BOUNDING THE ČEBYŠEV FUNCTIONAL FOR THE RIEMANN-STIELTJES INTEGRAL VIA A BEESACK INEQUALITY AND APPLICATIONS

P. CERONE AND S.S. DRAGOMIR

ABSTRACT. Lower and upper bounds of the Čebyšev functional for the Riemann-Stieltjes integral are given. Applications for the three point quadrature rules of functions that are $n-$time differentiable are also provided.

1. INTRODUCTION

In 1975, P.R. Beesack [1] showed that, if y, v, w are real valued functions defined on a compact interval $[a,b]$, where w is of bounded variation with total variation $\bigvee_a^b(w)$, and such that the Riemann-Stieltjes integrals $\int_a^b y(t)\,dv(t)$ and $\int_a^b w(t)\,y(t)\,dv(t)$ both exist, then

$$m\int_a^b y(t)\,dv(t) + \bigvee_a^b(w)\cdot \inf_{a\le\alpha<\beta\le b}\left[\int_\alpha^\beta y(t)\,dv(t)\right] \le \int_a^b w(t)\,y(t)\,dv(t) \le m\int_a^b y(t)\,dv(t) + \bigvee_a^b(w)\cdot \sup_{a\le\alpha<\beta\le b}\left[\int_\alpha^\beta y(t)\,dv(t)\right], \tag{1.1}$$

where $m := \inf_{t\in[a,b]}\{w(t)\}$.

The second of the inequalities above extends a result of R. Darst and H. Pollard [5] who dealt with the case $y(t)=1$, $t\in[a,b]$ and $v(t)$ continuous on $[a,b]$.

In [6], S.S. Dragomir has introduced the following *Čebyšev functional for the Riemann-Stieltjes integral*:

$$T(f,g;u) := \frac{1}{u(b)-u(a)}\int_a^b f(t)\,g(t)\,du(t) - \frac{1}{u(b)-u(a)}\int_a^b f(t)\,du(t)\cdot\frac{1}{u(b)-u(a)}\int_a^b g(t)\,du(t), \tag{1.2}$$

provided $u(b)\neq u(a)$ and the involved Riemann-Stieltjes integrals exist.

It has been shown in [6] that, if f,g are continuous, $m\le f(t)\le M$ for each $t\in[a,b]$ and u is of bounded variation, then the error in approximating the Riemann-Stieltjes integral of the product in terms of the product of integrals, as described

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in the definition of the Čebyšev functional (1.2), satisfies the inequality:

$$|T(f,g;u)| \le \frac{1}{2}(M-m)\cdot\frac{1}{|u(b)-u(a)|}\left\|g-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right\|_\infty \bigvee_a^b(u), \tag{1.3}$$

where the constant $\frac{1}{2}$ is best possible and $\|\cdot\|_\infty$ is the sup-norm.

Moreover, if f,g are continuous, $m \le f(t) \le M$ for $t \in [a,b]$ and u is monotonic nondecreasing on $[a,b]$, then:

$$|T(f,g;u)| \le \frac{1}{2}(M-m)\frac{1}{|u(b)-u(a)|}\cdot\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|du(t) \tag{1.4}$$

and the constant $\frac{1}{2}$ here is also sharp.

Finally, if f,g are Riemann integrable and u is Lipschitzian with the constant $L>0$ then also

$$|T(f,g;u)| \le \frac{1}{2}(M-m)\frac{L}{|u(b)-u(a)|}\cdot\int_a^b\left|g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right|dt. \tag{1.5}$$

The constant $\frac{1}{2}$ is also best possible in (1.5) (see [7] and [8]).

The main aim of the present paper is to provide other bounds for the Čebyšev functional $T(f,g;u)$ by utilising the Beesack inequality (1.1). Applications for three point quadrature rules of functions that are $(n-1)-$differentiable $(n \ge 1)$ with the derivative $f^{(n-1)}$ absolutely continuous are given as well.

