APPROXIMATING THE RIEMANN-STIELTJES INTEGRAL VIA SOME MOMENTS OF THE INTEGRAND

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Abstract. Error bounds in approximating the Riemann-Stieltjes integral in terms of some moments of the integrand are given. Applications for $p-$convex functions and in approximating the Finite Foureir Transform are pointed out as well.

1. Introduction

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the arguably simpler expression

$$\frac{u(b)-u(a)}{b-a}\cdot\int_a^b f(t)\,dt, \tag{1.1}$$

where $\int_a^b f(t)\,dt$ is the Riemann integral, Dragomir and Fedotov [8] considered in 1998 the following *Grüss type error functional:*

$$D(f,u;a,b) := \int_a^b f(t)\,du(t) - \frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt. \tag{1.2}$$

If the *integrand* f is Riemann integrable and $-\infty < m \le f(t) \le M < \infty$ for any $t\in[a,b]$ while the *integrator* u is $L-$*Lipschitzian*, namely,

$$|u(t)-u(s)| \le L|t-s| \quad \text{for each} \quad t,s\in[a,b], \tag{1.3}$$

then the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and the following bound holds:

$$|D(f,u;a,b)| \le \frac{1}{2}L(M-m)(b-a). \tag{1.4}$$

In (1.4) the constant $\frac{1}{2}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

A different bound for the Grüss error functional $D(f,u;a,b)$ in the case that f is $K-$Lipschitzian and u is of bounded variation has been obtained by the same authors in 2001, see [9], where they showed that

$$|D(f,u;a,b)| \le \frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{1.5}$$

Here $\bigvee_a^b(u)$ denotes the total variation of u on $[a,b]$. The constant $\frac{1}{2}$ is also best possible.

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For other results concerning different bounds for the functional $D(f, u; a, b)$ under various assumptions on f and u, see the recent papers [3], [4], [6] – [7], [13] and the references therein.

The main aim of the present paper is to provide error bounds in approximating the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the following expression containing moments of the function f, namely the expression

$$\frac{p}{(b-a)^p}\left[u(b)\cdot\int_a^b (t-a)^{p-1} f(t)\,dt - u(a)\cdot\int_a^b (b-t)^{p-1} f(t)\,dt\right],$$

where $p > 0$ and the involved integrals exist.

Some inequalities for monotonic integrands and $p-$convex integrators as well as where u is an integral of a given weight are provided. An application for approximating the Finite Fourier Transform is also given.

The case $p = 1$ reduces to the Grüss error functional and in this way some earlier results are recaptured as well.

2. General Results

The following identity holds.

Lemma 1. *Let $f, u : [a, b] \to \mathbb{R}$ such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integrals $\int_a^b (t-a)^{p-1} f(t)\,dt$, $\int_a^b (b-t)^{p-1} f(t)\,dt$ for $p > 0$ exist. Then*

$$\begin{aligned}
&\int_a^b f(t)\,du(t) \\
&= \frac{p}{(b-a)^p}\left[u(b)\cdot\int_a^b (t-a)^{p-1} f(t)\,dt - u(a)\cdot\int_a^b (b-t)^{p-1} f(t)\,dt\right] \\
&\qquad + \int_a^b \left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t)\right] df(t).
\end{aligned} \tag{2.1}$$

Proof. Integrating by parts of the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b \left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t)\right] df(t) \\
&= \left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t)\right] f(t)\Big|_a^b \\
&\qquad - \int_a^b f(t)\,d\left[\frac{(t-a)^p u(b) + (b-t)^p u(a)}{(b-a)^p} - u(t)\right] \\
&= [u(b) - u(b)]\,f(b) - [u(a) - u(a)]\,f(a) \\
&\qquad - \left[\frac{pu(b)}{(b-a)^p}\int_a^b (t-a)^{p-1} f(t)\,dt\right. \\
&\qquad\qquad \left. - \frac{pu(a)}{(b-a)^p}\int_a^b (b-t)^{p-1} f(t)\,dt - \int_a^b f(t)\,du(t)\right]
\end{aligned}$$

$$= -\frac{pu(b)}{(b-a)^p}\int_a^b (t-a)^{p-1} f(t)\,dt + \frac{pu(a)}{(b-a)^p}\int_a^b (b-t)^{p-1} f(t)\,dt + \int_a^b f(t)\,du(t),$$

