THE PERTURBED MEDIAN PRINCIPLE FOR INTEGRAL INEQUALITIES WITH APPLICATIONS

S.S. DRAGOMIR

Abstract. In this paper a perturbed version of the Median Principle introduced by the author in [1] is developed. Applications for various Riemann-Stieltjes integral and Lebesgue integral inequalities are also provided.

1. Introduction

In Analytic Inequalities Theory, there are many results involving the sup-norm of a function or of its derivative. To give only two examples, we recall the *Ostrowski inequality*:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\left\|f'\right\|_{[a,b],\infty} \tag{1.1}$$

for any $x\in[a,b]$, where f is absolutely continuous on $[a,b]$ and $f'\in L_\infty[a,b]$, i.e., $\left\|f'\right\|_{[a,b],\infty}:=ess\sup_{t\in[a,b]}\left|f'(t)\right|<\infty$, and the *Čebyšev inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right| \leq \frac{1}{12}(b-a)^2\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],\infty}, \tag{1.2}$$

provided that f and g are absolutely continuous with $f',g'\in L_\infty[a,b]$.

Since, in order to estimate $\left\|f^{(r)}\right\|_{[a,b],\infty}$, in practice it is usually necessary to find the quantities

$$M_r:=\sup_{t\in[a,b]}f^{(r)}(t)\quad\text{and}\quad m_r:=\inf_{t\in[a,b]}f^{(r)}(t)$$

(as, obviously, $\left\|f^{(r)}\right\|_{[a,b],\infty}=\max\{|M_r|,|m_r|\}$), the knowledge of $\left\|f^{(r)}\right\|_{[a,b],\infty}$ may be as difficult as the knowledge of M_r and m_r.

As pointed out in [1], it is natural, therefore, to try to establish inequalities where instead of $\left\|f^{(r)}\right\|_{[a,b],\infty}$ one would have the positive quantity M_r-m_r. This can be also useful since for functions whose derivatives $f^{(r)}$ have a "modest variation" the quantity M_r-m_r may be much smaller than $\left\|f^{(r)}\right\|_{[a,b],\infty}$.

In order to address this problem, the author has stated in [1] the "Median Principle" that can be formalised as follows:

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Theorem 1 (Median Principle). *Let $\mathcal{P}_n^\circ$ be the class of polynomials*

$$\left\{P_n | P_n(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,\ a_i \in \mathbb{R},\ i = \overline{1,n}\right\}$$

and $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a function so that $f^{(n-1)}$ is absolutely continuous and $f^{(n)} \in L_\infty [a,b]$. Assume that the following inequality holds:

$$L\left(f, f^{(1)}, \dots, f^{(n-1)}, f^{(n)}; a, b\right) \leq R\left(\left\|f^{(n)}\right\|_{[a,b],\infty}; a, b\right), \tag{1.3}$$

where $L(\cdot, \cdot, \dots, \cdot; a, b) : \mathbb{R}^{(n+1)} \to \mathbb{R}$ (the left hand side) is a general function while $R(\cdot; a, b) : [0, \infty) \to \mathbb{R}$ (the right hand side) is monotonic nondecreasing on $[0, \infty)$.

If $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous and

$$-\infty < \gamma \leq g^{(n)} \leq \Gamma < \infty \quad \text{on} \quad [a,b], \tag{1.4}$$

where γ, Γ are real numbers, then

$$\sup_{P_n \in \mathcal{P}_n^\circ} L\left(g - \frac{\gamma+\Gamma}{2}P_n, g^{(1)} - \frac{\gamma+\Gamma}{2}P_n^{(1)}, \dots, g^{(n)} - \frac{\gamma+\Gamma}{2}P_n^{(n)}; a, b\right) \leq R\left(\frac{\Gamma-\gamma}{2}; a, b\right).$$

In order to exemplify the above principle, the author has given various examples classified as "inequalities of the $0^{\text{th}}-$degree", "$1^{\text{st}}-$degree" and, in general, "$\text{n}^{\text{th}}-$degree", where n is the maximal order of the derivative involved in the original inequality. To motivate our further exploration, we mention here only some results from the class of "$0^{\text{th}}-$degree" and for the Riemann-Stieltjes integral as follows:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function with the property that*

$$-\infty < m \leq f(x) \leq M < \infty \quad \text{for any} \quad x \in [a,b] \tag{1.5}$$

and u a function of bounded variation such that $u(a) = u(b)$. Then

$$\left|\int_a^b f(t)\, du(t)\right| \leq \frac{1}{2}(M-m) \bigvee_a^b (u). \tag{1.6}$$

The constant $\frac{1}{2}$ is sharp in (1.6).