2. The Results

The following result may be stated.

Theorem 1. *Let $f,g,u:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann-Stieltjes integrals $\int_a^b f(t)g(t)\,du(t)$, $\int_a^b f(t)\,du(t)$ and $\int_a^b g(t)\,du(t)$ exist. Then*

$$\begin{aligned}
&\bigvee_a^b(f)\cdot\inf_{a\le\alpha<\beta\le b}\left[\int_\alpha^\beta g(t)\,du(t)-\frac{u(\beta)-u(\alpha)}{u(b)-u(a)}\cdot\int_a^b g(s)\,du(s)\right] \\
&\le \int_a^b f(t)g(t)\,du(t)-\frac{1}{u(b)-u(a)}\cdot\int_a^b f(t)\,du(t)\cdot\int_a^b g(t)\,du(t) \\
&\le \bigvee_a^b(f)\cdot\sup_{a\le\alpha<\beta\le b}\left[\int_\alpha^\beta g(t)\,du(t)-\frac{u(\beta)-u(\alpha)}{u(b)-u(a)}\cdot\int_a^b g(s)\,du(s)\right],
\end{aligned} \tag{2.1}$$

provided $u(b)\ne u(a)$.

Proof. We observe that the following identity holds true (see also [6])

$$[u(b)-u(a)]\,T(f,g;u) = \int_a^b f(t)\left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t). \tag{2.2}$$

Since f is of bounded variation, it follows that f is bounded below and if we denote by m the infimum of f on $[a,b]$, then on applying the Beesack inequality for the choices

$$w(t)=f(t), \qquad y(t)=g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)$$

and $v(t)=u(t)$, $t\in[a,b]$, we can write that:

$$\begin{aligned} & m\int_a^b \left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \\ &\quad + \bigvee_a^b(f)\cdot \inf_{a\le\alpha<\beta\le b}\left\{\int_\alpha^\beta \left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)\right\} \\ &\le [u(b)-u(a)]\,T(f,g;u) \\ &\le m\int_a^b \left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \\ &\quad + \bigvee_a^b(f)\cdot \sup_{a\le\alpha<\beta\le b}\left\{\int_\alpha^\beta \left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)\right\}. \end{aligned} \tag{2.3}$$

Since

$$\int_a^b \left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t)=0$$

and

$$\begin{aligned} &\int_\alpha^\beta \left[g(t)-\frac{1}{u(b)-u(a)}\int_a^b g(s)\,du(s)\right]du(t) \\ &\qquad = \int_\alpha^\beta g(t)\,du(t)-\frac{u(\beta)-u(\alpha)}{u(b)-u(a)}\cdot\int_a^b g(s)\,du(s), \end{aligned}$$

hence, by (2.3), we deduce the desired result (2.1). □

The following corollary for weighted integrals may be stated:

Corollary 1. *Let $f,g,w:[a,b]\to\mathbb{R}$ be such that f is of bounded variation and the Riemann integrals $\int_a^b f(t)\,g(t)\,w(t)\,dt$, $\int_a^b f(t)\,w(t)\,dt$ and $\int_a^b g(t)\,w(t)\,dt$ exist.*

Then

$$\bigvee_a^b (f) \cdot \inf_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) w(t) \, dt - \frac{\int_\alpha^\beta w(s) \, ds}{\int_a^b w(s) \, ds} \cdot \int_a^b g(t) w(t) \, dt \right] \tag{2.4}$$
$$\le \int_a^b f(t) g(t) w(t) \, dt - \frac{1}{\int_a^b w(s) \, ds} \cdot \int_a^b f(t) w(t) \, dt \cdot \int_a^b g(t) w(t) \, dt$$
$$\le \bigvee_a^b (f) \cdot \sup_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) w(t) \, dt - \frac{\int_\alpha^\beta w(s) \, ds}{\int_a^b w(s) \, ds} \cdot \int_a^b g(t) w(t) \, dt \right],$$

provided $\int_a^b w(s) \, ds \neq 0$.

