THE BEST BOUNDS IN WALLIS' INEQUALITY

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Abstract. For all natural number n, we have

$$\frac{1}{\sqrt{\pi(n+4/\pi-1)}} \leq \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+1/4)}}.$$

The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible. From this, the well-known Wallis' inequality is improved.

1. Introduction

Let

$$P_n = \frac{(2n-1)!!}{(2n)!!}, \tag{1}$$

then we have

$$\frac{1}{2\sqrt{n}} < \frac{\sqrt{2}}{\sqrt{(2n+1)\pi}} < P_n < \frac{2}{\sqrt{(4n+1)\pi}} < \frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n}} \tag{2}$$

for $n > 1$. The inequality (2) is called Wallis' inequality in [8, p. 103].

The lower and upper bounds of P_n in (2) are always cited and applied by mathematicians. The smallest upper bound $\frac{2}{\sqrt{(4n+1)\pi}}$ and the largest lower bound $\frac{\sqrt{2}}{\sqrt{(2n+1)\pi}}$ in (2), that is, the following inequalities

$$\frac{\sqrt{2}}{\sqrt{(2n+1)\pi}} < P_n < \frac{2}{\sqrt{(4n+1)\pi}} \tag{3}$$

are obtained by N. D. Kazarinoff. See [7, pp. 47–48 and pp. 65–67].

We can rewrite inequality (3) as

$$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < P_n < \frac{1}{\sqrt{\pi(n+\frac{1}{4})}} \tag{4}$$

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A48.

Key words and phrases. Wallis' inequality, best bound, gamma function, monotonicity.

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), Doctor Fund of Jiaozuo Institute of Technology, China.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX.

for $n \in \mathbb{N}$. See [4, p. 259].

It is well-known that factorials and their 'continuous' extension play an eminent role, for instance, in Combinatorics, Graph Theory, and Special Functions.

In this article, we will refine inequality (4). More precisely, we will ask for two best possible constants A and B such that the following double inequality

$$\frac{1}{\sqrt{\pi(n+A)}} \le P_n \le \frac{1}{\sqrt{\pi(n+B)}} \tag{5}$$

holds for all natural number n. In other words, the constants $A = \frac{4}{\pi} - 1$ and $B = \frac{1}{4}$ can not be replaced by smaller and larger numbers in (5) respectively.

2. Lemmas

Lemma 1. *For $x > 0$, we have*

$$\frac{2x+1}{x(4x+1)} < \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} - \frac{\Gamma'(x)}{\Gamma(x)}, \tag{6}$$

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + O\left(\frac{1}{x^2}\right), \quad x \to \infty. \tag{7}$$

The proof of inequality (6) is given in [2, 3, 9], and the proof of the asymptotic expansion (7) can be found in [6] and [10, p. 378]. See also [1, p. 257].

Remark 1. Replacing x by $x + \frac{1}{2}$ in (6) yields

$$\frac{4x+4}{(2x+1)(4x+3)} < \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\Gamma'(x+\frac{1}{2})}{\Gamma(x+\frac{1}{2})}. \tag{8}$$

Lemma 2. *For $x > 0$, we have*

$$\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})} < \frac{2x+1}{\sqrt{4x+3}}. \tag{9}$$

Proof. Define for positive real number x

$$f(x) = \ln(2x+1) - \frac{1}{2}\ln(4x+3) - \ln\Gamma(x+1) + \ln\Gamma\left(x+\frac{1}{2}\right).$$

Differentiation $f(x)$ gives us

$$f'(x) = \frac{2}{2x+1} - \frac{2}{4x+3} - \left[\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma(x+\frac{1}{2})}\right],$$

utilizing (8), we obtain

$$f'(x) > \frac{2}{2x+1} - \frac{2}{4x+3} - \frac{4x+4}{(2x+1)(4x+3)} = 0.$$

Therefore, $f(x)$ is strictly increasing in (0,+∞), and

$$f(x) > f(0) = \frac{1}{2}\ln\frac{\pi}{3} > 0,$$

which leads to the inequality (9). □

Corollary 1. *For all natural number n, we have*

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} < \frac{2n+1}{\sqrt{4n+3}}. \tag{10}$$

Corollary 2. *The sequence*

$$\{Q_n\}_{n=1}^{\infty} \triangleq \left\{\left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n\right\}_{n=1}^{\infty} \tag{11}$$

is strictly decreasing.

Proof. By standard argument, we can rewrite $Q_{n+1} < Q_n$ for any natural number n as inequality (10). Therefore, the monotonicity follows. □

3. Main results

Now we give the main results of this paper.

Theorem 1. *For all natural number n, we have*

$$\frac{1}{\sqrt{\pi\left(n+\frac{4}{\pi}-1\right)}} \le \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{12}$$

The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are the best possible.

Proof. Since

$$\Gamma(n+1) = n!, \quad \Gamma\Big(n+\frac{1}{2}\Big) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}, \quad 2^n n! = (2n)!!,$$

the double inequality (12) is equivalent to

$$\frac{1}{4} < Q_n = \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^2 - n \le \frac{4}{\pi} - 1. \tag{13}$$

From the monotonicity of the sequence Q_n provided in Corollary 2, it follows that

$$\lim_{n\to\infty} Q_n < Q_n \le Q_1 = \frac{4}{\pi} - 1.$$

Using the asymptotic formula (7), we conclude from

$$Q_n = n\left[n^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} - 1\right]\left[n^{-\frac{1}{2}}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} + 1\right]$$

that

$$\lim_{n\to\infty} Q_n = \frac{1}{4}.$$

Thus, inequality (13) follows. The proof is complete. □

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