New upper bounds for Mathieu–type series

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Abstract

The Mathieu's series $S(r)$ was considered firstly by É.L. Mathieu in 1890; its alternating variant $\widetilde{S}(r)$ has been recently introduced by Pogány *et al.* [12] where various bounds have been established for $S, \widetilde{S}$. In this note we obtain new upper bounds over $S(r), \widetilde{S}(r)$ with the help of Hardy–Hilbert double integral inequality.

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1. **Introduction and preliminaries**

The series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \tag{1}$$

is named after Émile Léonard Mathieu (1835–1890), who investigated it in his 1890 book [9] written on the elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two–dimensional rectangular domain, see [13, Eq. (54), p. 258]. The alternating version of $S(r)$, that is

$$\widetilde{S}(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2} \tag{2}$$

was introduced following certain Tomovski's ideas and recently discussed by Pogány *et al.* in [12]. Applications of alternating Mathieu series $\widetilde{S}(r)$ concerning ODE which solution is the Butzer–Flocke–Hauss Omega function were studied in [3], [11]. The integral representations of $S(r), \widetilde{S}(r)$ [6], [12] respectively, reads as follows:

$$S(r) = \frac{1}{r}\int_0^\infty \frac{x\,\sin(rx)}{e^x-1}\,\mathrm{d}x, \qquad \widetilde{S}(r) = \frac{1}{r}\int_0^\infty \frac{x\,\sin(rx)}{e^x+1}\,\mathrm{d}x\,. \tag{3}$$

These integral expressions will be the starting points of our investigations.

2. **Results required**

Let us consider a Hölder pair $(p,q), p^{-1}+q^{-1}=1, p>1$, two non–negative functions $f\in L^p(\mathbb{R}_+)$, $g\in L^q(\mathbb{R}_+)$, and let us denote $\|\cdot\|_{L_s(\mathbb{R}_+)} := \|\cdot\|_s$ the usual integral L_s–norm on the set of positive reals. The celebrated Hardy–Hilbert (or Hilbert) integral inequality [10] reads

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)\,\mathrm{d}x\mathrm{d}y}{x+y} \le \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q\,. \tag{4}$$

The inequality is strict unless at least one of f,g is zero, and the constant on the right in (4) is the best possible [10].

Consider the scaled parametric integral

$$\mathcal{I}_p = \int_0^\infty \frac{|\sin x|^p}{x^p} \qquad \big(p>1\big)\,. \tag{5}$$

We point out that in [2, p. 663] the following estimate has been proved:

$$\mathcal{I}_p \le \frac{\pi}{2}\sqrt{\frac{2}{p}} \qquad \big(p\ge 2\big). \tag{6}$$

However, we shall give another estimate over $\mathcal{I}_p$ when $p>1$.

Lemma 1. *For all $p>1$ the following estimate holds*

$$\mathcal{I}_p \le q \tag{7}$$

where q is the conjugate Hölder pair to p.

Proof. Let us write

$$\mathcal{I}_p := \int_0^1 \frac{|\sin x|^p}{x^p}\,\mathrm{d}x + \int_1^\infty \frac{|\sin x|^p}{x^p}\,\mathrm{d}x\,.$$

Then, by the estimate $\sin x\le x$, $x\in[0,1]$ and by the redundant $|\sin x|\le 1, x>1$ respectively, we easily deduce

$$\mathcal{I}_p \le \int_0^1 \mathrm{d}x + \int_1^\infty \frac{\mathrm{d}x}{x^p} = 1+\frac{1}{p-1} = q\,. \tag{8}$$

This finishes the proof of the Lemma. □

3. **Main results**

At first we establish an upper bound for both $S(r), \widetilde{S}(r)$ of magnitude $O\big(r^{-1/2}\big)$.