Remark 1. *For the particular case when* $w(t) = 1$, $t \in [a,b]$, *then we get from (2.4) the following inequality:*

$$\bigvee_a^b (f) \cdot \inf_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) \, dt - \frac{\beta - \alpha}{b - a} \cdot \int_a^b g(t) \, dt \right] \tag{2.5}$$
$$\le \int_a^b f(t) g(t) \, dt - \frac{1}{b-a} \int_a^b f(t) \, dt \cdot \int_a^b g(t) \, dt$$
$$\le \bigvee_a^b (f) \cdot \sup_{a \le \alpha < \beta \le b} \left[\int_\alpha^\beta g(t) \, dt - \frac{\beta - \alpha}{b - a} \cdot \int_a^b g(t) \, dt \right],$$

provided f *is of bounded variation and the involved Riemann integrals exist.*

3. Applications for Three Point Quadratures

Recall that in [4] (see also [9, p. 223]) P. Cerone and S.S. Dragomir established the following identity concerning a three point quadrature rule for $n-$time differentiable functions $f : [a,b] \to \mathbb{R}$:

$$\int_a^b f(t) \, dt = \sum_{k=1}^n \frac{1}{k!} \left\{ (1-\gamma)^k \left[(b-x)^k + (-1)^{k-1} (x-a)^k \right] f^{(k-1)}(x) \right. \tag{3.1}$$
$$\left. + \gamma^k \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1} (b-x)^k f^{(k-1)}(b) \right] \right\}$$
$$+ (-1)^n \int_a^b C_n(x,t) f^{(n)}(t) \, dt,$$

where the Peano kernel is given by:

$$C_n(x,t) := \begin{cases} \dfrac{[t - (\gamma x + (1-\gamma) a)]^n}{n!} & \text{if } t \in [a,x], \\ \\ \dfrac{[t - (\gamma x + (1-\gamma) b)]^n}{n!} & \text{if } t \in (x,b], \end{cases} \tag{3.2}$$

and $\gamma \in [0,1]$, $x \in [a,b]$.

We note that the above representation generalised the interior point quadrature rule obtained in 1999 by Cerone et al. in [2] for $\gamma = 0$ and the trapezoid type rule obtained in 2000 by Cerone et al. in [3] for $\gamma = 1$.

The function $C_n(x,\cdot)$ is of bounded variation for each fixed $x \in [a,b]$ and a simple calculation reveals that

$$
\begin{aligned}
(3.3)\quad & \bigvee_a^b \left((-1)^n C_n(x,\cdot)\right) \\
&= \int_a^x \left|\frac{dC_n(x,t)}{dt}\right| dt + \int_x^b \left|\frac{dC_n(x,t)}{dt}\right| dt \\
&= \int_a^x \frac{|t-(\gamma x+(1-\gamma)a)|^{n-1}}{(n-1)!} dt + \int_x^b \frac{|\gamma x+(1-\gamma)b-t|^{n-1}}{(n-1)!} dt \\
&= \frac{1}{n!}(x-a)^n\left[\gamma^n+(1-\gamma)^n\right] + \frac{1}{n!}(b-x)^n\left[\gamma^n+(1-\gamma)^n\right] \\
&= \frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]
\end{aligned}
$$

for any $x \in [a,b]$.

