HARDY-TYPE INEQUALITIES VIA AUXILIARY SEQUENCES

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ABSTRACT. We prove some Hardy-type inequalities via an approach that involves constructing auxiliary sequences.

1. Introduction

Suppose throughout that $p \neq 0, \frac{1}{p}+\frac{1}{q}=1$. Let l^p be the Banach space of all complex sequences $\mathbf{a}=(a_n)_{n\geq 1}$ with norm

$$||\mathbf{a}|| := (\sum_{n=1}^{\infty}|a_n|^p)^{1/p} < \infty.$$

The celebrated Hardy's inequality ([5, Theorem 326]) asserts that for $p>1$,

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n}\sum_{k=1}^{n}a_k\Big|^p \leq (\frac{p}{p-1})^p\sum_{k=1}^{\infty}|a_k|^p. \tag{1.1}$$

Hardy's inequality can be regarded as a special case of the following inequality:

$$\sum_{j=1}^{\infty}\Big|\sum_{k=1}^{\infty}c_{j,k}a_k\Big|^p \leq U\sum_{k=1}^{\infty}|a_k|^p,$$

in which $C=(c_{j,k})$ and the parameter p are assumed fixed ($p>1$), and the estimate is to hold for all complex sequences $\mathbf{a}$. The l^p operator norm of C is then defined as the p-th root of the smallest value of the constant U:

$$||C||_{p,p}=U^{\frac{1}{p}}.$$

Hardy's inequality thus asserts that the Cesáro matrix operator C, given by $c_{j,k}=1/j, k\leq j$ and 0 otherwise, is bounded on l^p and has norm $\leq p/(p-1)$. (The norm is in fact $p/(p-1)$.)

We say a matrix A is a summability matrix if its entries satisfy: $a_{j,k}\geq 0$, $a_{j,k}=0$ for $k>j$ and $\sum_{k=1}^{j}a_{j,k}=1$. We say a summability matrix A is a weighted mean matrix if its entries satisfy:

$$a_{j,k}=\lambda_k/\Lambda_j,\ 1\leq k\leq j; \Lambda_j=\sum_{i=1}^{j}\lambda_i, \lambda_i\geq 0, \lambda_1>0.$$

Hardy's inequality (1.1) now motivates one to determine the l^p operator norm of an arbitrary summability matrix A. For examples, the following two inequalities were claimed to hold by Bennett ([1, p. 40-41]; see also [2, p. 407]):

$$\sum_{n=1}^{\infty}\Big|\frac{1}{n^{\alpha}}\sum_{i=1}^{n}(i^{\alpha}-(i-1)^{\alpha})a_i\Big|^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p, \tag{1.2}$$

$$\sum_{n=1}^{\infty}\Big|\frac{1}{\sum_{i=1}^{n}i^{\alpha-1}}\sum_{i=1}^{n}i^{\alpha-1}a_i\Big|^p \leq \Big(\frac{\alpha p}{\alpha p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p, \tag{1.3}$$

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whenever $\alpha > 0, p > 1, \alpha p > 1$.

No proofs of the above two inequalities were supplied in [1]-[2] and recently, the author [4] and Bennett himself [3] proved inequalities (1.2) for $p > 1, \alpha \geq 1, \alpha p > 1$ and (1.3) for $p > 1, \alpha \geq 2$ or $0 < \alpha \leq 1, \alpha p > 1$ independently.

We point out here that Bennett in fact was able to prove (1.2) for $p \geq 1, \alpha > 0, \alpha p > 1$ (see [3, Theorem 1] with $\beta = 1$ there) which now leaves the case $p > 1, 1 < \alpha < 2$ of inequality (1.3) the only case open to us. For this, Bennett expects inequality (1.3) to hold for $1 + 1/p < \alpha < 2$ (see page 830 of [3]) and as a support, Bennett [3, Theorem 18] has shown that inequality (1.3) holds for $\alpha = 1 + 1/p, p \geq 1$.

In this paper, we will study inequality (1.3) using a method of Knopp [6] which involves constructing auxiliary sequences. We will partially resolve the remaining case $p > 1, 1 < \alpha < 2$ of inequality (1.3) by proving in Section 2 the following:

Theorem 1.1. *Inequality* (1.3) *holds for* $p \geq 2, 1 \leq \alpha \leq 1 + 1/p$ *or* $1 < p \leq 4/3, 1 + 1/p \leq \alpha \leq 2$.