Theorem 1. *Let (p,q), $p>1$ be a Hölder pair. Then we have*

$$S(r) \le \frac{16\sqrt{\pi}\, q^{1/(2p)} p^{1/(2q)}}{\sqrt{r}\, \sin^{1/2}(\pi/p)} \qquad \text{and} \qquad \widetilde{S}(r) \le \frac{16\sqrt{\pi}\, q^{1/(2p)} p^{1/(2q)}}{\sqrt{r}\, \sin^{1/2}(\pi/p)}\,. \tag{9}$$

Proof. It is sufficient to prove the inequality on the left in (9) since the right one can be proved similarly. First, we give two elementary inequalities:

$$\frac{x}{e^x+1} \le \frac{x}{e^x-1} \le \frac{2}{e^{x/2}} \qquad \big(x \ge 0\big) \tag{10}$$

$$\frac{xy(x+y)}{64} \le \exp\Big\{\frac{x}{4}+\frac{y}{4}+\frac{x+y}{4}\Big\} = \exp\Big\{\frac{x+y}{2}\Big\} \qquad \big(x, y \ge 0\big)\,. \tag{11}$$

Thus, we have

$$\begin{aligned}
\big(S(r)\big)^2 &= \frac{1}{r^2}\int_0^\infty\int_0^\infty \frac{xy\sin(rx)\sin(ry)}{(e^x-1)(e^y-1)}\,\mathrm{d}x\mathrm{d}y \\
&\le \frac{4}{r^2}\int_0^\infty\int_0^\infty |\sin(rx)\sin(ry)|e^{-(x+y)/2}\mathrm{d}x\mathrm{d}y && \text{(by (10))}\\
&\le \frac{256}{r^2}\int_0^\infty\int_0^\infty \frac{|\sin(rx)\sin(ry)|}{xy(x+y)}\,\mathrm{d}x\mathrm{d}y\,. && \text{(by (11))}
\end{aligned}$$

Taking $f(x) = x^{-1}|\sin(rx)| = g(x)$ we apply the Hardy–Hilbert inequality to the last expression, such that one transforms into

$$\begin{aligned}
\big(S(r)\big)^2 &\le \frac{256\pi}{r^2\sin(\pi/p)}\left(\int_0^\infty \frac{|\sin(rx)|^p}{x^p}\,\mathrm{d}x\right)^{1/p}\left(\int_0^\infty \frac{|\sin(ry)|^q}{y^q}\,\mathrm{d}y\right)^{1/q}\\
&= \frac{256\pi\, r^{(p-1)/p+(q-1)/q}}{r^2\,\sin(\pi/p)}\,(\mathcal{I}_p)^{\frac1p}(\mathcal{I}_q)^{\frac1q} && (12)\\
&\le \frac{256\pi\, q^{\frac1p}\cdot p^{\frac1q}}{r\,\sin(\pi/p)} && \text{(by (7))}
\end{aligned}$$

This is equivalent to the asserted result (9). □

Now, we will extend this result, scaling the exponent of r in the upper bound (9). The achieved magnitude should be $O\big(r^{-1/(2p)}\big)$, $p>1$.

Theorem 2. *Let $(p,q), p>1$ be a Hölder pair. Then for all $r>0, v>1$ we have*

$$S(r) \le \frac{C(p,v)}{r^{1/(2p)}} \qquad \text{and} \qquad \widetilde{S}(r) \le \frac{C(p,v)}{r^{1/(2p)}} \tag{13}$$

where

$$C(p,v) := \frac{2^{(5q+1)/(2q)}\ \max\{2^{1/(2p)}, 2^{1/(2q)}\}\,\big(\pi p\big)^{1/(2p)}\,\big(\Gamma(q)\Gamma(2q)\big)^{1/(2q)}}{q^{3/2}\,\big(\sin(\pi/p)\,(p-1/v)^{1/v}(p-1+1/v)^{1-1/v}\big)^{1/(2p)}}\,. \tag{14}$$

Proof. For a given Hölder pair $(p,q), p>1$ and for some $r>0$ consider

$$\begin{aligned}\big(S(r)\big)^2 &= \frac{1}{r^2}\int_0^\infty\int_0^\infty \frac{xy\sin(rx)\sin(ry)}{(e^x-1)(e^y-1)}\,\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{r^6}\int_0^\infty\int_0^\infty \frac{\sin(x)\sin(y)}{xy(x+y)^{1/p}}\cdot\frac{x^2y^2(x+y)^{1/p}}{(e^{x/r}-1)(e^{y/r}-1)}\,\mathrm{d}x\mathrm{d}y\,. \qquad (15)\end{aligned}$$