Also,

$$
\begin{aligned}
(3.4)\quad & \int_a^b C_n(x,t)\,dt \\
&= \frac{1}{n!}\int_a^x \left[t-(\gamma x+(1-\gamma)a)\right]^n dt + \frac{1}{n!}\int_x^b \left[t-(\gamma x+(1-\gamma)b)\right]^n dt \\
&= \frac{1}{(n+1)!}\Big\{\left[x-(\gamma x+(1-\gamma)a)\right]^{n+1} - \left[a-(\gamma x+(1-\gamma)a)\right]^{n+1} \\
&\qquad + \left[b-(\gamma x+(1-\gamma)b)\right]^{n+1} - \left[x-(\gamma x+(1-\gamma)b)\right]^{n+1}\Big\} \\
&= \frac{1}{(n+1)!}\Big\{(1-\gamma)^{n+1}(x-a)^{n+1} - (-1)^{n+1}\gamma^{n+1}(x-a)^{n+1} \\
&\qquad + \gamma^{n+1}(b-x)^{n+1} - (-1)^{n+1}(1-\gamma)^{n+1}(b-x)^{n+1}\Big\} \\
&= \frac{1}{(n+1)!}\Big\{(b-x)^{n+1}\left[\gamma^{n+1}+(-1)^n(1-\gamma)^{n+1}\right] \\
&\qquad + (-1)^n\left[\gamma^{n+1}+(-1)^n(1-\gamma)^{n+1}\right](x-a)^{n+1}\Big\} \\
&= \frac{1}{(n+1)!}\left[(b-x)^{n+1}+(-1)^n(x-a)^{n+1}\right]\left[\gamma^{n+1}+(-1)^n(1-\gamma)^{n+1}\right]
\end{aligned}
$$

for any $x \in [a,b]$.

We can state the following result:

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an $(n-1)-$differentiable function $(n\geq 1)$ with the derivative $f^{(n-1)}$ absolutely continuous on $[a,b]$. Then we have*

$$\int_a^b f(t)\,dt=\sum_{k=1}^{n}\frac{1}{k!}\Big\{(1-\gamma)^k\left[(b-x)^k+(-1)^{k-1}(x-a)^k\right]f^{(k-1)}(x) \tag{3.5}$$
$$+\gamma^k\left[(x-a)^k f^{(k-1)}(a)+(-1)^{k-1}(b-x)^k f^{(k-1)}(b)\right]\Big\}$$
$$+\frac{1}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\left[(b-x)^{n+1}+(-1)^n(x-a)^{n+1}\right]$$
$$\times\left[(-1)^n\gamma^{n+1}+(1-\gamma)^{n+1}\right]+E_n(f,x,\gamma;a,b),$$

where the remainder $E_n(f,x,\gamma;a,b)$ (which is defined implicitly by (3.5))satisfies the bounds:

$$\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right] \tag{3.6}$$
$$\leq E_n(f,x,\gamma;a,b)$$
$$\leq\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right]$$

and

$$\delta_n(f;\alpha,\beta)=f^{(n-1)}(\beta)-f^{(n-1)}(\alpha)-\frac{\beta-\alpha}{b-a}\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right], \tag{3.7}$$

where $\gamma\in[0,1]$ and $x\in[a,b]$.

Proof. Apply the inequality (2.5) for the functions $f=(-1)^n C_n(x,\cdot)$ and $g=f^{(n)}$ to get

$$\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right] \tag{3.8}$$
$$\leq\ (-1)^n\int_a^b C_n(x,t)f^{(n)}(t)\,dt-\frac{1}{b-a}(-1)^n\int_a^b C_n(x,t)\,dt\cdot\int_a^b f^{(n)}(t)\,dt$$
$$\leq\ \frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_n(f;\alpha,\beta)\right].$$

Since, by (3.3)

$$\bigvee_a^b\left((-1)^n C_n(x,\cdot)\right)=\frac{1}{n!}\left[\gamma^n+(1-\gamma)^n\right]\left[(b-x)^n+(x-a)^n\right]$$

and by (3.4)

$$(-1)^n\int_a^b C_n(x,t)\,dt$$
$$=\frac{1}{(n+1)!}\left[(b-x)^{n+1}+(-1)^n(x-a)^{n+1}\right]\left[(-1)^n\gamma^{n+1}+(1-\gamma)^{n+1}\right],$$