We shall leave the explanation of Knopp's approach in detail in Section 2 by pointing out here that it can be applied to prove other types of inequalities similar to that of Hardy's. As an example, we note that Theorem 359 of [5] states:

Theorem 1.2. *For* $0 < p < 1$ *and* $a_n \geq 0$*,*

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=n}^{\infty} a_k \Big)^p \geq p^p \sum_{n=1}^{\infty} a_n^p.$$

The constant p^p in Theorem 1.2 is not best possible and this was fixed by Levin and Stečkin [7, Theorem 61] for $0 < p \leq 1/3$ in the following

Theorem 1.3. *For* $0 < p \leq 1/3$ *and* $a_n \geq 0$*,*

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=n}^{\infty} a_k \Big)^p \geq \Big(\frac{p}{1-p} \Big)^p \sum_{n=1}^{\infty} a_n^p.$$

We shall give another proof of this result in Section 3 using Knopp's approach. We point out here for each $1/3 < p < 1$, Levin and Stečkin also gave a better constant than the one p^p given in Theorem 1.2. For example, when $p = 1/2$, they gave $\sqrt{3}/2$ instead of $1/\sqrt{2}$. In Section 4, we shall further improve this constant by proving the following

Theorem 1.4. *For* $a_n \geq 0$*,*

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n} \sum_{k=n}^{\infty} a_k \Big)^{1/2} \geq \sqrt{7}/3 \sum_{n=1}^{\infty} a_n^{1/2}.$$

In our proofs of Theorems 1.1-1.2, certain auxiliary sequences are constructed and there can be many ways to construct such sequences. In Section 5, we give an example regarding these possibilities by answering a question of Bennett.

2. Proof of Theorem 1.1

We begin this section by explaining Knopp's idea [6] on proving Hardy's inequality (1.1). In fact, we will explain this more generally for the case involving weighted mean matrices. For real numbers $\lambda_1 > 0, \lambda_i \geq 0, i \geq 2$, we write $\Lambda_n = \sum_{i=1}^{n} \lambda_i$ and we are looking for a positive constant U such that

$$\sum_{n=1}^{\infty} \Big| \frac{1}{\Lambda_n} \sum_{k=1}^{n} \lambda_k a_k \Big|^p \leq U \sum_{k=1}^{\infty} |a_k|^p \tag{2.1}$$

holds for all complex sequences $\mathbf{a}$ with $p > 1$ being fixed. Knopp's idea is to find an auxiliary sequence $\mathbf{w} = \{w_i\}_{i=1}^{\infty}$ of positive terms such that by Hölder's inequality,

$$\begin{aligned}\Big(\sum_{k=1}^{n}\lambda_k|a_k|\Big)^p &= \Big(\sum_{k=1}^{n}\lambda_k|a_k|w_k^{-\frac{1}{p^*}}\cdot w_k^{\frac{1}{p^*}}\Big)^p \\ &\leq \Big(\sum_{k=1}^{n}\lambda_k^p|a_k|^p w_k^{-(p-1)}\Big)\Big(\sum_{j=1}^{n}w_j\Big)^{p-1}\end{aligned}$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty}\Big|\frac{1}{\Lambda_n}\sum_{k=1}^{n}\lambda_k a_k\Big|^p &\leq \sum_{n=1}^{\infty}\frac{1}{\Lambda_n^p}\Big(\sum_{k=1}^{n}\lambda_k^p|a_k|^p w_k^{-(p-1)}\Big)\Big(\sum_{j=1}^{n}w_j\Big)^{p-1} \\ &= \sum_{k=1}^{\infty}w_k^{-(p-1)}\lambda_k^p\Big(\sum_{n=k}^{\infty}\frac{1}{\Lambda_n^p}\Big(\sum_{j=1}^{n}w_j\Big)^{p-1}\Big)|a_k|^p.\end{aligned}$$

Suppose now one can find for each $p > 1$ a positive constant U, a sequence $\mathbf{w}$ of positive terms with w_n^{p-1}/λ_n^p decreasing to 0, such that for any integer $n \geq 1$,

$$(w_1 + \cdots + w_n)^{p-1} < U\Lambda_n^p\Big(\frac{w_n^{p-1}}{\lambda_n^p} - \frac{w_{n+1}^{p-1}}{\lambda_{n+1}^p}\Big), \tag{2.2}$$

then it is easy to see that inequality (2.1) follows from this. When $\lambda_n = 1$ for all n, Knopp's choice for $\mathbf{w}$ is given by $w_n = \binom{n-1-1/p}{n-1}$ and one can show that (2.2) holds in this case with $U = (p^*)^p$ and Hardy's inequality (1.1) follows from this.