which is equivalent with the desired identity (2.1). □

Remark 1. *For $p=1$ we get the identity:*

$$\int_a^b f(t)\,du(t) = \frac{u(b)-u(a)}{b-a}\cdot\int_a^b f(t)\,dt + \int_a^b \left[\frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)\right] df(t) \tag{2.2}$$

that has been obtained in [5]*, see also* [6]*.*

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ by the quadrature

$$\frac{p}{(b-a)^p}\left[u(b)\cdot\int_a^b (t-a)^{p-1} f(t)\,dt - u(a)\cdot\int_a^b (b-t)^{p-1} f(t)\,dt\right], \tag{2.3}$$

we consider the error functional:

$$F(f,u,p;a,b) := \int_a^b f(t)\,du(t) - \frac{p}{(b-a)^p}\left[u(b)\cdot\int_a^b (t-a)^{p-1} f(t)\,dt - u(a)\cdot\int_a^b (b-t)^{p-1} f(t)\,dt\right]. \tag{2.4}$$

The following result may be stated.

Theorem 1. *Let $f,u:[a,b]\to\mathbb{R}$ be as in Lemma 1. For $p>0$, define*

$$\Delta_p(u;t,a,b) := \frac{(t-a)^p\,u(b)+(b-t)^p\,u(a)}{(b-a)^p} - u(t), \tag{2.5}$$

where $t\in[a,b]$.

If $F(f,u,p;a,b)$ is the error functional defined by (2.4), then:

$$|F(f,u,p;a,b)| \le \begin{cases} \sup\limits_{t\in[a,b]} |\Delta_p(u;t,a,b)| \bigvee_a^b(f) & \text{if } f \text{ is of bounded variation;} \\[2ex] L\int_a^b |\Delta_p(u;t,a,b)|\,dt & \text{if } f \text{ is } L-\text{Lipschitzian;} \\ \int_a^b |\Delta_p(u;t,a,b)|\,df(t) & \text{if } f \text{ is monotonic nondecreasing.} \end{cases} \tag{2.6}$$

Proof. It is well known that for the Riemann-Stieltjes integral $\int_a^b w(t)\,dv(t)$ we have the bounds

$$\left|\int_a^b w(t)\,dv(t)\right| \leq \begin{cases} \sup\limits_{t\in[a,b]} |w(t)| \bigvee_a^b (v) \\ \qquad \text{if } v \text{ is of bounded variation;} \\ \\ L\int_a^b |w(t)|\,dt \\ \qquad \text{if } v \text{ is } L-\text{Lipschitzian;} \\ \int_a^b |w(t)|\,dv(t) \\ \qquad \text{if } v \text{ is monotonic nondecreasing.} \end{cases} \tag{2.7}$$

Now, on utilising the representation (2.1) and applying (2.7) for $w(t) := \Delta_p(u;t,a,b)$, $t\in[a,b]$ and $v=f$, we deduce the desired result. □

Remark 2. *For $p=1$, by denoting*

$$\Delta(u;t,a,b) = \Delta_1(u;t,a,b) = \frac{(t-a)\,u(b)+(b-t)\,u(a)}{b-a} - u(t)$$

and

$$F(f,u;a,b) = \int_a^b f(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\cdot\int_a^b f(t)\,dt,$$

we get from (2.6)

$$|F(f,u;a,b)| \leq \begin{cases} \sup\limits_{t\in[a,b]} |\Delta(u;t,a,b)| \bigvee_a^b (f) \\ \qquad \textit{if } f \textit{ is of bounded variation;} \\ \\ L\int_a^b |\Delta(u;t,a,b)|\,dt \\ \qquad \textit{if } f \textit{ is } L-\textit{Lipschitzian;} \\ \int_a^b |\Delta(u;t,a,b)|\,df(t) \\ \qquad \textit{if } f \textit{ is monotonic nondecreasing.} \end{cases} \tag{2.8}$$

The inequality (2.8) has been obtained in [5].