then, on utilising the inequality (3.8), we have

$$(3.9)\quad \frac{1}{n!}\left[\gamma^{n}+(1-\gamma)^{n}\right]\left[(b-x)^{n}+(x-a)^{n}\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_{n}(f;\alpha,\beta)\right]$$
$$\leq(-1)^{n}\int_{a}^{b}C_{n}(x,t)\,f^{(n)}(t)\,dt$$
$$-\frac{1}{(n+1)!}\left[(b-x)^{n+1}+(-1)^{n}(x-a)^{n+1}\right]\left[(-1)^{n}\gamma^{n+1}+(1-\gamma)^{n+1}\right]$$
$$\times\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]$$
$$\leq\frac{1}{n!}\left[\gamma^{n}+(1-\gamma)^{n}\right]\left[(b-x)^{n}+(x-a)^{n}\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_{n}(f;\alpha,\beta)\right].$$

Now, due to the fact that, by the representation (3.1) we have

$$(3.10)\quad (-1)^{n}\int_{a}^{b}C_{n}(x,t)\,f^{(n)}(t)\,dt$$
$$=\int_{a}^{b}f(t)\,dt-\sum_{k=1}^{n}\frac{1}{k!}\Big\{(1-\gamma)^{k}\left[(b-x)^{k}+(-1)^{k-1}(x-a)^{k}\right]f^{(k-1)}(x)$$
$$+\gamma^{k}\left[(x-a)^{k}f^{(k-1)}(a)+(-1)^{k-1}(b-x)^{k}f^{(k-1)}(b)\right]\Big\}$$

then, on making use of remainder's representation $E_n(f,x,\gamma;a,b)$ (which is defined implicitly by (3.5)), we deduce from (3.9) the desired result (3.6). □

Remark 2. *For $\gamma=0$, we get from Theorem 2:*

$$(3.11)\quad \int_{a}^{b}f(t)\,dt=\sum_{k=1}^{n}\frac{1}{k!}\left[(b-x)^{k}+(-1)^{k-1}(x-a)^{k}\right]f^{(k-1)}(x)$$
$$+\frac{1}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\left[(b-x)^{n+1}+(-1)^{n}(x-a)^{n+1}\right]$$
$$+F_{n}(f,x;a,b),$$

where the remainder satisfies the bounds

$$(3.12)\quad \frac{1}{n!}\left[(b-x)^{n}+(x-a)^{n}\right]\inf_{a\leq\alpha<\beta\leq b}\left[\delta_{n}(f;\alpha,\beta)\right]$$
$$\leq F_{n}(f,x;a,b)$$
$$\leq\frac{1}{n!}\left[(b-x)^{n}+(x-a)^{n}\right]\sup_{a\leq\alpha<\beta\leq b}\left[\delta_{n}(f;\alpha,\beta)\right]$$

for $x\in[a,b]$.

For $\gamma = \frac{1}{2}$, we get from Theorem 2 that:

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{2^k k!} \left\{ \left[(b-x)^k + (-1)^{k-1}(x-a)^k \right] f^{(k-1)}(x) + \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b) \right] \right\} + \frac{[1+(-1)^n]}{2^{n+1}(n+1)!} \times \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right] + G_n(f,x;a,b), \tag{3.13}$$

where the remainder satisfies the inequality:

$$\frac{1}{2^{n-1}n!}\left[(b-x)^n + (x-a)^n\right] \inf_{a\le\alpha<\beta\le b}\left[\delta_n(f;\alpha,\beta)\right] \le G_n(f,x;a,b) \le \frac{1}{2^{n-1}n!}\left[(b-x)^n + (x-a)^n\right] \sup_{a\le\alpha<\beta\le b}\left[\delta_n(f;\alpha,\beta)\right], \tag{3.14}$$

for $x \in [a,b]$.

