GENERALIZATIONS OF BERNOULLI'S NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, the concepts of Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$, the generalized Bernoulli numbers $B_n(a,b)$ are further generalized to one which is called as the generalized Bernoulli polynomials, and many basic properties and some relationships between B_n, $B_n(x)$ and $B_n(a,b)$ are established.

1. Introduction

It is well known that Bernoulli's numbers and polynomials play important roles in mathematics, they are main objects in the Theory of Special Functions [4]. Their definitions can be given as follows.

Definition 1. The numbers B_n, $0 \le n \le \infty$, are called Bernoulli numbers if

$$\phi(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n, \quad |t| < 2\pi. \tag{1}$$

Definition 2. The functions $B_n(x)$, $0 \le n \le \infty$, are called Bernoulli polynomials if they satisfy

$$\phi(x;t) = \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n, \quad |t| < 2\pi, \quad x \in \mathbb{R}. \tag{2}$$

The usual definition of Bernoulli polynomials is

$$\frac{t^\sigma e^{ut}}{(e^t - 1)^\sigma} = \sum_{n=0}^{\infty} \frac{B_n^\sigma(u)}{n!} t^n, \quad |t| < 2\pi. \tag{3}$$

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In [2], the last two authors generalized the concept of Bernoulli numbers as follows.

Definition 3. Let $a, b > 0$ and $a \neq b$. The generalized Bernoulli numbers $B_n(a,b)$ are defined by

$$\phi(t; a, b) = \frac{t}{b^t - a^t} = \sum_{n=0}^{\infty} \frac{B_n(a,b)}{n!} t^n, \quad |t| < \frac{2\pi}{|\ln b - \ln a|}. \tag{4}$$

Among other things, some basic properties and relationships between B_n, $B_n(x)$ and $B_n(a,b)$ were also studied in [2] initially and originally.

In this article, we first give the following definition of the generalized Bernoulli polynomials, which generalizes the concepts stated above, and then research their basic properties and relationships to Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$ and the generalized Bernoulli numbers $B_n(a,b)$.

2. Definitions and Properties of Generalized Bernoulli Polynomials

It is easy to see that the following definition is natural and essential generalizations of the concepts of Bernoulli numbers B_n, Bernoulli polynomials $B_n(x)$ and the generalized Bernoulli numbers $B_n(a,b)$.

Definition 4. Let $a, b, c > 0$ and $a \neq b$. The generalized Bernoulli polynomials $B_n(x; a, b, c)$ for nonnegative integer n are defined by

$$\phi(x; t; a, b, c) = \frac{tc^{xt}}{b^t - a^t} = \sum_{n=0}^{\infty} \frac{B_n(x; a, b, c)}{n!} t^n, \quad |t| < \frac{2\pi}{|\ln b - \ln a|}, \quad x \in \mathbb{R}. \tag{5}$$

The generalized Bernoulli polynomials $B_n(x; a, b, c)$ have the following properties, which are stated as theorems below.

Theorem 1. *Let $a, b, c > 0$ and $a \neq b$. For $x \in \mathbb{R}$ and $n \geq 0$, we have*

$$\begin{aligned} &B_n(x; 1, e, e) = B_n(x), \quad B_n(0; a, b, c) = B_n(a, b), \quad B_n(0; 1, e, e) = B_n, \\ &B_n(x; a, b, 1) = B_n(a, b), \quad B_n(x; 1, e, 1) = B_n, \end{aligned} \tag{6}$$

$$B_n(x; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} [\ln c]^{n-k} B_k(a, b) x^{x-k}, \tag{7}$$

$$B_n(x; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} [\ln c]^{n-k} [\ln b - \ln a]^{k-1} B_k \left(\frac{\ln a}{\ln a - \ln b} \right) x^{x-k}, \tag{8}$$

$$B_n(x; a, b, c) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{n}{k} \binom{k}{j} [\ln c]^{n-k} [\ln a]^{k-j} \left[\ln \frac{b}{a} \right]^{j-1} B_j x^{x-k}. \tag{9}$$

Proof. Applying Definition 3 to the term $\frac{t}{b^t-a^t}$ and expanding the exponential function c^{xt} at $t=0$ yields

$$\begin{aligned}\frac{tc^{xt}}{b^t-a^t}&=\left(\sum_{k=0}^{\infty}\frac{B_k(a,b)}{k!}t^k\right)\left(\sum_{i=0}^{\infty}\frac{x^i(\ln c)^i}{i!}t^i\right)\\&=\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(\ln c)^{k-i}}{i!(k-i)!}B_i(a,b)x^{k-i}t^k\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k(a,b)x^{n-k}\right)\frac{t^n}{n!}.\end{aligned}\tag{10}$$

Combining (10) and (5) and equating their coefficients of t^n produces formula (7).

The following two formulae were provided in [2]:

$$B_n(a,b)=(\ln b-\ln a)^{n-1}B_n\left(\frac{\ln a}{\ln a-\ln b}\right),\tag{11}$$

$$B_n(a,b)=\sum_{i=0}^{n}(-1)^{n-i}(\ln b-\ln a)^{i-1}(\ln a)^{n-i}\binom{n}{i}B_i.\tag{12}$$

Substituting (11) and (12) into (7) leads to (8) and (9).

