\right)^{\frac{1}{r+1}} \cdot \left(1+\frac{1}{r}\right)^{\frac{1}{r+1}}; \\ \\ \frac{2\|f'\|_{I,1}}{m(I)} + \frac{H}{2^r(r+1)}[m(I)]^r \\ \qquad \text{if } 0 < m(I) < 2\left(\frac{\|f'\|_{I,1}}{H}\right)^{\frac{1}{r+1}} \left(1+\frac{1}{r}\right)^{\frac{1}{r+1}}. \end{cases} \tag{3.6}$$

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