Finally, for $\gamma = 1$, we obtain from Theorem 2 that:

$$\int_a^b f(t)\,dt = \sum_{k=1}^{n} \frac{1}{k!} \left[(x-a)^k f^{(k-1)}(a) + (-1)^{k-1}(b-x)^k f^{(k-1)}(b) \right] + \frac{(-1)^n}{(n+1)!} \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \left[(b-x)^{n+1} + (-1)^n (x-a)^{n+1} \right] + H_n(f,x;a,b) \tag{3.15}$$

where the remainder $H_n(f,x;a,b)$ satisfies the bounds:

$$\frac{1}{n!}\left[(b-x)^n + (x-a)^n\right] \inf_{a\le\alpha<\beta\le b}\left[\delta_n(f;\alpha,\beta)\right] \le H_n(f,x;a,b) \le \frac{1}{n!}\left[(b-x)^n + (x-a)^n\right] \sup_{a\le\alpha<\beta\le b}\left[\delta_n(f;\alpha,\beta)\right] \tag{3.16}$$

for $x \in [a,b]$.

The following particular case may be useful in applications:

If $n = 1$ and $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ then we have the representation:

$$\int_a^b f(t)\,dt = (1-\gamma)(b-a) f(x) + \gamma\left[(x-a) f(a) + (b-x) f(b)\right] + \left[f(b) - f(a)\right]\left(\frac{a+b}{2} - x\right)(1-2\gamma) + E(f,x,\gamma;a,b) \tag{3.17}$$

and the remainder $E(f,x,\gamma;a,b)$ satisfies the bounds

$$(b-a) \inf_{a\le\alpha<\beta\le b}\left[\delta(f;\alpha,\beta)\right] \le E(f,x,\gamma;a,b) \le (b-a) \sup_{a\le\alpha<\beta\le b}\left[\delta(f;\alpha,\beta)\right] \tag{3.18}$$

where

$$\delta\left(f;\alpha,\beta\right):=f\left(\beta\right)-f\left(\alpha\right)-\frac{\beta-\alpha}{b-a}\left[f\left(b\right)-f\left(a\right)\right],$$

and $x\in[a,b]$ while $\gamma\in[0,1]$.

One must observe that for $n=1$ the bounds for the error are independent of x and γ. However, this quality is not inherited for the quadrature rules with $n\geq 2$.

References

[1] P.R. BEESACK, Bounds for Riemann-Stieltjes integrals, *Rocky Mountain J. Math.,* **5**(1) (1975), 75-78.

[2] P. CERONE, S.S. DRAGOMIR and J. ROUMELIOTIS, Some Ostrowski type inequalities for $n-$time differentiable mappings and applications, *Demonstratio Math.,* **32**(2) (1999), 697-712.

[3] P. CERONE, S.S. DRAGOMIR, J. ROUMELIOTIS and J. SŬNDE, A new generalisation of the trapezoid formula for $n-$time differentiable mappings and applications, *Demonstratio Math.,* **33**(4) (2000), 719-736.

[4] P. CERONE and S.S. DRAGOMIR, Three point identities and inequalities for $n-$time differentiable functions, *SUT J. of Math.* (Japan), **36**(2) (2000), 351-383.

[5] R. DARST and H. POLLARD, An inequality for the Riemann-Stieltjes integral, *Proc. Amer. Math. Soc.*, 25 (1970), 912-913.

[6] S.S. DRAGOMIR, Sharp bounds of Čebyšev functional for Stieltjes integrals and applications, *Bull. Austral. Math. Soc.,* **67** (2003), 257-266.

[7] S.S. DRAGOMIR, New estimates of the Čebyšev functional for Stieltjes integrals and applications, *J. Korean Math. Soc.,* **41**(2) (2004), 249-264.

[8] S.S. DRAGOMIR, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.,* **26** (2004), 89-122.

[9] S.S. DRAGOMIR and Th.M. RASSIAS, *Ostrowski Type Inequalities and Applications in Numerical Integration,* Kluwer Academic Publishers, Dordrecht, 2002.

School of Computer Science and Mathematics, Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia.

E-mail address: pietro.cerone@vu.edu.au

URL: http://rgmia.vu.edu.au/cerone

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.vu.edu.au/dragomir