By the Hölder inequality we conclude

$$\begin{aligned}\big(S(r)\big)^2 &\le \frac{1}{r^6}\left(\int_0^\infty\int_0^\infty \frac{|\sin(x)\sin(y)|^p}{x^py^p(x+y)}\,\mathrm{d}x\mathrm{d}y\right)^{1/p} \\ &\qquad\times\left(\int_0^\infty\int_0^\infty \frac{x^{2q}y^{2q}(x+y)^{q-1}}{(e^{x/r}-1)^q(e^{y/r}-1)^q}\,\mathrm{d}x\mathrm{d}y\right)^{1/q}. \qquad (16)\end{aligned}$$

Choosing this time w as the Hölder conjugate pair to given $v>1$ and specifying

$$f(x)=g(x)=x^{-p}|\sin(x)|^p\,,$$

we evaluate by the Hardy–Hilbert inequality (4) the first integral from above:

$$\begin{aligned}\mathcal{J} &= \int_0^\infty\int_0^\infty \frac{|\sin(x)\sin(y)|^p}{x^py^p(x+y)}\,\mathrm{d}x\mathrm{d}y \\ &\le \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty \frac{|\sin(x)|^{pv}}{x^{pv}}\,\mathrm{d}x\right)^{1/v}\left(\int_0^\infty \frac{|\sin(y)|^{pw}}{y^{pw}}\,\mathrm{d}y\right)^{1/w}. \qquad (17)\end{aligned}$$

Estimating (17) by (7) we deduce

$$\mathcal{J} \le \frac{\pi}{\sin(\pi/p)}\,\frac{pv}{(pv-1)^{1/v}((p-1)v+1)^{1-1/v}}\,. \qquad (18)$$

The second integral in (16) we evaluate in the following way:

$$\begin{aligned}\mathcal{K} &= \int_0^\infty\int_0^\infty \frac{x^{2q}y^{2q}(x+y)^{q-1}}{(e^{x/r}-1)^q(e^{y/r}-1)^q}\,\mathrm{d}x\mathrm{d}y \\ &= r^{5q+1}\int_0^\infty \frac{x^{2q}y^{2q}(x+y)^{q-1}}{(e^x-1)^q(e^y-1)^q}\,\mathrm{d}x\mathrm{d}y \\ &\le r^{5q+1}\,\max\{2,\,2^{q-1}\}\int_0^\infty \frac{x^{3q-1}\,\mathrm{d}x}{(e^x-1)^q}\int_0^\infty \frac{y^{2q}\,\mathrm{d}y}{(e^y-1)^q} \qquad (19) \\ &\le (2r)^{5q+1}q^{-3q}\max\{2,\,2^{q-1}\}\,\Gamma(q)\Gamma(2q)\,. \qquad (20)\end{aligned}$$

where in (19) we make use of the estimate (such that follows by (10)):

$$\int_0^\infty \frac{x^\alpha}{(e^x-1)^q}\,\mathrm{d}x \le 2^q\int_0^\infty x^{\alpha-q}e^{-qx/2}\,\mathrm{d}x = \frac{2^{\alpha+1}}{q^{\alpha-q+1}}\,\Gamma(\alpha-q+1)\,, \qquad (21)$$

specified for $\alpha=3q-1,\,2q$ respectively. So, the upper bound over $S(r)$ in (13) is proved.

Repeating the previous procedure, now for $\widetilde{S}$, we clearly deduce

$$\big(\widetilde{S}(r)\big)^2 \le \frac{\max\{2^{1/p}, 2^{1/q}\}\,\mathcal{J}^{1/p}}{r^{1/p}} \left(\int_0^\infty \int_0^\infty \frac{x^{3q-1} y^{2q}}{(e^x+1)^q (e^y+1)^q}\, \mathrm{d}x \mathrm{d}y \right)^{1/q}. \tag{22}$$

