RELATIONSHIPS BETWEEN GENERALIZED BERNOULLI NUMBERS AND POLYNOMIALS AND GENERALIZED EULER NUMBERS AND POLYNOMIALS

QIU-MING LUO AND FENG QI

Abstract. In this paper, concepts of the generalized Bernoulli and Euler numbers and polynomials are introduced, and some relationships between them are established.

1. Introduction

It is well-known that the Bernoulli and Euler polynomials are two classes of special functions. They play important roles and have made many unexpected appearances in Numbers Theory, Theory of Functions, Theorical Physics, and the like. There has been much literature about Bernoulli and Euler polynomials, for some examples, please refer to references in this paper.

The Bernoulli numbers and polynomials and Euler numbers and polynomials are generalized to the generalized Bernoulli numbers and polynomials and to the generalized Euler numbers and polynomials in [2, 3, 4] in recent years.

In this article, we first restate the definitions of the generalized Bernoulli and Euler numbers and polynomials, and then discuss the relationships between the generalized Bernoulli and Euler numbers and polynomials. These results generalize, reinforce, and deepen those in [1, 5, 8, 10, 11].

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2. Definitions of Bernoulli and Euler numbers and polynomials

In this section, we will restate definitions of (generalized) Bernoulli numbers, (generalized) Bernoulli polynomials, (generalized) Euler numbers, and (generalized) Euler polynomials as follows. For more details, please see [1, 2, 3, 4, 10].

Definition 2.1 ([1, 10]). The Bernoulli numbers B_k and Euler numbers E_k are defined respectively by

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}B_k, \quad |t|<2\pi; \tag{2.1}$$

$$\frac{2e^t}{e^{2t}+1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k, \quad |t|\le\pi. \tag{2.2}$$

Definition 2.2. [1, 10] The Bernoulli polynomials $B_k(x)$ and Euler polynomials $E_k(x)$ are defined respectively by

$$\frac{te^{xt}}{e^t-1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}B_k(x), \quad |t|<2\pi, \quad x\in\mathbb{R}; \tag{2.3}$$

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x), \quad |t|\le\pi, \quad x\in\mathbb{R}. \tag{2.4}$$

Note that $B_k=B_k(0)$, $E_k=2^kE_k(\frac{1}{2})$.

Definition 2.3 ([2, 4]). Let a,b,c be positive numbers, the generalized Bernoulli numbers $B_k(a,b)$ and the generalized Euler numbers $E_k(a,b,c)$ are defined by

$$\frac{t}{b^t-a^t}=\sum_{k=0}^{\infty}\frac{t^k}{k!}B_k(a,b), \quad |t|<\frac{2\pi}{\ln b-\ln a}; \tag{2.5}$$

$$\frac{2c^t}{b^{2t}+a^{2t}}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(a,b,c). \tag{2.6}$$

It is easy to see that $B_k=B_k(1,e)$, $E_k=E_k(1,e,e)$.

Definition 2.4 ([2, 4]). The generalized Bernoulli polynomials $B_k(x;a,b,c)$ and the generalized Euler polynomials $E_k(x;a,b,c)$ are defined by

$$\frac{tc^{xt}}{b^t-a^t}=\sum_{k=0}^{\infty}\frac{t^k}{k!}B_k(x;a,b,c), \quad |t|<\frac{2\pi}{|\ln b-\ln a|}, \quad x\in\mathbb{R}; \tag{2.7}$$

$$\frac{2c^{xt}}{b^t+a^t}=\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x;a,b,c), \quad x\in\mathbb{R}. \tag{2.8}$$

It is not difficult to see that $B_k(x)=B_k(x;1,e,e)$, $E_k(x)=E_k(x;1,e,e)$, $B_k(a,b)=B_k(0;a,b,c)$, and $E_k(a,b,c)=2^kE_k(\frac{1}{2};a,b,c)$, where $x\in\mathbb{R}$.

3. Relationships between generalized Bernoulli and Euler numbers

In this section, we will discuss some relationships between the generalized Bernoulli numbers and the generalized Euler numbers.

Theorem 3.1. *Let $k \in \mathbb{N}$ and $r \in \mathbb{R}$, then we have*

$$k\sum_{j=0}^{k-1}\binom{k-1}{j}(r-1)^{k-j-1}(\ln c)^{k-j-1}E_j(a,b,c) = \sum_{j=0}^{k}\binom{k}{j}2^{2j-1}\left[(2\ln b + r\ln c)^{k-j} - (2\ln a + r\ln c)^{k-j}\right]B_j(a,b), \quad (3.1)$$

where a, b, c are positive numbers.