Theorem 3. *Let $f, l : [a,b] \to \mathbb{R}$ be two continuous functions on $[a,b]$. If f satisfies (1.5) and u is of bounded variation such that*

$$\int_a^b l(t)\, du(t) = 0, \tag{1.7}$$

then

$$\left|\int_a^b f(t)\, l(t)\, du(t)\right| \leq \frac{1}{2}(M-m)\|l\|_{[a,b],\infty} \bigvee_a^b (u). \tag{1.8}$$

The constant $\frac{1}{2}$ is sharp in (1.8).

As a corollary of the above, we stated in [1] that the following Grüss type inequality obtained in [2] can be stated:

Corollary 1. *Let f,g be continuous on $[a,b]$, f satisfies (1.5) and u is of bounded variation and such that $u(b)\neq u(a)$. Then*

$$\left|\frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t)\right| \leq \frac{1}{2}(M-m)\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_{[a,b],\infty}\bigvee_a^b(u). \tag{1.9}$$

The constant $\frac{1}{2}$ is best possible in (1.9).

2. A Perturbed Version of the Median Principle

For two real numbers $\delta,\Delta\in\mathbb{R}$ with $\Delta\neq\delta$, we consider the set of all functions $f,g:[a,b]\to\mathbb{R}$ given by:

$$\mathcal{M}_{[a,b]}(\delta,\Delta) := \left\{(f,g)\,\middle|\,\left|f(x)-\frac{\delta+\Delta}{2}g(x)\right| \leq \frac{1}{2}|\Delta-\delta|\,|g(x)| \quad \text{for any } \ x\in[a,b]\right\}. \tag{2.1}$$

We observe, for instance, if $\Delta>\delta$ and $g(x)\neq 0$ for $x\in[a,b]$, then $\mathcal{M}_{[a,b]}(\delta,\Delta)$ contains the pair of functions (f,g) satisfying the condition

$$\left|\frac{f(x)}{g(x)}-\frac{\delta+\Delta}{2}\right| \leq \frac{1}{2}(\Delta-\delta), \quad \text{for all } \ x\in[a,b] \tag{2.2}$$

or, equivalently, the condition:

$$\delta\leq\frac{f(x)}{g(x)}\leq\Delta, \quad \text{for all } \ x\in[a,b]. \tag{2.3}$$

Moreover, if we assume that $g(x)>0$ for any $x\in[a,b]$, then the condition (2.3) is equivalent with

$$\delta g(x)\leq f(x)\leq\Delta g(x), \quad \text{for all } \ x\in[a,b]. \tag{2.4}$$

In practical application, the assumptions (2.3) or (2.4) are natural to impose. However, for the sake of generality, we consider, in the following, the class $\mathcal{M}_{[a,b]}(\delta,\Delta)$ which is larger and can be extended to the complex case as well.

We are able now to state a perturbed version for the Median Principle of $0^{\text{th}}-$degree:

Lemma 1. *Assume that the following inequality holds:*

$$L\left(h,h^{(1)},...,h^{(n)};a,b\right)\leq R\left(\left|h^{(n)}\right|;a,b\right), (n\geq 0) \tag{2.5}$$

where $L(\cdot,...,\cdot;a,b):\mathbb{R}^{n+1}\to\mathbb{R}$ (the left hand side) is a general function and $R(\cdot;a,b):[0,\infty)\to\mathbb{R}$ (the right hand side) is monotonic nondecreasing $[0,\infty)$.

If f,g are such that $f^{(n-1)},g^{(n-1)}$ are absolutely continuous and $\left(f^{(n)},g^{(n)}\right)\in\mathcal{M}_{[a,b]}\left(\delta_n,\Delta_n\right)$, for some real numbers $\delta_n\neq\Delta_n$, then

$$L\left(f-\frac{\delta_n+\Delta_n}{2}g,...,f^{(n)}-\frac{\delta_n+\Delta_n}{2}g^{(n)};a,b\right)\leq R\left(\frac{1}{2}\left|\Delta_n-\delta_n\right|\left|g^{(n)}\right|;a,b\right). \tag{2.6}$$