We now want to apply Knopp's approach to prove Theorem 1.1. For this, we replace $\alpha - 1$ by α and rewrite (1.3) as

$$\sum_{n=1}^{\infty}\Big|\frac{1}{\sum_{i=1}^{n}i^{\alpha}}\sum_{i=1}^{n}i^{\alpha}a_i\Big|^p \leq \Big(\frac{(\alpha+1)p}{(\alpha+1)p-1}\Big)^p\sum_{n=1}^{\infty}|a_n|^p.$$

Note that we are interested in the case $0 \leq \alpha \leq 1$ here. From our discussions above, we are looking for a sequence $\mathbf{w}$ of positive terms with w_n^{p-1}/λ_n^p decreasing to 0, such that for any integer $n \geq 1$,

$$(w_1 + \cdots + w_n)^{p-1} < \Big(\frac{(\alpha+1)p}{(\alpha+1)p-1}\Big)^p\Big(\sum_{i=1}^{n}i^{\alpha}\Big)^p\Big(\frac{w_n^{p-1}}{n^{\alpha p}} - \frac{w_{n+1}^{p-1}}{(n+1)^{\alpha p}}\Big). \tag{2.3}$$

Following Knopp's choice, we define a sequence $\mathbf{w}$ such that

$$w_{n+1} = \frac{n+\alpha-1/p}{n}w_n, \quad n \geq 1. \tag{2.4}$$

Note that the above sequence is uniquely determined for any given positive w_1 and therefore we may assume $w_1 = 1$ here. We note further that we need $\alpha > -1/p^*$ in order for $w_n > 0$ for all n and we also point out that it is easy to show by induction that

$$\sum_{i=1}^{n}w_i = \frac{n+\alpha-1/p}{1+\alpha-1/p}w_n. \tag{2.5}$$

Moreover, one can easily check that

$$\frac{w_n^{p-1}}{n^{\alpha p}} = O(n^{-\alpha-1/p^*}),$$

so that w_n^{p-1}/λ_n^p decreases to 0 as n approaches infinity as long as $\alpha > -1/p^*$.

Now we need a lemma on sums of powers, which is due to Levin and Stečkin [7, Lemma 1, 2, p.18]:

Lemma 2.1. *For an integer $n \geq 1$,*

$$\sum_{i=1}^{n} i^r \geq \frac{1}{r+1} n(n+1)^r, \quad 0 \leq r \leq 1, \tag{2.6}$$

$$\sum_{i=1}^{n} i^r \geq \frac{r}{r+1} \frac{n^r(n+1)^r}{(n+1)^r - n^r}, \quad r \geq 1. \tag{2.7}$$

Inequality (2.7) *reverses when* $-1 < r \leq 1$.

We note here only the case $r \geq 0$ for (2.7) was proved in [7] but one checks easily that the proof extends to the case $r > -1$.

As we are interested in $0 \leq \alpha \leq 1$ here, we can now combine (2.4)-(2.6) to deduce that inequality (2.3) will follow from

$$(1 + \frac{\alpha - 1/p}{n})^{p-1} < \frac{n}{1+\alpha-1/p}\Big((1+\frac{1}{n})^{\alpha p} - (1+\frac{\alpha-1/p}{n})^{p-1}\Big).$$

We can simplify the above inequality further by recasting it as

$$\Big(1 + \frac{\alpha + 1/p^*}{n}\Big)^{1/p}\Big(1 + \frac{\alpha - 1/p}{n}\Big)^{1/p^*} < \Big(1+\frac{1}{n}\Big)^{\alpha}. \tag{2.8}$$

Now we define for fixed $n \geq 1, p > 1$,

$$f(x) = x\ln(1+1/n) - \frac{1}{p}\ln(1+\frac{x+1/p^*}{n}) - \frac{1}{p^*}\ln(1+\frac{x-1/p}{n}).$$

It is easy to see here that inequality (2.8) is equivalent to $f(\alpha) > 0$. It is also easy to see that $f(x)$ is a convex function of x for $0 \leq x \leq 1$ and that $f(1/p) = 0$. It follows from this that if $f'(1/p) \leq 0$ then $f(x) > 0$ for $0 \leq x < 1/p$ and if $f'(1/p) \geq 0$ then $f(x) > 0$ for $1/p < x \leq 1$. We have

$$f'(1/p) = \ln(1+1/n) - \frac{1}{n} + \frac{1}{pn(n+1)}.$$

We now use Taylor expansion to conclude for $x > 0$,

$$x - x^2/2 < \ln(1+x) < x - x^2/2 + x^3/3. \tag{2.9}$$

It follows from this that for $p \geq 2$, $n \geq 2$,

$$f'(1/p) < -\frac{1}{2n^2} + \frac{1}{3n^3} + \frac{1}{pn(n+1)} \leq -\frac{1}{2n^2} + \frac{1}{3n^3} + \frac{1}{2n(n+1)} = \frac{1}{3n^3} - \frac{1}{2n^2(n+1)} \leq 0.$$

and for $n = 1$,

$$f'(1/p) = \ln 2 - 1 + \frac{1}{2p} \leq \ln 2 - 1 + \frac{1}{4} < 0,$$

It's also easy to check that for $1 < p \leq 4/3$, $n = 1$,

$$f'(1/p) = \ln 2 - 1 + \frac{1}{2p} > 0.$$

For $n \geq 2, 1 < p \leq 4/3$, by using the first inequality of (2.9) we get

$$f'(1/p) > -\frac{1}{2n^2} + \frac{1}{pn(n+1)} \geq 0.$$

This now enables us to conclude the proof of Theorem 1.1.