Remark 3. *If $u(t)=\int_a^t w(s)\,ds$, $t\in[a,b]$, then from (2.1) we get the representation*

$$\int_a^b f(t)\,w(t)\,dt = \frac{p}{(b-a)^p}\int_a^b w(s)\,ds\int_a^b (t-a)^{p-1} f(t)\,dt \tag{2.9}$$
$$+\frac{1}{(b-a)^p}\cdot\int_a^b \left[(t-a)^p\int_a^b w(s)\,ds-(b-a)^p\int_a^t w(s)\,ds\right]df(t)$$

for any $p>0$, provided that the involved integrals exist.

For $p=1$, we obtain the identity due to Cerone in [2].

3. Further Bounds for Monotonic Integrands

In this section some bounds for the error functional $F(f,u,p;a,b)$ where the integrator f is monotonic nondecreasing are given.

Theorem 2. *Let $f,u:[a,b]\to\mathbb{R}$ be such that f is monotonic nondecreasing, u satisfies the bounds:*

$$-\infty<n\leq u(t)\leq N<\infty \quad \text{for any} \quad t\in[a,b] \tag{3.1}$$

and the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ exists. Then

$$\begin{aligned}
&\frac{np}{(b-a)^p}\left[\int_a^b (b-t)^{p-1} f(t)\,dt-\int_a^b (t-a)^{p-1} f(t)\,dt\right] \\
&\qquad\qquad -(N-n)[f(b)-f(a)] \\
&\leq F(f,u,p;a,b) \\
&\leq \frac{Np}{(b-a)^p}\left[\int_a^b (b-t)^{p-1} f(t)\,dt-\int_a^b (t-a)^{p-1} f(t)\,dt\right] \\
&\qquad\qquad -(N-n)[f(b)-f(a)],
\end{aligned} \tag{3.2}$$

where $F(f,u,p;a,b)$ is given by (2.4).

Proof. From (3.1) we obviously have:

$$\begin{aligned}
n(t-a)^p &\leq u(b)\leq N(t-a)^p, \\
n(b-t)^p &\leq u(a)\leq N(b-t)^p, \\
-N(b-a)^p &\leq -u(t)(b-a)^p\leq -n(b-a)^p
\end{aligned}$$

for any $t\in[a,b]$. Summing the above three inequalities, we have that

$$n\cdot\frac{(t-a)^p+(b-t)^p}{(b-a)^p}-N\leq \Delta_p(u;t,a,b)\leq N\cdot\frac{(t-a)^p+(b-t)^p}{(b-a)^p}-n$$

for any $t\in[a,b]$.

Now, integrating over the monotonic nondecreasing function f we have

$$\begin{aligned}
&\frac{n}{(b-a)^p}\left[\int_a^b (t-a)^p\,df(t)+\int_a^b (b-t)^p\,df(t)\right]-N[f(b)-f(a)] \\
&\leq F(f,u,p;a,b) \\
&\leq \frac{N}{(b-a)^p}\left[\int_a^b (t-a)^p\,df(t)+\int_a^b (b-t)^p\,df(t)\right]-n[f(b)-f(a)].
\end{aligned} \tag{3.3}$$

Integrating by parts, we also have

$$\int_a^b (t-a)^p\,df(t)=(b-a)^p f(b)-p\int_a^b (t-a)^{p-1} f(t)\,dt$$

and

$$\int_a^b (b-t)^p\,df(t)=-(b-a)^p f(a)+p\int_a^b (b-t)^{p-1} f(t)\,dt.$$

Then

$$\begin{aligned}
(3.4)\qquad & \frac{n}{(b-a)^p}\left[\int_a^b (t-a)^p\,df(t)+\int_a^b (b-t)^p\,df(t)\right]-N\left[f(b)-f(a)\right]\\
&=\frac{n}{(b-a)^p}\left\{(b-a)^p\left[f(b)-f(a)\right]+p\left[\int_a^b (b-t)^{p-1}f(t)\,dt\right.\right.\\
&\qquad\left.\left.-\int_a^b (t-a)^{p-1}f(t)\,dt\right]\right\}-N\left[f(b)-f(a)\right]\\
&=\frac{np}{(b-a)^p}\left[\int_a^b (b-t)^{p-1}f(t)\,dt-\int_a^b (t-a)^{p-1}f(t)\,dt\right]\\
&\qquad-(N-n)\left[f(b)-f(a)\right]
\end{aligned}$$