Proof. Since $\left(f^{(n)},g^{(n)}\right)\in\mathcal{M}_{[a,b]}\left(\delta_n,\Delta_n\right)$, then, by (2.1),

$$\left|f^{(n)}-\frac{\delta+\Delta}{2}g^{(n)}\right|\leq\frac{1}{2}\left|\Delta_n-\delta_n\right|\left|g^{(n)}\right| \tag{2.7}$$

which implies, by making use of (2.5) for $h=f-\frac{\delta_n+\Delta_n}{2}g$, that

$$L\left(f-\frac{\delta_n+\Delta_n}{2}g,...,f^{(n)}-\frac{\delta_n+\Delta_n}{2}g^{(n)};a,b\right)\leq R\left(\left|f^{(n)}-\frac{\delta_n+\Delta_n}{2}g^{(n)}\right|;a,b\right).$$

Further, by utilising the monotonicity property of $R(\cdot;a,b)$ and (2.7) we deduce the desired result (2.6). ∎

3. Some Examples for $0^{\text{th}}-$Degree Inequalities

The main aim of the results obtained below are to provide various examples for how the above principle can be utilised in practice in order to derive inequalities when the upper bounds are expressed in terms of the difference between the supremum and the infimum of a function. These inequalites are called the "$0^{\text{th}}-$degree" inequalities.

The following result concerning the Riemann-Stieltjes integral holds.

Theorem 4. *Let $(f,g)\in\mathcal{M}_{[a,b]}(\delta,\Delta)$ and $u:[a,b]\to\mathbb{R}$ be such that the Riemann-Stieltjes integrals $\int_a^b f(t)\,du(t)$ and $\int_a^b g(t)\,du(t)$ exist. Then*

$$\left|\int_a^b f(t)\,du(t)-\frac{\delta+\Delta}{2}\int_a^b g(t)\,du(t)\right| \tag{3.1}$$
$$\leq\frac{1}{2}\left|\Delta-\delta\right|\times\begin{cases}\|g\|_{[a,b],\infty}\bigvee_a^b(u) & \text{provided } g\in C[a,b] \text{ and } u\in BV[a,b];\\[2mm] \int_a^b|g(x)|\,du(x) & \text{provided } g\in C[a,b] \text{ and } u\in\mathcal{M}^{\nearrow}[a,b];\\[2mm] L\|g\|_{[a,b],1} & \text{provided } g\in R[a,b] \text{ and } u\in Lip_L[u,b];\end{cases}$$

where $C[a,b]$ is the class of continuous functions on $[a,b]$, $BV[a,b]$ the class of bounded variation functions on $[a,b]$, $\mathcal{M}^{\nearrow}[a,b]$ is the class of monotonic nondecreasing functions on $[a,b]$ while $Lip_L[u,b]$ is the class of Lipschitzian functions with the constant $L>0$. All the inequalities in (3.1) are sharp.

Proof. The following result is well known for the Riemann-Stieltjes integral: If $p:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b p(x)\,dv(x)$ exists and

$$\left|\int_a^b p(x)\,dv(x)\right|\leq\max_{x\in[a,b]}|p(x)|\bigvee_a^b(u), \tag{3.2}$$

where $\bigvee_a^b (v)$ denotes the total variation of u on $[a,b]$.

Now, since $(f,g) \in \mathcal{M}_{[a,b]}(\delta, \Delta)$, then on applying Lemma 1 for the inequality (3.2) we can state

$$\begin{aligned}\left|\int_a^b f(t)\,du(t) - \frac{\delta+\Delta}{2}\int_a^b g(t)\,du(t)\right| &= \left|\int_a^b \left[f(t) - \frac{\delta+\Delta}{2} g(t)\right] du(t)\right| \\ &\leq \sup_{t\in[a,b]} \left|f(t) - \frac{\delta+\Delta}{2} g(t)\right| \bigvee_a^b (u) \\ &\leq \frac{1}{2}\left|\Delta - \delta\right| \|g\|_{[a,b],\infty} \bigvee_a^b (u)\end{aligned}$$

and the first inequality in (3.1) is proved.

If $p : [a,b] \to \mathbb{R}$ is continuous and $v : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then the Riemann-Stieltjes integral $\int_a^b p(x)\,dv(x)$ obviously exists and

$$\left|\int_a^b p(x)\,dv(x)\right| \leq \int_a^b |p(x)|\,dv(x). \tag{3.3}$$

Also, if $p : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to \mathbb{R}$ is $l-$Lipschitzian, then the Riemann-Stieltjes integral $\int_a^b p(x)\,dv(x)$ exists and

$$\left|\int_a^b p(x)\,du(x)\right| \leq L\int_a^b |p(x)|\,dx. \tag{3.4}$$

Now, on applying Lemma 1 for these two inequalities we obtain the desired results.

