# ON AN OPEN PROBLEM BY FENG QI REGARDING AN INTEGRAL INEQUALITY

S. MAZOUZI AND FENG QI

ABSTRACT. In the article, a functional inequality in abstract spaces is established, which gives a new affirmative answer to an open problem posed by Feng Qi in *Several integral inequalities* appeared in J. INEQUAL. PURE APPL. MATH. **1** (2000), no. 2, Art. 19 (Available online at `http://jipam.vu.edu.au/v1n2/001_00.html`) and RGMIA RES. REP. COLL. **2** (1999), no. 7, Art. 9, 1039–1042 (Available online at `http://rgmia.vu.edu.au/v2n7.html`). Moreover, some integral inequalities and a discrete inequality involving sums are deduced.

## 1. INTRODUCTION

Under what condition does the inequality

$$\int_a^b [f(x)]^t \,\mathrm{d}x \geq \left(\int_a^b f(x)\,\mathrm{d}x\right)^{t-1} \tag{1}$$

hold for $t > 1$?

This problem was proposed by the second author, F. Qi, in [8] after the following inequality was proved:

$$\int_a^b \big[f(x)\big]^{n+2} \,\mathrm{d}x \geq \left(\int_a^b f(x)\,\mathrm{d}x\right)^{n+1}, \tag{2}$$

where $f(x)$ has continuous derivative of the $n$-th order on the interval $[a,b]$, $f^{(i)}(a) \geq 0$ for $0 \leq i \leq n-1$, and $f^{(n)}(x) \geq n!$.

In the joint paper [12], K.-W. Yu and F. Qi obtained an answer to the above problem by using the integral version of Jensen's inequality and a property of

2000 *Mathematics Subject Classification.* Primary 39B62; Secondary 26D15.

*Key words and phrases.* functional inequality, integral inequality, Jensen's inequality.

The second author was supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), Doctor Fund of Jiaozuo Institute of Technology, CHINA.

This paper was typeset using $\mathcal{AMS}$-LaTeX.

convexity: Inequality (1) is valid for all $f \in C([a,b])$ such that $\int_a^b f(x)\,\mathrm{d}x \geq (b-a)^{t-1}$ for given $t > 1$.

Let $[x]$ denote the greatest integer less than or equal to $x$, $f^{(-1)}(x) = \int_a^x f(s)\,\mathrm{d}s$, $f^{(0)}(x) = f(x)$, $\gamma(t) = t(t-1)(t-2)\cdots[t-(n-1)]$ for $t \in (n, n+1]$, and $\gamma(t) = 1$ for $t < 1$, where $n$ is a positive integer. In [11], N. Towghi provided other sufficient conditions for inequality (1) to be valid: If $f^{(i)}(a) \geq 0$ for $i \leq [t-2]$ and $f^{[t-2]}(x) \geq \gamma(t-1)(x-a)^{(t-[t])}$, then $\int_a^b f(x)\,\mathrm{d}x \geq (b-a)^{t-1}$ and inequality (1) holds.

T. K. Pogány in [7], by avoiding the assumptions of differentiability used in [8, 11] and the convexity criteria used in [12], and instead using the classical integral inequalities due to Hölder, Nehari, Barnes and their generalizations by Godunova and Levin, established some inequalities which generalize, reverse, or weight inequality (1).

In this paper, by employing a functional inequality introduced in [5], which is an abstract generalization of the classical Jensen's inequality [9], we further establish the following functional inequality (4) from which inequality (1), some integral inequality, and an interesting discrete inequality involving sums can be deduced.

**Theorem 1.** *Let $\mathcal{L}$ be a linear vector space of real-valued functions, $p$ and $q$ be two real numbers such that $p \geq q \geq 1$. Assume that $f$ and $g$ are two positive functions in $\mathcal{L}$ and $G$ is a positive linear form on $\mathcal{L}$ such that*

(1) *$G(g) > 0$,*
(2) *$fg$ and $gf^p \in \mathcal{L}$.*

*If*

$$[G(g)]^{p-1} \leq [G(gf)]^{p-q}, \tag{3}$$

*then*

$$[G(gf)]^q \leq G(gf^p). \tag{4}$$

The new inequality (4) has the feature that it is stated for summable functions defined on a finite measure space $(E, \Sigma, \mu)$ whose $L^1$-norms are bounded from below by some constant involving the measure of the whole space $E$ as well as the exponents $p$ and $q$.