Bearing in mind (10) we easily conclude by (21) that

$$\int_0^\infty \frac{x^\alpha}{(e^x+1)^q}\, \mathrm{d}x \le \int_0^\infty \frac{x^\alpha}{(e^x-1)^q}\, \mathrm{d}x \le \frac{2^{\alpha+1}}{q^{\alpha-q+1}}\, \Gamma(\alpha-q+1)\,.$$

Now, transforming the right–hand expression in (22), we easily arrive at the upper bound concerning $\widetilde{S}$ in (13). □

4. **Discussion**

A. In this research note we derive upper bounds for $S(r), \widetilde{S}(r)$, such that possess the form

$$S(r) \le \frac{\Phi(\theta)}{r^\alpha} \qquad \big(\alpha > 0\big)\,.$$

Here $\Phi(\theta)$ is an absolute constant and θ denotes the vector of scaling parameters. We obtain our main results (9) and (13) *via* the Hardy–Hilbert integral inequality.

At first, we recall some ancestor results such that will be compared to our bounds for small r. In [9] Mathieu posed his famous conjecture $S(r) < r^{-2}$, $r > 0$. The conjecture was proved after more then 60 years by Berg [1] and by Makai [8]. Actually they showed more:

$$\frac{1}{r^2+1/2} < S(r) < \frac{1}{r^2} \qquad \big(r > 0\big)\,. \tag{23}$$

Another proof of this upper bound has been given by van der Corput and Heflinger [4]. Diananda [5] improved Mathieu's bound to

$$S(r) \le \frac{1}{r^2} - \frac{1}{(2r^2+2r+1)(8r^2+3r+3)} \qquad \big(r > 0\big)\,. \tag{24}$$

Here has to be mentioned Guo's bound of magnitude $O(r^{-2})$, [7, Eq. (10)].

B. We obtain easily an upper bound, such that is superior to Mathieu's bound r^{-2} for small r. Indeed, starting with the integral expressions for $S(r)$ and $\widetilde{S}(r)$ in (3) we have

$$S(r) \le \frac{1}{r} \int_0^\infty \frac{x\,\mathrm{d}x}{e^x-1} = \frac{\pi^2}{6r} =: S^\star(r) \qquad \text{and} \qquad \widetilde{S}(r) \le \frac{1}{r} \int_0^\infty \frac{x\,\mathrm{d}x}{e^x+1} = \frac{\pi^2}{12r}\,. \tag{25}$$

So, when $r \in (0, 6/\pi]$, it follows $S^\star(r) \le r^{-2}$.

C. Let us denote $S_1(r), S_2(r)$ the upper bounds listed in Theorem 1, 2 respectively. Comparing Mathieu's bound with $S_1(r)$, solving the equation $S_1(r) = r^{-2}$ we find that

$$S_1(r) \le \frac{1}{r^2} \qquad \left(0 < r \le \frac{\sin^{2/3}(\pi/p)}{4\sqrt[3]{4\pi}\, p^{1/(3q)} q^{1/(3p)}} := r_1(p) < 1\right).$$

Therefore, $S_1(r)$ is obviously superior to bounds with $O(r^{-2})$, r small. The similar comparison involving $S_2(r)$ and/or Diananda's (24) and Guo's bounds, we leave to the interested reader. These analyses show that our bounds (9), (13) mainly improve the earlier ones.

D. Let us compare $S_1(r)$ and $S_2(r)$. It is not hard to see that

$$r_0 := r_0(p,v) = \frac{2^{3q-1}\pi\, p^{2-q}q^{4q-1}\big((p-1/v)^{1/v}(p-1+1/v)^{1-1/v}\big)^{q-1}}{\sin(\pi/p)\, \max\{2, 2^{q-1}\}\, \Gamma(q)\Gamma(2q)}$$

is the unique positive solution of $S_1(r) = S_2(r)$. Accordingly, it follows that

$$S_2(r) < S_1(r) \qquad \big(r \in (0, r_0)\big)\,,$$

while for $r > r_0$ the reversed conclusion holds. We point out that r_0 can easily skip 1; for instance $r_0(2,2) = 512\,\pi$.

E. Because the alternating Mathieu series has been introduced recently in [12], the here established bounds are unique until now. However, for r large the bounding inequalities presented also in [12] are sharper than the here presented ones.

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