To prove the sharpness of the inequalities in (3.1), we assume, for instance, that $f(x) = \Delta g(x)$, $x \in [a,b]$.

In this situation, the inequality (3.1) is obviously equivalent with

$$\left|\int_a^b g(x)\,du(x)\right| \leq \begin{cases} \|g\|_{[a,b],\infty} \bigvee_a^b (u); \\ \int_a^b g(x)\,du(x); \\ L\int_a^b |g(x)|\,dx \end{cases}$$

which are obviously sharp inequalities. ∎

The above result has many particular instances of interest.

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be such that there exist the constants $M > m$ with*

$$-\infty < m \leq f(t) \leq M < \infty \quad \textit{for any} \quad t \in [a,b]. \tag{3.5}$$

Then

$$\left|\int_a^b f(t)\,du(t) - \frac{m+M}{2}[u(b)-u(a)]\right| \tag{3.6}$$
$$\leq \frac{1}{2}(M-m)\times\begin{cases}\bigvee_a^b(u) & \text{if } f\in C[a,b] \text{ and } u\in BV[a,b];\\[2mm] [u(b)-u(a)] & \text{if } f\in C[a,b] \text{ and } u\in\mathcal{M}^{\nearrow}[a,b];\\[2mm] (b-a)L & \text{if } f\in R[a,b] \text{ and } u\in Lip_L[u,b].\end{cases}$$

Another simple result is the following one.

Corollary 3. *Assume that there exist the constants l,L such that:*

$$-\infty < l \leq \frac{f(t)-f(x)}{t-x} \leq L < \infty \tag{3.7}$$

for $t,x\in[a,b]$ with $t\neq x$. Then

$$\left|\int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)] - \frac{l+L}{2}\left[(x-a)u(a)+(b-x)u(b)-\int_a^b u(t)\,dt\right]\right| \tag{3.8}$$
$$\leq \frac{1}{2}(L-l)\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u) & \text{if } u\in BV[a,b];\\[2mm] \Big[bu(b)+au(a)-x[u(a)+u(b)] \\ \qquad +\int_a^b \operatorname{sgn}(x-t)\,u(t)\,dt\Big] & \text{if } u\in\mathcal{M}^{\nearrow}[a,b];\\[2mm] K\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 & \text{if } u\in Lip_L[u,b].\end{cases}$$

Proof. The condition (3.7) obviously implies that

$$\left|f(t)-f(x)-\frac{l+L}{2}(t-x)\right| \leq \frac{1}{2}(L-l)|t-x|$$

for $t,x\in[a,b]$.

If we fix $x\in[a,b]$ and apply Theorem 4 for $f_x(t)=f(t)-f(x)$, $g_x(t)=t-x$ and $\delta=\gamma$, $\Delta=\Gamma$, we get

$$\left|\int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)] - \frac{l+L}{2}\left[(x-a)u(a)+(b-x)u(b)-\int_a^b u(t)\,dt\right]\right| \tag{3.9}$$

$$\leq \frac{1}{2}(L-l) \times \begin{cases} \sup_{t\in[a,b]} |t-x| \bigvee_a^b (u) & \text{if } u \in BV[a,b]; \\ \int_a^b |t-x|\, du(t) & \text{if } u \in \mathcal{M}^{\nearrow}[a,b]; \\ K\int_a^b |t-x|\, dt & \text{if } u \in Lip_L[u,b]. \end{cases}$$

$$= \frac{1}{2}(L-l) \times \begin{cases} \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (u); \\ \Big[bu(b) + au(a) - x[u(a)+u(b)] \\ \qquad + \int_a^b \operatorname{sgn}(x-t)\, u(t)\, dt\Big]; \\ K\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)^2 \end{cases}$$

since, a simple integration by parts shows that

$$\begin{aligned} \int_a^b |t-x|\, du(t) &= \int_a^x (x-t)\, du(t) + \int_x^b (t-x)\, du(t) \\ &= bu(b) + au(a) - x[u(a)+u(b)] + \int_a^b \operatorname{sgn}(x-t)\, u(t)\, dt. \end{aligned}$$

On the other hand,

$$\int_a^b (t-x)\, du(t) = (b-x)\, u(b) + (x-a)\, u(a) - \int_a^b u(t)\, dt,$$

which, together with (3.9) produces the desired result (3.8). ∎

Remark 1. *If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on $[a,b]$ and if $\gamma = \inf_{t\in[a,b]} f'(t)$, $\Gamma = \sup_{t\in[a,b]} f'(t)$, then by Lagrange's theorem, we have*

$$\gamma \leq \frac{f(t)-f(x)}{t-x} \leq \Gamma$$

for any $x,t\in[a,b]$ with $t\neq x$, and the inequality (3.8) holds true for these γ and Γ.

