GENERALIZATIONS OF THE HERMITE-HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE s-CONVEX

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Abstract. Some new results related to the right-hand side of the Hermite-Hadamard type inequality for the class of functions whose derivatives at certain powers are s-convex functions in the second sense are obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and a, $b \in I$, with $a < b$. The following double inequality is well known in the literature as the *Hermite-Hadamard inequality* [10]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

For recent results, refinements, counterparts, generalizations of the Hermite-Hadamard inequality see [7] – [12] and [14] – [19].

Dragomir and Agarwal [8] established the following result connected with the right-hand side of (1.1).

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{8}\left[|f'(a)| + |f'(b)|\right]. \tag{1.2}$$

In [13], Hudzik and Maligranda considered among others the class of functions which are s-convex in the second sense. This class is defined in the following way: a function $f : \mathbb{R}^+ \to \mathbb{R}$, where $\mathbb{R}^+ = [0, \infty)$, is said to be *$s$-convex in the second sense* if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of s-convex functions in the second sense is usually denoted by K_s^2.

It can be easily seen that for $s = 1$, s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

For recent results and generalizations concerning s-convex functions see [1] – [6] and [14].

In [5], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s–convex functions in the second sense.

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Theorem 2. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s–convex function in the second sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^1[0,1]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.3). The above inequalities are sharp.

New inequalities of Hermite-Hadamard type for differentiable functions based on concavity and s-convexity established by U.S. Kirmaci et al. [14], are presented below:

Theorem 3. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q\geq1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\qquad\leq\frac{b-a}{2}\left(\frac{1}{2}\right)^{\frac{q-1}{q}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}}\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{1.4}$$

Theorem 4. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq\frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{\frac{q-1}{q}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\ &\quad\times\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}\right] \\ &\leq\frac{b-a}{2}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(a)|^q\right)^{\frac{1}{q}}\right].\end{aligned} \tag{1.5}$$

Theorem 5. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\leq\frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{\frac{q-1}{q}}2^{\frac{s-1}{q}}\left(\left|f'\left(\frac{a+3b}{2}\right)\right|+\left|f'\left(\frac{3a+b}{2}\right)\right|\right) \\ &\leq\frac{b-a}{2}\left(\left|f'\left(\frac{a+3b}{2}\right)\right|+\left|f'\left(\frac{3a+b}{2}\right)\right|\right).\end{aligned} \tag{1.6}$$

The main aim of this paper is to establish new inequalities of Hermite-Hadamard type for the class of functions whose derivatives at certain powers are s-convex functions in the second sense.

2. Inequalities for Functions whose Derivatives are s-convex

In order to prove our main results we consider the following lemma:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° where $a, b \in I$ with $a < b$. Then the following equality holds:*

$$\frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{r+1}\int_0^1 [(r+1)t - 1]\, f'(tb + (1-t)a)\,dt \tag{2.1}$$

for some fixed $r \in [0, 1]$.

Proof. We note that

$$\begin{aligned} I &= \int_0^1 [(r+1)t - 1]\, f'(tb + (1-t)a)\,dt \\ &= [(r+1)t - 1]\frac{f(tb + (1-t)a)}{b-a}\Big|_0^1 - \frac{r+1}{b-a}\int_0^1 f(tb + (1-t)a)\,dt \\ &= \frac{rf(b) + f(a)}{b-a} - \frac{r+1}{b-a}\int_0^1 f(tb + (1-t)a)\,dt. \end{aligned}$$

Setting $x = tb + (1-t)a$, and $dx = (b-a)dt$ gives

$$I = \frac{f(a) + rf(b)}{b-a} - \frac{r+1}{(b-a)^2}\int_a^b f(x)\,dx.$$

Therefore,

$$\left(\frac{b-a}{r+1}\right) I = \frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)\,dx$$

which gives the desired representation (2.1). ∎

The next theorem gives a new refinement of the upper Hermite-Hadamard inequality for s-convex functions.

Theorem 6. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a) + rf(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\le \frac{(b-a)}{(r+1)(s+1)(s+2)}\left[\left(r(s+1) + 2\left(\frac{1}{r+1}\right)^{s+1} - 1\right)|f'(b)| \right. \\ &\qquad \left. + \left(s - r + 2(r+1)\left(\frac{r}{r+1}\right)^{s+2} + 1\right)|f'(a)|\right], \end{aligned} \tag{2.2}$$

for some fixed $r \in [0, 1]$.