## 2. Lemma and proof of Theorem 1

To prove our main result, Theorem 1, it is necessary to recall a functional inequality in [5], which can be stated as follows.

**Lemma 1.** *Let $\mathcal{L}$ be a linear vector space of real valued functions and $f, g \in \mathcal{L}$ with $g \geq 0$. Assume that $F$ is a positive linear form on $\mathcal{L}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function such that*

(1) *$F(g) = 1$,*
(2) *$fg$ and $(\varphi \circ f)g \in \mathcal{L}$.*

*Then*

$$\varphi(F(fg)) \leq F((\varphi \circ f)g). \tag{5}$$

Notice that Lemma 1 is in fact a form of the classical Jensen inequality. There is a vast literature on this subject, see, e.g., [1, 2, 3, 4, 6, 10] and references therein.

*Proof of Theorem 1.* Define a positive linear form $F(u) = \frac{G(u)}{G(g)}$, then, we obviously have $F(g) = 1$. From Lemma 1, if we take as a convex function $\varphi(x) = x^p$ for $p \geq 1$, then

$$[F(gf)]^p \leq F(gf^p), \tag{6}$$

that is,

$$\left[\frac{G(gf)}{G(g)}\right]^p \leq \frac{G(gf^p)}{G(g)}$$

which gives

$$\frac{[G(gf)]^{p-q}}{[G(g)]^{p-1}}[G(gf)]^q \leq G(gf^p).$$

Since inequality (3) holds, thus inequality (4) follows. □

## 3. Corollaries and remarks

As a new positve and concrete answer to F. Qi's problem mentioned at the beginning of this paper, we have the following

**Corollary 1.** *Let $(E, \Sigma, \mu)$ be a finite measure space and let $\mathcal{L}$ be the space of all integrable functions on $E$. If $p$ and $q$ are two real numbers such that $p \geq q \geq 1$, and $f$ and $g$ are two positive functions of $\mathcal{L}$ such that*

(1) *$\int_E g \,\mathrm{d}\mu > 0$,*
(2) *$fg$ and $gf^p \in \mathcal{L}$,*

*then*

$$\left(\int_E gf\,\mathrm{d}\mu\right)^q \le \int_E gf^p\,\mathrm{d}\mu, \tag{7}$$

*provided that* $\left(\int_E gf\,\mathrm{d}\mu\right)^{p-q} \ge \left(\int_E g\,\mathrm{d}\mu\right)^{p-1}$.

*Proof.* This follows from Theorem 1 by taking $G(u) = \int_E u\,\mathrm{d}\mu$ as a positive linear form. □

*Remark* 1. We observe that if $p = q$ and $G(g) \le 1$, then inequality (3) is always fulfilled, and accordingly, we have

$$[G(gf)]^p \le G(gf^p)$$

for all $p \ge 1$.

*Remark* 2. If $\mathcal{L}$ contains the constant functions, then for

$$f = \begin{cases} 0, & p \ge q \ge 1, \\ [G(g)]^{p-q}, & p > q \ge 1, \\ 1, & p = q,\ G(g) = 1, \end{cases} \tag{8}$$

equality occurs in (4)

*Remark* 3. In fact, inequality (4) holds even if inequality (3), as merely a sufficient condition, is not satisfied. Let $p > q \ge 1$, $m = \frac{q-1}{p-q}$ and $c = \left[q\left(\frac{p-q}{p-1}\right)^{q-1}\right]^{1/(p-q)}$. If $E = [a,b]$ is a finite interval of $\mathbb{R}$ and $f(x) = c(x-a)^m$, then $\left(\int_a^b f\,\mathrm{d}x\right)^q = \int_a^b f^p\,\mathrm{d}x$. On the other hand, inequality (3) is no longer satisfied if $q\left(\frac{p-q}{p-1}\right)^{p-1} < 1$. This is due to the fact that $\left(\int_a^b f\,\mathrm{d}x\right)^{p-q} = q\left(\frac{p-q}{p-1}\right)^{p-1}\left(b-a\right)^{p-1}$.