The following result may be stated as well:

Corollary 4. *Assume that there exists the constants γ,Γ so that:*

$$\left| f(t) - f(x) - \frac{\gamma+\Gamma}{2} |t-x|^{\alpha} \right| \leq \frac{1}{2} |\Gamma-\gamma|\, |t-x|^{\alpha} \tag{3.10}$$

for a given $x \in [a,b]$ *and* $\alpha > 0$ *for each* $t \in [a,b]$. *Then*

$$\begin{aligned}(3.11)\quad &\left|\int_a^b f(t)\,du(t) - f(x)[u(b)-u(a)]\right.\\ &\left.-\frac{\gamma+\Gamma}{2}\left[(x-a)^\alpha u(a) + (b-x)^\alpha u(b) - \alpha\int_a^b |t-x|^{\alpha-1} u(t)\,dt\right]\right|\\ &\leq \frac{1}{2}|\Gamma-\gamma| \times \begin{cases} \left[\frac{1}{2}(b-a) + \left|x-\frac{a+b}{2}\right|\right]^\alpha \bigvee_a^b(u) & \text{if } u \in BV[a,b];\\ \left[(x-a)^\alpha u(a) + (b-x)^\alpha u(b)\right. \\ \quad \left.-\alpha\int_a^b (t-x)^{\alpha-1} u(t)\,dt\right] & \text{if } u \in \mathcal{M}^{\nearrow}[a,b];\\ \frac{L}{\alpha+1}\left[(b-x)^{\alpha+1} + (x-a)^{\alpha+1}\right] & \text{if } u \in Lip_L[u,b].\end{cases}\end{aligned}$$

Proof. Follows by applying Theorem 4 for $\delta = \gamma$, $\Delta = \Gamma$ and $g_x(t) = |t-x|^\alpha$ with x fixed in $[a,b]$. The details are omitted. ∎

Remark 2. *The above result contains some cases of interest, for instance when* $x = a$, $x = b$ *or* $x = \frac{a+b}{2}$. *To be more specific, we choose only one case to exemplify. Let us assume that there exist* $\phi, \Phi \in \mathbb{R}$ *such that*

$$\left|f(t) - f\left(\frac{a+b}{2}\right) - \frac{\phi+\Phi}{2}\left|t-\frac{a+b}{2}\right|^\alpha\right| \leq \frac{1}{2}|\Phi-\phi|\left|t-\frac{a+b}{2}\right|^\alpha$$

for any $t \in [a,b]$, *where* $\alpha > 0$ *is given. Then*

$$\begin{aligned}(3.12)\quad &\left|\int_a^b f(t)\,du(t) - f\left(\frac{a+b}{2}\right)[u(b)-u(a)]\right.\\ &\left.-\frac{\phi+\Phi}{2}\left[\frac{(b-a)^\alpha}{2^\alpha}[u(b)+u(a)] - \alpha\int_a^b \left|t-\frac{a+b}{2}\right|^{\alpha-1} u(t)\,dt\right]\right|\\ &\leq \frac{1}{2}|\Phi-\phi| \times \begin{cases} \frac{(b-a)^\alpha}{2^\alpha}\bigvee_a^b(u) & \text{if } u \in BV[a,b];\\ \left[\frac{(b-a)^\alpha}{2^\alpha}[u(b)+u(a)]\right. \\ \quad \left.-\alpha\cdot\int_a^b \left|t-\frac{a+b}{2}\right|^{\alpha-1} u(t)\,dt\right] & \text{if } u \in \mathcal{M}^{\nearrow}[a,b];\\ \frac{L(b-a)^{\alpha+1}}{2^\alpha(\alpha+1)} & \text{if } u \in Lip_L[u,b].\end{cases}\end{aligned}$$

The following result can be stated as well.