Formulae in (6) are obvious. □

Now we give some results about derivatives and integrals of the generalized Bernoulli polynomials $B_n(x;a,b,c)$ as follows.

Theorem 2. *Let $a,b,c>0$, $a\ne b$, $n\ge 0$, and $x\in\mathbb{R}$. For any nonnegative integer ℓ and real numbers α and β, we have*

$$\frac{\partial^\ell B_n(x;a,b,c)}{\partial x^\ell}=\frac{n!}{(n-\ell)!}(\ln c)^\ell B_{n-\ell}(x;a,b,c),\tag{13}$$

$$\int_\alpha^\beta B_n(t;a,b,c)\,\mathrm{d}t=\frac{1}{(n+1)\ln c}[B_{n+1}(\beta;a,b,c)-B_{n+1}(\alpha;a,b,c)],\tag{14}$$

where $B_0(x;a,b,c)=\frac{1}{\ln b-\ln a}$.

Proof. Formula (13) follows from standard arguments and induction.

Integrating on both sides of (13) with respect to variable x for $\ell=1$ turns out formula (14). □

Theorem 3. *Let $a,b,c>0$, $a\ne b$, $n\ge 0$ and $x\in\mathbb{R}$. Then*

$$B_n(x+1;a,b,c)=\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k(x;a,b,c),\tag{15}$$

$$B_n(x+1;a,b,c)=B_n\left(x;\frac{a}{c},\frac{b}{c},c\right),\tag{16}$$

and, for $m \ge 2$,

$$B_m(x+1;a,b,c)=B_m(x;a,b,c)+m(\ln c)^{m-1}x^{m-1} \\ +\sum_{k=0}^{m-1}\binom{m}{k}[(\ln a)^{m-k}-(\ln b)^{m-k}+(\ln c)^{m-k}]B_k(x;a,b,c). \tag{17}$$

Proof. By definition of the generalized Bernoulli polynomials, we have

$$\frac{tc^{(x+1)t}}{b^t-a^t}=\sum_{n=0}^{\infty}\frac{B_n(x+1;a,b,c)}{n!}t^n \tag{18}$$

and

$$\begin{aligned}\frac{tc^{(x+1)t}}{b^t-a^t}&=\frac{tc^{xt}}{b^t-a^t}\cdot c^t\\ &=\left(\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n\right)\left(\sum_{k=0}^{\infty}\frac{(\ln c)^k}{k!}t^k\right)\\ &=\sum_{n=0}^{\infty}\frac{\sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k(x;a,b,c)}{n!}t^n.\end{aligned} \tag{19}$$

Combining (18) and (19) and equating their coefficients of t^n leads to formula (15).

Similarly, since

$$\frac{tc^{(x+1)t}}{b^t-a^t}=\frac{tc^{xt}}{\left(\frac{b}{c}\right)^t-\left(\frac{a}{c}\right)^t}=\sum_{n=0}^{\infty}\frac{B_n\left(x;\frac{a}{c},\frac{b}{c},c\right)}{n!}t^n, \tag{20}$$

equating the coefficients of t^n in (18) and (20) leads to formula (16).

Straightforward computation gives us that

$$\begin{aligned}\frac{tc^{(x+1)t}}{b^t-a^t}&=tc^{xt}+\frac{tc^{xt}(a^t-b^t+c^t)}{b^t-a^t}\\ &=\sum_{n=0}^{\infty}\frac{(\ln c)^n x^n}{n!}t^{n+1}\\ &\quad+\left(\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n\right)\left(\sum_{\ell=0}^{\infty}\frac{[(\ln a)^{\ell}-(\ln b)^{\ell}+(\ln c)^{\ell}]}{\ell!}t^{\ell}\right)\\ &=\sum_{n=0}^{\infty}\frac{(\ln c)^n x^n}{n!}t^{n+1}\\ &\quad+\sum_{n=0}^{\infty}\left[\sum_{\ell=0}^{n}\binom{n}{\ell}\left[(\ln a)^{n-\ell}-(\ln b)^{n-\ell}+(\ln c)^{n-\ell}\right]B_\ell(x;a,b,c)\right]\frac{t^n}{n!}\end{aligned} \tag{21}$$

$$
\begin{aligned}
&= B_0(x;a,b,c) + \big[1 + B_1(x;a,b,c) + B_0(x;a,b,c)(\ln a - \ln b + \ln c)\big]t \\
&\quad + \sum_{n=2}^{\infty}\big[n(\ln c)^{n-1}x^{n-1} + B_n(x;a,b,c)\big]\frac{t^n}{n!} \\
&\quad + \sum_{n=2}^{\infty}\left\{\sum_{\ell=0}^{n-1}\binom{n}{\ell}\big[(\ln a)^{n-\ell} - (\ln b)^{n-\ell} + (\ln c)^{n-\ell}\big]B_\ell(x;a,b,c)\right\}\frac{t^n}{n!}.
\end{aligned}
$$

Equating (5) and (21) yields (17). □

Corollary 1. *For $n \ge 1$, $b > 0$ and $x \in \mathbb{R}$, we have*

$$B_n(x+1;1,b,b) = B_n(x;1,b,b) + n(\ln b)^{n-1}x^{n-1}. \tag{22}$$

Remark 1. If taking $b = e$ in (22), the following well known result is deduced

$$B_n(x+1) = B_n(x) + nx^{n-1}, \quad n \ge 1. \tag{23}$$

Similarly, from (13) it follows that

$$B_i'(t) = iB_{i-1}(t), \quad B_0(t) = 1. \tag{24}$$

Actually, the Bernoulli polynomials $B_i(t)$, $i \in \mathbb{N}$, are uniquely determined by formulae (23) and (24), see [1, 23.1.5 and 23.1.6] or [4].