Proof. From (2.5), Cauchy multiplication, and the power series identity theorem, we have

$$\begin{aligned}\frac{2tc^{rt}}{b^{2t}+a^{2t}} &= \frac{2t}{b^{4t}-a^{4t}}\left[(b^2c^r)^t - (a^2c^r)^t\right]\\ &= \left[\sum_{k=0}^{\infty}2^{2k-1}B_k(a,b)\frac{t^k}{k!}\right]\left[\sum_{k=0}^{\infty}\left[(\ln(b^2c^r))^k - (\ln(a^2c^r))^k\right]\frac{t^k}{k!}\right] \quad (3.2)\\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\binom{k}{j}2^{2j-1}\left[(2\ln b + r\ln c)^{k-j} - (2\ln a + r\ln c)^{k-j}\right]B_j(a,b)\right]\frac{t^k}{k!}.\end{aligned}$$

From (2.6), Cauchy multiplication, and the power series identity theorem, we have

$$\begin{aligned}\frac{2tc^{rt}}{b^{2t}+a^{2t}} &= \frac{2tc^t}{b^{2t}+a^{2t}}\cdot c^{(r-1)t}\\ &= \left[\sum_{k=0}^{\infty}E_k(a,b,c)\frac{t^{k+1}}{k!}\right]\left[\sum_{k=0}^{\infty}\left[(r-1)^k(\ln c)^k\right]\frac{t^k}{k!}\right] \quad (3.3)\\ &= \sum_{k=0}^{\infty}\left[k\sum_{j=0}^{k}\binom{k-1}{j}(r-1)^{k-j-1}(\ln c)^{k-j-1}E_j(a,b,c)\right]\frac{t^k}{k!}.\end{aligned}$$

Equating coefficients of the terms $\frac{t^k}{k!}$ in (3.2) and (3.3) leads to (3.1). □

Taking $r = 1$ in (3.1) and defining $0^0 = 1$, we have

Corollary 3.1.1. *For nonnegative integer k and positive numbers a, b, c, we have*

$$kE_{k-1}(a,b,c) = \sum_{j=0}^{k}\binom{k}{j}2^{2j-1}\left[(2\ln b + \ln c)^{k-j} - (2\ln a + \ln c)^{k-j}\right]B_j(a,b). \quad (3.4)$$

Furthermore, Setting $a = 1$ and $b = c = e$ in (3.4), we hve

Corollary 3.1.2. *For nonnegative integer k, we have*

$$kE_{k-1} = \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1}(3^{k-j} - 1)B_j. \tag{3.5}$$

Remark 3.1. In [7, p. 943] and [9], the following formulae were given respectively:

$$(1 + 2^{2k})(1 - 2^{2k-1})B_{2k} = \sum_{j=0}^{k} \binom{2k}{2j} B_{2j}E_{2k-2j}, \tag{3.6}$$

$$(2 - 2^{2k})B_{2k} = \sum_{j=0}^{k} \binom{2k}{2j} 2^{2j} B_{2j}E_{2k-2j}. \tag{3.7}$$

Replacing k by $2k$ in (3.5), we have

Corollary 3.1.3. *For nonnegative integer k, we have*

$$\sum_{j=0}^{2k} \binom{2k}{j} 2^{2j-1}(3^{2k-j} - 1)B_j = 0. \tag{3.8}$$

Taking $r = 2$ in (3.1) leads to the following

Corollary 3.1.4. *For positive integer k and positive numbers a, b, c, we have*

$$\begin{aligned} &k\sum_{j=0}^{k-1} \binom{k-1}{j} (\ln c)^{k-j-1} E_j(a, b, c) \\ &= \sum_{j=0}^{k} \binom{k}{j} 2^{k+j-1} \left[(\ln b + \ln c)^{k-j} - (\ln a + \ln c)^{k-j}\right] B_j(a, b). \end{aligned} \tag{3.9}$$

Taking $a = 1$ and $b = c = e$ in (3.9) yields

Corollary 3.1.5. *For positive integer k, we have*

$$k\sum_{j=0}^{k-1} \binom{k-1}{j} E_j = 2^{k-1} \sum_{j=0}^{k} \binom{k}{j} (2^k - 2^j)B_j. \tag{3.10}$$

Remark 3.2. The result in (3.10) is equivalent to Lemma 2 in [5, p. 6].