**Corollary 2.** *Let* $f \in \mathbb{L}^1(a,b)$*, the space of integrable functions on the interval* $(a,b)$ *with respect to the Lebesgue measure, such that* $|f(x)| \ge k(x)$ *a.e. for* $x \in (a,b)$*, where*

$$(b-a)^{(p-1)/(p-q)} \le \int_a^b k(x)\,\mathrm{d}x < \infty \tag{9}$$

*for some* $p > q \ge 1$*, then*

$$\left(\int_a^b |f(x)|\,\mathrm{d}x\right)^q \le \int_a^b |f(x)|^p\,\mathrm{d}x. \tag{10}$$

*Proof.* This follows from Lemma 1 easily. □

We now apply Corollary 2 to deduce F. Qi's main result, Proposition 1.3 in [8], in detail.

**Corollary 3.** *Suppose that $f \in C^n([a,b])$ satisfies $f^{(i)}(a) \geq 0$ and $f^{(n)}(x) \geq n!$ for $x \in [a,b]$, where $0 \leq i \leq n-1$ and $n \in \mathbb{N}$, the set of all positive integers, then*

$$\int_a^b \big[f(x)\big]^{n+2}\,\mathrm{d}x \geq \left(\int_a^b f(x)\,\mathrm{d}x\right)^{n+1}. \tag{11}$$

*Proof.* Since $f^{(n)}(x) \geq n!$, then successive integrations over $[a,x]$ give

$$f^{(n-k)}(x) \geq \frac{n!}{k!}(x-a)^k, \quad k = 0,1,\dots,n-1,$$

hence

$$(x-a)^{n-k} f^{(n-k)}(x) \geq \frac{n!}{k!}(x-a)^n, \quad k = 0,1,\dots,n-1.$$

On the other hand, Taylor's expansion applied to $f$ with Lagrange remainder states that

$$\begin{aligned} f(x) &= f(a) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(\xi)}{n!}(x-a)^n \\ &\geq \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(x-a)^n \\ &= 2^n (x-a)^n, \end{aligned}$$

where $\xi \in (a,x)$. But since $x$ is arbitrary and $2^n \geq n+1$ for all $n \in \mathbb{N}$, then

$$f(x) \geq (n+1)(x-a)^n \geq 0$$

for all $x \in (a,b)$. Therefore

$$\int_a^b f(x)\,\mathrm{d}x \geq (b-a)^{n+1},$$

and inequality (11) follows by virtue of Corollary 2. □

*Remark* 4. The following function

$$f : [a,b] \to \mathbb{R}^+,\ x \mapsto f(x) = \frac{(x-a)^{n+1}}{(n+2)^n}$$

for a fixed $n \in \mathbb{N}$ satisfies $f \in C^n([a,b])$ and $f^{(i)}(a) \geq 0$, for $0 \leq i \leq n-1$, but $f^{(n)}(x) = \frac{(n+1)!}{(n+2)^n}(x-a)$ for $x \in [a,b]$. This means that the condition $f^{(n)} \geq n!$ on $[a,b]$ is no longer fulfilled. However, we have

$$\left(\int_a^b f\,\mathrm{d}x\right)^{n+2} = \int_a^b f^{n+3}\,\mathrm{d}x = \frac{(b-a)^{(n+2)^2}}{(n+2)^{(n+1)(n+2)}}.$$

Finally, let us apply Corollary 1 to derive a discrete inequality.

**Corollary 4.** *Let $E = \{a_1, \ldots, a_N\}$, $f : E \to \mathbb{R}^+$ defined by $f(a_i) = b_i$ for $i = 1, \ldots, N$, and let $\mu$ be a discrete positive measure given by $\mu(\{a_i\}) = \alpha_i > 0$ for $i = 1, \ldots, N$. If, for $p \geq q \geq 1$*

$$\left(\sum_{i=1}^{N} \alpha_i\right)^{p-1} \leq \left(\sum_{i=1}^{N} \alpha_i b_i\right)^{p-q}, \tag{12}$$