and

$$\begin{aligned}
(3.5)\qquad & \frac{N}{(b-a)^p}\left[\int_a^b (t-a)^p\,df(t)+\int_a^b (b-t)^p\,df(t)\right]-n\left[f(b)-f(a)\right]\\
&=\frac{N}{(b-a)^p}\left\{(b-a)^p\left[f(b)-f(a)\right]+p\left[\int_a^b (b-t)^{p-1}f(t)\,dt\right.\right.\\
&\qquad\left.\left.-\int_a^b (t-a)^{p-1}f(t)\,dt\right]\right\}-n\left[f(b)-f(a)\right]\\
&=\frac{Np}{(b-a)^p}\left[\int_a^b (b-t)^{p-1}f(t)\,dt-\int_a^b (t-a)^{p-1}f(t)\,dt\right]\\
&\qquad+(N-n)\left[f(b)-f(a)\right].
\end{aligned}$$

Now, on utilising (3.3) – (3.5) we deduce the desired result (3.2). □

Remark 4. *In the particular case when $p=1$, the inequality (3.2) reduces to*

$$|F(f,u,p;a,b)|\le (N-n)\left[f(b)-f(a)\right]. \tag{3.6}$$

4. An Inequality for Integrators that are $s-$Convex in the Second Sense

Following Hudzik and Maligranda [12] (see also [11, p. 286]) we say that the function $g:\mathbb{R}_+\to\mathbb{R}$ is $p-$*convex in the second sense*, where $p>0$ is fixed, if:

$$g(tu+(1-t)v)\le t^p g(u)+(1-t)^p g(v) \tag{4.1}$$

for any $u,v\ge 0$ and $t\in[0,1]$.

For different properties of this class of functions, see [12] and [11, pp. 286 – 293].

The following inequality of Hermite-Hadamard type is due to Dragomir and Fitzpatrick [10]:

Theorem 3. *Let g be a $p-$convex function in the second sense on an interval $I\subset[0,\infty)$ with $p\in(0,1]$ and let $a,b\in I$ with $a<b$. Then:*

$$2^{p-1}g\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b g(t)\,dt\le\frac{g(a)+g(b)}{p+1}. \tag{4.2}$$

We can state and prove now the following result about the Riemann-Stieltjes integral:

Theorem 4. *Let u be $p-$convex with $p > 0$, f be monotonic nondecreasing on $[a, b]$ and such that the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integrals $\int_a^b (t-a)^{p-1} f(t)\,dt$, $\int_a^b (b-t)^{p-1} f(t)\,dt$ exist. Then*

$$\int_a^b f(t)\,du(t) \geq \frac{p}{(b-a)^p}\left[u(b)\int_a^b (t-a)^{p-1} f(t)\,dt - u(a)\int_a^b (b-t)^{p-1} f(t)\,dt\right]. \tag{4.3}$$

Proof. Since u is $p-$convex, then:

$$u(t) = u\left(\frac{t-a}{b-a}\cdot b + \frac{b-t}{b-a}\cdot a\right) \leq \left(\frac{t-a}{b-a}\right)^p u(b) + \left(\frac{b-t}{b-a}\right)^p u(a),$$

which shows, upon using the notations of (2.5), that

$$\Delta_p(u, t; a, b) \geq 0 \quad \text{for any} \quad t \in [a, b].$$

Since f is monotonic nondecreasing on $[a, b]$, we have then

$$\int_a^b \Delta_p(u, t; a, b)\,df(t) \geq 0,$$

which, via the representation (2.1), is equivalent with the desired inequality (4.3). □

Remark 5. *The case $p = 1$, i.e., where the function u is convex in the usual sense, produces the following inequality*

$$\int_a^b f(t)\,du(t) \geq \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\,dt, \tag{4.4}$$

where f is monotonic nondecreasing, which has been obtained in [5]*, see also* [6].

5. Approximating the Finite Fourier Transform

The *Fourier Transform* is one of the most important mathematical tools in a wide variety of fields in science and engineering [1, p. xi].