Theorem 4. *Let $a,b,c > 0$, $a \ne b$, $n \ge 0$, and $x \in \mathbb{R}$. Then*

$$B_n(1-x;a,b,c) = (-1)^n B_n\left(x;\frac{c}{b},\frac{c}{a},c\right) \tag{25}$$

$$= B_n\left(-x;\frac{a}{c},\frac{b}{c},\frac{1}{c}\right), \tag{26}$$

$$B_n(x+y;a,b,c) = \sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k(x;a,b,c)y^{n-k} \tag{27}$$

$$= \sum_{k=0}^{n}\binom{n}{k}(\ln c)^{n-k}B_k(y;a,b,c)x^{n-k}. \tag{28}$$

Proof. From Definition 5, it follows that

$$\frac{tc^{(1-x)t}}{b^t - a^t} = \sum_{n=0}^{\infty}\frac{B_n(1-x;a,b,c)}{n!}t^n. \tag{29}$$

Meanwhile, we have

$$\frac{tc^{(1-x)t}}{b^t - a^t} = \frac{tc^{-xt}}{\left(\frac{b}{c}\right)^t - \left(\frac{a}{c}\right)^t} = \sum_{n=0}^{\infty}\frac{B_n\left(-x;\frac{a}{c},\frac{b}{c},c\right)}{n!}t^n, \tag{30}$$

$$\frac{tc^{(1-x)t}}{b^t - a^t} = \frac{-tc^{x(-t)}}{\left(\frac{c}{a}\right)^{-t} - \left(\frac{c}{b}\right)^{-t}} = \sum_{n=0}^{\infty}(-1)^n\frac{B_n\left(x;\frac{c}{b},\frac{c}{a},c\right)}{n!}t^n. \tag{31}$$

Therefore, formulae (25) and (26) follow from equating series-expansions in (29), (30) and (31).

Similarly, we have

$$\frac{tc^{(x+y)t}}{b^t-a^t}=\sum_{n=0}^{\infty}\frac{B_n(x+y;a,b,c)}{n!}t^n, \tag{32}$$

$$\begin{aligned}\frac{tc^{(x+y)t}}{b^t-a^t}&=\frac{tc^{xt}}{b^t-a^t}\cdot c^{yt}\\&=\left(\sum_{n=0}^{\infty}\frac{B_n(x;a,b,c)}{n!}t^n\right)\left(\sum_{i=0}^{\infty}\frac{y^i(\ln c)^i}{i!}t^i\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}y^{n-k}(\ln c)^{n-k}B_k(x;a,b,c)\right)\frac{t^n}{n!},\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{tc^{(x+y)t}}{b^t-a^t}&=\frac{tc^{yt}}{b^t-a^t}\cdot c^{xt}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(\ln c)^{n-k}B_k(y;a,b,c)\right)\frac{t^n}{n!}.\end{aligned} \tag{34}$$

Hence, formulae (27) and (28) follow from equating series-expansions in (32), (33) and (34). The proof is complete. □

Theorem 5. *Let m,n be natural numbers. Then, for any positive number b, the following identity holds:*

$$\begin{aligned}\sum_{j=1}^{m}j^n&=\frac{1}{(n+1)(\ln b)^n}\big[B_{n+1}(m+1;1,b,b)-B_{n+1}(0;1,b,b)\big]\\&=\frac{1}{(n+1)(\ln b)^n}\big[B_{n+1}(m+1;1,b,b)-B_{n+1}(1;1,b,b)\big].\end{aligned} \tag{35}$$

Proof. Rewritting formula (22) yields

$$x^{n-1}=\frac{1}{n(\ln b)^{n-1}}[B_n(x+1;1,b,b)-B_n(x;1,b,b)], \tag{36}$$

which implies

$$j^n=\frac{1}{(n+1)(\ln b)^n}[B_{n+1}(j+1;1,b,b)-B_{n+1}(j;1,b,b)]. \tag{37}$$

Summing up on both sides of (37) from 0 to m or from 1 to m with respect to j leads to formula (35) easily. □

Remark 2. The calculation of values of $\sum_{j=1}^{m}j^n$ is an interesting problem, and there has been a rich literature, for example [3].

Remark 3. At last, it is pointed out that the Bernoulli's and Euler's numbers and the Bernoulli's and Euler's polynomoals can be further generalized to more general results in this manner. These conclusions will be published in some subsequent papers.

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