Proof. From Lemma 1, we have

$$\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{b-a}{r+1}\int_0^1 |(r+1)t-1|\,|f'(tb+(1-t)a)|\,dt \\
&= \frac{b-a}{r+1}\int_0^{\frac{1}{r+1}} (1-(r+1)t)\,|f'(tb+(1-t)a)|\,dt \\
&\qquad + \frac{b-a}{r+1}\int_{\frac{1}{r+1}}^1 ((r+1)t-1)\,|f'(tb+(1-t)a)|\,dt \\
&\leq \frac{b-a}{r+1}\int_0^{\frac{1}{r+1}} (1-(r+1)t)\left[t^s|f'(b)|+(1-t)^s|f'(a)|\right]dt \\
&\qquad + \frac{b-a}{r+1}\int_{\frac{1}{r+1}}^1 ((r+1)t-1)\left[t^s|f'(b)|+(1-t)^s|f'(a)|\right]dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{b-a}{r+1}\left[\frac{\left(\frac{1}{r+1}\right)^{s+1}}{(s+1)(s+2)}|f'(b)|+\frac{s+2+(r+1)\left[\left(\frac{r}{r+1}\right)^{s+2}-1\right]}{(s+1)(s+2)}|f'(a)|\right] \\
&\qquad + \frac{b-a}{r+1}\left[\frac{r(s+1)+\left(\frac{1}{r+1}\right)^{s+1}-1}{(s+1)(s+2)}|f'(b)|+\frac{(r+1)\left(\frac{r}{r+1}\right)^{s+2}}{(s+1)(s+2)}|f'(a)|\right] \\
&= \frac{(b-a)}{(r+1)(s+1)(s+2)}\left[\left(r(s+1)+2\left(\frac{1}{r+1}\right)^{s+1}-1\right)|f'(b)|\right. \\
&\qquad \left. + \left(s-r+2(r+1)\left(\frac{r}{r+1}\right)^{s+2}+1\right)|f'(a)|\right],
\end{aligned}$$

which completes the proof. ∎

Therefore, we can deduce the following results.

Corollary 1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on I° and $a,b \in I$ with $a<b$. Assume that $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$.*

(1) *If $r=1$ in (2.2), then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(s+2^{-s})(b-a)}{2(s+1)(s+2)}\left[|f'(b)|+|f'(a)|\right]. \tag{2.3}$$

(2) *If $r=0$ in (2.2), then the following inequality holds:*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{(s+1)(s+2)}\left[|f'(b)|+(s+1)|f'(a)|\right]. \tag{2.4}$$

Proof. It is obvious from Theorem 6. ∎

Remark 1. *We note that inequality (2.3) with $s=1$ gives an improvement for the inequality (1.2).*

A similar result is embodied in the following theorem:

Theorem 7. *Let $f: I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[(r+1)(1+p)\right]^{\frac{1}{p}}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+r^{(p+1)/p}\left(|f'(b)|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}
\tag{2.5}
$$

for some fixed $r \in [0,1]$, where $q = p/(p-1)$.

Proof. Suppose that $p > 1$. From Lemma 1 and using the Hölder inequality, we have

$$
\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{b-a}{r+1}\int_0^{\frac{1}{r+1}}(1-(r+1)t)\left|f'(tb+(1-t)a)\right|dt \\
&\qquad+\frac{b-a}{r+1}\int_{\frac{1}{r+1}}^1((r+1)t-1)\left|f'(tb+(1-t)a)\right|dt \\
&\leq \frac{b-a}{r+1}\left(\int_0^{\frac{1}{r+1}}(1-(r+1)t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{r+1}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad+\frac{b-a}{r+1}\left(\int_{\frac{1}{r+1}}^1((r+1)t-1)^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{r+1}}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $|f'|^{\,q}$ is convex, we have

$$
\int_0^{\frac{1}{r+1}}\left|f'(tb+(1-t)a)\right|^q dt \leq \frac{|f'(a)|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q}{s+1}
$$

and

$$
\int_{\frac{1}{r+1}}^1\left|f'(tb+(1-t)a)\right|^q dt \leq \frac{|f'(b)|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q}{s+1}.
$$

Therefore, we have

$$\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[(r+1)(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+r^{(p+1)/p}\left(\left|f'(b)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$, which is required. ∎

Corollary 2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. Assume that $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$.*

(1) *If $r = 1$ in (2.5), then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.6}$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

(2) *If $r = 0$ in (2.5), then the following inequality holds:*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}(1+p)^{\frac{1}{p}}}\left(\left|f'(a)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}, \tag{2.7}$$

where $q = \frac{p}{p-1}$.