Throughout this section $f : [a, b] \to \mathbb{R}$ will be a Riemann integrable function defined on the finite interval $[a, b]$ and $\mathcal{F}(g)$ will be its *Finite Fourier Transform.* That is,

$$\mathcal{F}(f)(t) := \int_a^b f(s)\,e^{-2\pi i t s}\,ds.$$

Consider also the *exponential mean* of two complex numbers z, w defined by

$$E(z, w) := \begin{cases} \frac{e^z - e^w}{z - w} & \text{if } z \neq w, \\ \exp(w) & \text{if } z = w, \end{cases} \qquad z, w \in \mathbb{C}.$$

Now, for $w(s) = \exp(-2\pi its)$, on applying the identity (2.9), which holds for complex-valued functions as well, we get the following representation of the Finite Fourier Transform

$$\mathcal{F}(f)(t) = \frac{p}{(b-a)^{p-1}} \cdot E(-2\pi itb, -2\pi ita) \cdot \int_a^b (s-a)^{p-1} f(s)\, ds + \mathcal{R}(f)(t), \tag{5.1}$$

where the remainder $\mathcal{R}(f)$ has the representation

$$\begin{aligned}\mathcal{R}(f)(t) &= \frac{1}{(b-a)^{p-1}} \int_a^b (s-a) \Big[(s-a)^{p-1} E(-2\pi itb, -2\pi ita) \\ &\quad - (b-a)^{p-1} E(-2\pi its, -2\pi ita) \Big] df(s).\end{aligned}$$

In order to provide a composite rule in approximating the Finite Fourier Transform in terms of moments for the function f, we consider a division $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and the quadrature rule

$$\begin{aligned}A(I_n, t) &:= \sum_{i=0}^{n-1} \frac{p}{(x_{i+1} - x_i)^{p-1}} \cdot E(-2\pi itx_{i+1}, -2\pi itx_i) \\ &\quad \times \int_{x_i}^{x_{i+1}} (s - x_i)^{p-1} f(s)\, ds\end{aligned} \tag{5.2}$$

that has been obtained from (5.1) applied on each subinterval $[x_i, x_{i+1}]$ and the results were summed over i from 0 to $n-1$. It is an open question as to whether or not $A(I_n, t)$ is uniformly convergent to $\mathcal{F}(f)(t)$ on $[a, b]$ and what the order of convergence is ?

The following numerical experiment obtained by implementing the quadrature (5.2) for the function $f(s) = s + 1$ and $p = 2$ shows the behavior of the absolute error value

$$E_n(t) := |\mathcal{F}(f)(t) - A(I_n, t)|, t \in [a, b], \tag{5.3}$$

for a division with 10 points (Figure 1) respectively 100 points (Figure 2).

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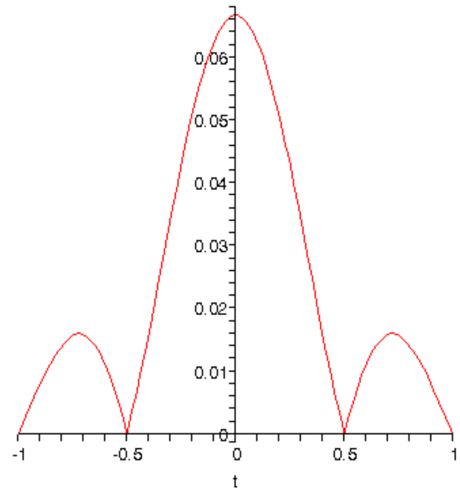

FIGURE 1. The behaviour of $E_n(t)$ from (5.3) for $n = 10$.

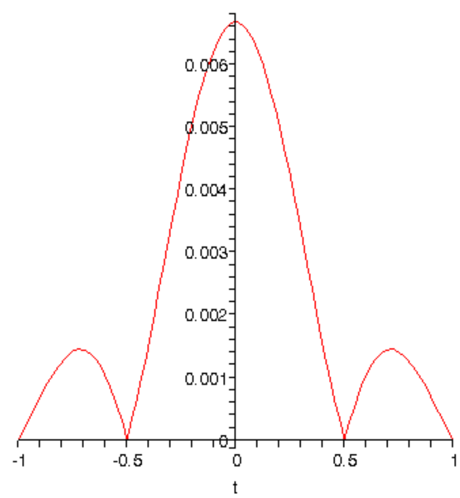

FIGURE 2. The behaviour of $E_n(t)$ from (5.3) for $n = 100$.

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