Setting $a = 1$ and $b = c = e$ in (3.1) gives us

Corollary 3.1.6. *Let $r \in \mathbb{R}$, then we have*

$$k\sum_{j=0}^{k-1} \binom{k-1}{j} (r-1)^{k-j-1}E_j = \sum_{j=0}^{k} \binom{k}{j} 2^{2j-1} \left[(2 + r)^{k-j} - r^{k-j}\right] B_j. \tag{3.11}$$

Theorem 3.2. *For positive numbers a, b, c and nonnegative integer k, we have*

$$\sum_{j=0}^{k}\binom{k}{j}B_j(a,b)E_{k-j}(a,b,c) = \sum_{j=0}^{k}\binom{k}{j}2^{2j-1}\left[(\ln b+\ln c)^{k-j}+(\ln a+\ln c)^{k-j}\right]B_j(a,b). \tag{3.12}$$

Proof. By (2.5), Cauchy multiplication, and the power series identity theorem, we have

$$\begin{aligned}\frac{2tc^t}{(b^t-a^t)(b^{2t}+a^{2t})} &= \frac{2t}{b^{4t}-a^{4t}}\left[(bc)^t+(ac)^t\right]\\ &= \left[\sum_{k=0}^{\infty}2^{2k-1}B_k(a,b)\frac{t^k}{k!}\right]\left[\sum_{k=0}^{\infty}\left[(\ln(bc))^k+(\ln(ac))^k\right]\frac{t^k}{k!}\right]\\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\binom{k}{j}2^{2j-1}\left[(\ln b+\ln c)^{k-j}-(\ln a+\ln c)^{k-j}\right]B_j(a,b)\right]\frac{t^k}{k!}.\end{aligned} \tag{3.13}$$

By (2.6), Cauchy multiplication, and the power series identity theorem, we have

$$\begin{aligned}&\frac{2tc^t}{(b^t-a^t)(b^{2t}+a^{2t})}\\ &= \left[\sum_{k=0}^{\infty}B_k(a,b)\frac{t^{k+1}}{k!}\right]\left[\sum_{k=0}^{\infty}E_k(a,b,c)\frac{t^k}{k!}\right]\\ &= \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\binom{k}{j}B_j(a,b)E_{k-j}(a,b,c)\right]\frac{t^k}{k!}.\end{aligned} \tag{3.14}$$

Equating coefficients of $\frac{t^k}{k!}$ in (3.13) and (3.14) leads to (3.12). □

Taking $a = 1$, $b = c = e$ in (3.12), we have

Corollary 3.2.1. *For nonnegative integer k, we have*

$$\sum_{j=0}^{k}\binom{k}{j}B_jE_{k-j} = \sum_{j=0}^{k}\binom{k}{j}(2^{k+j-1}+2^{2j-1})B_j. \tag{3.15}$$

4. Relations between generalized Bernoulli and Euler polynomials

In this section, we will discuss some relationships between the generalized Bernoulli polynomials and the generalized Euler polynomials.

Theorem 4.1. *For positive numbers a,b,c, nonnegative integer k, and $x\in\mathbb{R}$, we have*

$$kE_{k-1}(x;a,b,c)=\sum_{j=0}^{k}\binom{k}{j}2^j\Big[(\ln b)^{k-j}-(\ln a)^{k-j}\Big]B_j\left(\frac{x}{2};a,b,c\right). \tag{4.1}$$

Proof. By (2.7), Cauchy multiplication, and the power series identity theorem, we have

$$\begin{aligned}\frac{2tc^{xt}}{b^t+a^t}&=\frac{2tc^{xt}(b^t-a^t)}{b^{2t}-a^{2t}}\\ &=\left[\sum_{k=0}^{\infty}\Big[2^kB_k\left(\frac{x}{2};a,b,c\right)\Big]\frac{t^k}{k!}\right]\left[\sum_{k=0}^{\infty}\Big[(\ln b)^k-(\ln a)^k\Big]\frac{t^k}{k!}\right]\\ &=\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}\binom{k}{j}2^j\Big[(\ln b)^{k-j}-(\ln a)^{k-j}\Big]B_j\left(\frac{x}{2};a,b,c\right)\right]\frac{t^k}{k!}.\end{aligned} \tag{4.2}$$

By (2.8), Cauchy multiplication, and the power series identity theorem, we have

$$\frac{2tc^{xt}}{b^t+a^t}=t\sum_{k=0}^{\infty}\frac{t^k}{k!}E_k(x;a,b,c)=\sum_{k=0}^{\infty}\Big[kE_{k-1}(x;a,b,c)\Big]\frac{t^k}{k!}. \tag{4.3}$$