Proof. It follows directly from Theorem 7. ∎

Remark 2. *We observe that the inequality (2.6) is better than the inequality (1.5).*

Our next result gives a new refinement for the upper Hermite-Hadamard inequality:

Theorem 8. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on I° and $a, b \in I$ with $a < b$. If $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{2.8}$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}(r+1)^{\frac{1}{p}+\frac{s}{q}}(p+1)^{1+p}}\left[\left(\left[(r+1)^s+1\right]\left|f'(a)\right|^q+r^s\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(r^s\left|f'(a)\right|^q+\left[(r+1)^s+1\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],$$

for some fixed $r \in [0,1]$, where $q = \frac{p}{p-1}$.

Proof. We consider the inequality (2.5), that is

$$\begin{aligned}
&\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[(r+1)(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\qquad\left.+r^{(p+1)/p}\left(\left|f'(b)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, then we have

$$\left|f'\left(\frac{a+rb}{r+1}\right)\right|^q \leq \left(\frac{1}{r+1}\right)^s \left|f'(a)\right|^q+\left(\frac{r}{r+1}\right)^s\left|f'(b)\right|^q,$$

which gives

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[(r+1)(1+p)\right]^{\frac{1}{p}}}\left[\left(\left|f'(a)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+r^{(p+1)/p}\left(\left|f'(b)\right|^q+\left|f'\left(\frac{b+ra}{r+1}\right)\right|^q\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}(r+1)^{\frac{1}{p}+\frac{s}{q}}(p+1)^{1+p}}\left[\left(\left[(r+1)^s+1\right]\left|f'(a)\right|^q+r^s\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(r^s\left|f'(a)\right|^q+\left[(r+1)^s+1\right]\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

which completes the proof. ∎

Corollary 3. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be an absolutely continuous function on I° and $a,b\in I$ with $a<b$. If $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\quad\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(\left(1+2^{-s}\right)\left|f'(a)\right|^q+2^{-s}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\qquad\left.+\left(2^{-s}\left|f'(a)\right|^q+\left(1+2^{-s}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{2.9}$$

where $q=\frac{p}{p-1}$.

Proof. We consider the inequality (2.6), that is

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Since $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, then $\left|f'\left(\frac{a+b}{2}\right)\right|^q \leq \frac{|f'(a)|^q+|f'(b)|^q}{2^s}$, which gives

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(\left(1+2^{-s}\right)|f'(a)|^q+2^{-s}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(2^{-s}|f'(a)|^q+\left(1+2^{-s}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right],$$

where $q=\frac{p}{p-1}$, which completes the proof. ∎

Corollary 4. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be an absolutely continuous function on I° and $a,b\in I$ with $a<b$. If $|f'|^{p/(p-1)}$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{\left(1+2^{1-s}\right)^{\frac{1}{q}}(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[|f'(a)|+|f'(b)|\right], \tag{2.10}$$

where $q=\frac{p}{p-1}$.

Proof. We consider the inequality (2.9), i.e.,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(\left(1+2^{-s}\right)|f'(a)|^q+2^{-s}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(2^{-s}|f'(a)|^q+\left(1+2^{-s}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Now, let $a_1=\left(1+2^{-s}\right)|f'(a)|^q$, $b_1=2^{-s}|f'(b)|^q$, $a_2=2^{-s}|f'(a)|^q$ and $b_2=\left(1+2^{-s}\right)|f'(b)|^q$.

Here, $0<\frac{1}{q}<1$, for $q\geq 1$. Using the fact that $\sum\limits_{i=1}^{n}(a_i+b_i)^k\leq\sum\limits_{i=1}^{n}a_i^k+\sum\limits_{i=1}^{n}b_i^k$, for $0<k<1$, $a_1,a_2,...,a_n\geq 0$ and $b_1,b_2,...,b_n\geq 0$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[\left(\left(1+2^{-s}\right)|f'(a)|^q+2^{-s}|f'(b)|^q\right)^{\frac{1}{q}}\right. \\
&\qquad\left.+\left(2^{-s}|f'(a)|^q+\left(1+2^{-s}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\left(1+2^{1-s}\right)^{\frac{1}{q}}(b-a)}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{\frac{1}{p}}}\left[|f'(a)|+|f'(b)|\right],
\end{aligned}$$

where $q=\frac{p}{p-1}$, which is required. ∎

Remark 3. *1. Using the technique in Corollary 4, one can obtain in a similar manner another result by considering the inequality (2.8). However, the details are left to the interested reader.*

2. All of the above inequalities obviously hold for convex functions. Simply choose $s=1$ in each of those results to get the desired results.