Corollary 5. *Assume that there exist the constants* $n, N \in \mathbb{R}$ *such that:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(s)\,ds - \frac{n+N}{2}\cdot g(x)\right| \leq \frac{1}{2}|N-n|\,|g(x)|$$

for any $x \in [a,b]$. *Then*

$$\left|\int_a^b f(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(s)\,ds - \frac{n+N}{2}\int_a^b g(t)\,du(t)\right| \tag{3.13}$$
$$\leq \frac{1}{2}|N-n| \times \begin{cases} \|g\|_{[a,b],\infty}\bigvee_a^b(u) & \text{provided } g\in C[a,b] \text{ and } u\in BV[a,b]; \\ \int_a^b |g(x)|\,du(x) & \text{provided } g\in C[a,b] \text{ and } u\in \mathcal{M}^{\nearrow}[a,b]; \\ L\|g\|_{[a,b],1} & \text{provided } g\in R[a,b] \text{ and } u\in Lip_L[u,b]. \end{cases}$$

4. Inequalities of the 1^{ST}−Degree

An inequality that contains at most the first derivative of the involved functions will be called an inequality of the 1^{st}−degree.

If one would like examples of such inequalities for two functions, the following Ostrowski's inequality obtained in [3] is the most suitable

$$|C(h,l;a,b)| \leq \frac{1}{8}(b-a)(M-m)\|h'\|_{[a,b],\infty}, \tag{4.1}$$

where $C(h,l;a,b)$ is the *Čebyšev functional* given by

$$C(h,l;a,b) := \frac{1}{b-a}\int_a^b h(x)\,l(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b l(x)\,dx$$

and $-\infty < m \leq h(x) \leq M < \infty$ for a.e. $x\in[a,b]$ while l is absolutely continuous and such that $l' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is sharp.

Another example of such an inequality is the Čebyšev one

$$|C(h,l;a,b)| \leq \frac{1}{12}(b-a)^2\|h'\|_{[a,b],\infty}\|l'\|_{[a,b],\infty}, \tag{4.2}$$

provided h,l are absolutely continuous and $h',l' \in L_\infty[a,b]$. The constant $\frac{1}{12}$ here is sharp.

The following result holds:

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be a Lebesgue integrable function such that* $-\infty < m \leq f(x) \leq M < \infty$ *for a.e.* $x\in[a,b]$. *If* g *and* h *are absolutely continous and such that there exists the constants* δ *and* Δ *such that* $(g',h')\in\mathcal{M}_{[a,b]}(\delta,\Delta)$, *then we have the following perturbed version of the Ostrowski's inequality:*

$$\left|C(f,g;a,b) - \frac{\Delta+\delta}{2}C(f,h;a,b)\right| \leq \frac{1}{16}(b-a)(M-m)(\Delta-\delta)\|h'\|_{[a,b],\infty}. \tag{4.3}$$

Proof. By applying Lemma 1 to the Ostrowski's inequality (4.1) we have

$$\left|C\left(f, g-\frac{\Delta+\delta}{2}h; a,b\right)\right| \leq \frac{1}{8}(b-a)(M-m)\left\|\frac{1}{2}(\Delta-\delta)h'\right\|_{[a,b],\infty}$$

which is clearly equivalent with the desired result (4.3). ∎

Finally, we have the following perturbed version of the Čebyšev inequality

Theorem 6. *Let f,g,h,l be absolutely continuous on $[a,b]$. If there exists the constants φ,Φ, δ and Δ such that $(f',l') \in \mathcal{M}_{[a,b]}(\varphi,\Phi)$ and $(g',h') \in \mathcal{M}_{[a,b]}(\delta,\Delta)$, then*

$$\begin{aligned} &\left| C(f,g;a,b) - \frac{\Delta+\delta}{2} C(f,h;a,b) \right. \\ &\left. - \frac{\Phi+\varphi}{2} C(l,g;a,b) + \frac{\Delta+\delta}{2} \cdot \frac{\Phi+\varphi}{2} C(h,l;a,b) \right| \\ &\leq \frac{1}{48}(b-a)^2 (\Phi-\varphi)(\Delta-\delta) \|h'\|_{[a,b],\infty} \|l'\|_{[a,b],\infty}. \end{aligned} \tag{4.4}$$

Similar results can be obtained for other inequalities involving the derivatives up to an order $n \geq 1$. However, the details are left to the interested reader.

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Research Group in Mathematical Inequalities & Applications, School of Engineering and Science, Victoria University, PO Box 14428, Melbourne City, VIC, Australia. 8001

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://www.staff.vu.edu.au/rgmia/dragomir/`