Equating coefficients of $\frac{t^k}{k!}$ in (4.2) and (4.3) leads to (4.1). □

Letting $a=1$ and $b=c=e$ in (4.1) and defining $0^0=1$, we have

Corollary 4.1.1. *For $k\in\mathbb{N}$ and $x\in\mathbb{R}$, we have*

$$kE_{k-1}(x)=\sum_{j=0}^{k-1}\binom{k}{j}2^jB_j\left(\frac{x}{2}\right). \tag{4.4}$$

From (2.3), Cauchy multiplication, and the power series identity theorem, it follows that

Corollary 4.1.2. *For $k\in\mathbb{N}$ and $x\in\mathbb{R}$, we have*

$$2^k\left[B_k\left(\frac{x+1}{2}\right)-B_k\left(\frac{x}{2}\right)\right]=\sum_{j=0}^{k-1}\binom{k}{j}2^jB_j\left(\frac{x}{2}\right). \tag{4.5}$$

Combining (4.4) with (4.5), we have

Corollary 4.1.3. *For $k\in\mathbb{N}$ and $x\in\mathbb{R}$, we have*

$$kE_{k-1}(x)=2^k\left[B_k\left(\frac{x+1}{2}\right)-B_k\left(\frac{x}{2}\right)\right]. \tag{4.6}$$

Remark 4.1. The formula (4.6) is the same as Lemma 3 in [5, p. 6].

Using (2.7), (2.8), Cauchy multiplication, and the power series identity theorem, we obtain

Theorem 4.2. *For positive numbers a, b, c, nonnegative integer k, and $x \in \mathbb{R}$,*

$$2^k B_k(x; a, b, c) = \sum_{j=0}^{k} \binom{k}{j} B_j(x; a, b, c) E_{k-j}(x; a, b, c). \tag{4.7}$$

Taking $a = 1$ and $b = c = e$ in (4.7), we have

Corollary 4.2.1. *Let $x \in \mathbb{R}$ and k be nonnegative integer, then*

$$2^k B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j(x) E_{k-j}(x). \tag{4.8}$$

Theorem 4.3. *For positive numbers a, b, c, nonnegative integer k, and $x \in \mathbb{R}$, we have*

$$kE_{k-1}(x; a, b, c) = 2B_k(x; a, b, c) - 2\sum_{j=0}^{k} \binom{k}{j} 2^j (\ln a)^{k-j} B_j\left(\frac{x}{2}; a, b, c\right). \tag{4.9}$$

Proof. By (2.7), Cauchy multiplication, and the power series identity theorem, we obtain

$$\begin{aligned} \frac{2tc^{xt}}{b^t + a^t} &= \frac{2tc^{xt}}{b^t - a^t} - \frac{4tc^{xt}a^t}{b^{2t} - a^{2t}} \\ &= 2\sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x; a, b, c) - 2\left\{\sum_{k=0}^{\infty} \left[2^k B_k\left(\frac{x}{2}; a, b, c\right)\right] \frac{t^k}{k!}\right\} \left\{\sum_{k=0}^{\infty} \frac{t^k}{k!} (\ln a)^k\right\} \\ &= \sum_{k=0}^{\infty} \left[2B_k(x; a, b, c) - 2\sum_{j=0}^{k} \binom{k}{j} 2^j (\ln a)^{k-j} B_j\left(\frac{x}{2}; a, b, c\right)\right] \frac{t^k}{k!} \end{aligned} \tag{4.10}$$

By (2.8), Cauchy multiplication, and the power series identity theorem, we have

$$\frac{2tc^{xt}}{b^t + a^t} = t\sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x; a, b, c) = \sum_{k=0}^{\infty} \left[kE_{k-1}(x; a, b, c)\right] \frac{t^k}{k!} \tag{4.11}$$

Equating coefficients of $\frac{t^k}{k!}$ in (4.10) and (4.11) leads to (4.9). □

If having $a = 1$ and $b = c = e$ in (4.9), then

Corollary 4.3.1. *For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have*

$$kE_{k-1} = 2\left[B_k(x) - 2^k B_k\left(\frac{x}{2}\right)\right]. \tag{4.12}$$

Remark 4.2. The formula (4.12) is the same as that in [10, p. 48].

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(Luo) Department of Broadcast-Television-Teaching, Jiaozuo University, Jiaozuo City, Henan 454002, China

(Qi) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, China

E-mail address: qifeng@jzit.edu.cn or qifeng618@hotmail.com

URL: http://rgmia.vu.edu.au/qi.html