3. Interchanging a and b in Lemma 1, we obtain the following equality

$$\frac{1}{b-a}\int\limits_a^b f(x)dx-\frac{rf(a)+f(b)}{r+1}=\frac{b-a}{r+1}\int\limits_0^1[(r+1)t-1]f'((1-t)b+ta).$$

For this reason, if we interchanging a and b in all above results, we can write new results using the above equality.

3. Applications to Special Means

We consider the means for arbitrary real numbers α,β $(\alpha\neq\beta)$ as follows:

(1) *Arithmetic mean*:

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in\mathbb{R}.$$

(2) *Generalized log-mean*:

$$L_s(\alpha,\beta)=\left[\frac{\beta^{s+1}-\alpha^{s+1}}{(s+1)(\beta-\alpha)}\right]^{\frac{1}{s}},\ s\in\mathbb{R}\backslash\{-1,0\},\ \alpha,\beta\in\mathbb{R},\ \alpha\neq\beta.$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

In [13], the following example is given:

Let $s\in(0,1)$ and $a,b,c\in\mathbb{R}$. We define a function $f:[0,\infty)\to\mathbb{R}$ as

$$f(t)=\begin{cases} a, & t=0 \\ bt^s+c, & t>0.\end{cases}$$

If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$. Hence, for $a = c = 0$, $b = 1$, we have $f : [0,1] \to [0,1]$, $f(t) = t^s$, $f \in K_s^2$.

Proposition 1. *Let $a, b \in I^\circ$, $a < b$ and $0 < s < 1$. Then, we have*

$$|L_s^s(a,b) - A(a^s, b^s)| \leq s(b-a)\frac{s+2^{-s}}{2(s+1)(s+2)}\left(|a|^{s-1} + |b|^{s-1}\right) \tag{3.1}$$

and

$$|L_s^s(a,b) - |a|^s| \leq \frac{s(b-a)}{(s+1)(s+2)}\left((s+1)|a|^{s-1} + |b|^{s-1}\right). \tag{3.2}$$

Proof. The assertion follows from Corollary 1 applied to the s-convex mapping $f : [0,1] \to [0,1]$, $f(x) = x^s$. ∎

Proposition 2. *Let $a, b \in I^\circ$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have*

$$\begin{aligned} &|L_s^s(a,b) - A(a^s, b^s)| \\ &\leq s\frac{b-a}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{1/p}}\left[\left(|a|^{q(s-1)} + \left|\frac{a+b}{2}\right|^{q(s-1)}\right)^{1/q}\right. \\ &\qquad \left. + \left(|b|^{(s-1)q} + \left|\frac{a+b}{2}\right|^{q(s-1)}\right)^{1/q}\right] \end{aligned} \tag{3.3}$$

and

$$|L_s^s(a,b) - |a|^s| \leq \frac{s(b-a)}{(s+1)^{1+\frac{1}{q}}(1+p)^{1/p}}\left(|a|^{(s-1)q} + |b|^{(s-1)q}\right)^{1/q}. \tag{3.4}$$

Proof. The assertion follows from Corollary 2 applied to the s-convex mapping $f : [0,1] \to [0,1]$, $f(x) = x^s$. ∎

Proposition 3. *Let $a, b \in I^\circ$, $a < b$ and $0 < s < 1$. Then, for all $q > 1$, we have*

$$|L_s^s(a,b) - A(a^s, b^s)| \leq s(b-a)\frac{(1+2^{1-s})^{1/q}}{(s+1)^{1+\frac{1}{q}}\left[2(1+p)\right]^{1/p}}\left(|a|^{s-1} + |b|^{s-1}\right). \tag{3.5}$$

Proof. The assertion follows from Corollary 4 applied to the s-convex mapping $f : [0,1] \to [0,1]$, $f(x) = x^s$. ∎

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