*then we have*

$$\left(\sum_{i=1}^{N} \alpha_i b_i\right)^{q} \leq \sum_{i=1}^{N} \alpha_i b_i^p. \tag{13}$$

*If, in particular, $\alpha_1 = \cdots = \alpha_N = c > 0$ satisfies*

$$c^{q-1} \leq \frac{1}{N^{p-1}} \left(\sum_{i=1}^{N} b_i\right)^{p-q}, \tag{14}$$

*then*

$$\left(\sum_{i=1}^{N} b_i\right)^{q} \leq \frac{1}{c^{q-1}} \sum_{i=1}^{N} b_i^p. \tag{15}$$

*Proof.* We observe that

$$\begin{aligned}
\left(\int_E f \, \mathrm{d}\mu\right)^{p-q} &= \left(\sum_{i=1}^{N} f(a_i)\mu(\{a_i\})\right)^{p-q} \\
&= \left(\sum_{i=1}^{N} \alpha_i b_i\right)^{p-q} \\
&\geq \left(\sum_{i=1}^{N} \alpha_i\right)^{p-1} \\
&\equiv [\mu(E)]^{p-1}.
\end{aligned}$$

and thus, the sufficient condition is satisfied. We conclude by Corollary 1 that

$$\left(\int_E f \, \mathrm{d}\mu\right)^{q} = \left(\sum_{i=1}^{N} \alpha_i b_i\right)^{q} \leq \int_E f^p \, \mathrm{d}\mu = \sum_{i=1}^{N} \alpha_i b_i^p.$$

The proof of inequality (15) is a particular case of the above argument, and thus we leave it to the reader. □

## References


[1] P. R. Beesack and J. E. Pecaric, *On Jensen's inequality for convex functions*, J. Math. Anal. Appl. **110** (1985), 536–552.

[2] P. R. Beesack and J. E. Pecaric, *On Jensen's inequality for convex functions, II*, J. Math. Anal. Appl. **118** (1986), 125–144.

[3] P. R. Beesack and J. E. Pecaric, *On Jensen's inequality for convex functions, III*, J. Math. Anal. Appl. **156** (1991), 231–239.

[4] S. S. Dragomir, C. E. M. Pearce, and J. Pecaric, *On Jensen's and related inequalities for isotonic sublinear functionals*, Acta Sci. Math (Szeged) **61**(1995), 373–382.

[5] S. Mazouzi, *A functional inequality*, Magh. Math. Rev. **3** (1994), no. 1, 83–87.

[6] J. E. Pečarić and I. Rasa, *On Jensen's inequality*, Acta Sci. Math. (Szeged) **56** (1992), 305–309.

[7] T. K. Pogány, *On an open problem of F. Qi*, J. Inequal. Pure Appl. Math. **3** (2002), no. 1, Art. 4. Available online at `http://jipam.vu.edu.au/v3n4/016_01.html`.

[8] F. Qi, *Several integral inequalities*, J. Inequal. Pure Appl. Math. **1** (2000), no. 2, Art. 19. Available online at `http://jipam.vu.edu.au/v1n2/001_00.html`. RGMIA Res. Rep. Coll. **2** (1999), no. 7, Art. 9, 1039–1042. Available online at `http://rgmia.vu.edu.au/v2n7.html`.

[9] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, Inc., New York, 1974.

[10] J. Sándor, *On the Jensen-Hadamard inequality*, Studia Univ. Babes-Bolyai Math. **36** (1991), no. 1, 9–15.

[11] N. Towghi, *Notes on integral inequalities*, RGMIA Res. Rep. Coll. **4** (2001), no. 2, Art. 12, 277–278. Available online at `http://rgmia.vu.edu.au/v4n2.html`.

[12] K.-W. Yu and F. Qi, *A short note on an integral inequality*, RGMIA Res. Rep. Coll. **4** (2001), no. 1, Art. 4, 23–25. Available online at `http://rgmia.vu.edu.au/v4n1.html`.

(S. Mazouzi) Department of Mathematics, Faculty of Sciences, University of Annaba, P. O. Box 12, Annaba 23000, ALGERIA

*E-mail address*: `mazouzi.s@voila.fr`

(F. Qi) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

*E-mail address*: `qifeng@jzit.edu.cn`