3. Another Proof of Theorem 1.3

We use the idea of Levin and Stečkin in the proof of Theorem 62 in [7] to find an auxiliary sequence $\mathbf{w} = \{w_i\}_{i=1}^{\infty}$ of positive terms so that for any finite summation from $n = 1$ to N with $N \geq 1$, we have

$$\sum_{n=1}^{N} a_n^p = \sum_{n=1}^{N} \frac{a_n^p}{\sum_{i=1}^{n} w_i} \sum_{k=1}^{n} w_k = \sum_{n=1}^{N} w_n \sum_{k=n}^{N} \frac{a_k^p}{\sum_{i=1}^{k} w_i}.$$

On letting $N \to \infty$, we then have

$$\sum_{n=1}^{\infty} a_n^p = \sum_{n=1}^{\infty} w_n \sum_{k=n}^{\infty} \frac{a_k^p}{\sum_{i=1}^{k} w_i}.$$

By Hölder's inequality, we have

$$\sum_{k=n}^{\infty} \frac{a_k^p}{\sum_{i=1}^{k} w_i} \leq \Big(\sum_{k=n}^{\infty} \Big(\sum_{i=1}^{k} w_i\Big)^{-1/(1-p)}\Big)^{1-p} \Big(\sum_{k=n}^{\infty} a_k\Big)^p.$$

Suppose now one can find a sequence $\mathbf{w}$ of positive terms with $w_n^{-1/(1-p)} n^{-p/(1-p)}$ decreasing to 0 for each $0 < p \leq 1/3$, such that for any integer $n \geq 1$,

$$(w_1 + \cdots + w_n)^{-1/(1-p)} \leq \Big(\frac{1-p}{p}\Big)^{p/(1-p)} \Big(\frac{w_n^{-1/(1-p)}}{n^{p/(1-p)}} - \frac{w_{n+1}^{-1/(1-p)}}{(n+1)^{p/(1-p)}}\Big), \tag{3.1}$$

then it is easy to see that Theorem 1.3 follows from this.

We now define our sequence $\mathbf{w}$ to be

$$w_{n+1} = \frac{n + 1/p - 2}{n} w_n, \quad n \geq 1. \tag{3.2}$$

Note that the above sequence is uniquely determined for any given positive w_1 and therefore we may assume $w_1 = 1$ here. We note further that $w_n > 0$ for all n as $0 < p \leq 1/3$ and it is easy to show by induction that

$$\sum_{i=1}^{n} w_i = \frac{n + 1/p - 2}{1/p - 1} w_n. \tag{3.3}$$

Moreover, one can easily check that

$$\frac{w_n^{-1/(1-p)}}{n^{p/(1-p)}} = O(n^{-(1-p)/p}),$$

so that $w_n^{-1/(p-1)} n^{-p/(1-p)}$ decreases to 0 as n approaches infinity.

We now combine (3.2)-(3.3) to recast inequality (3.1) as

$$(n + 1/p - 2)^{-1/(1-p)} \leq \frac{p}{1-p}\Big(n^{-p/(1-p)} - (n+1)^{-p/(1-p)} n^{1/(1-p)} (n + 1/p - 2)^{-1/(1-p)}\Big).$$

We further rewrite the above inequality as

$$\begin{aligned} \frac{1-p}{p} &\leq n^{-p/(1-p)}(n + 1/p - 2)^{1/(1-p)} - (n+1)^{-p/(1-p)} n^{1/(1-p)} \\ &= n\Big(\Big(1 + \frac{1/p - 2}{n}\Big)^{1/(1-p)} - \Big(1 + \frac{1}{n}\Big)^{-p/(1-p)}\Big). \end{aligned}$$

It is easy to see that the above inequality follows from $f(1/n) \geq 0$ where we define for $x \geq 0$,

$$f(x) = \Big(1 + (1/p - 2)x\Big)^{1/(1-p)} - \Big(1 + x\Big)^{-p/(1-p)} - \frac{1-p}{p} x.$$

We now prove that $f(x) \geq 0$ for $x \geq 0$ for $0 < p \leq 1/3$ and this will conclude the proof of Theorem 1.3. We note that

$$\begin{aligned} f'(x) &= \frac{1/p-2}{1-p}\Big(1+(1/p-2)x\Big)^{p/(1-p)}+\frac{p}{1-p}\Big(1+x\Big)^{-p/(1-p)-1}-\frac{1-p}{p}, \\ f''(x) &= \frac{p(1/p-2)^2}{(1-p)^2}\Big(1+(1/p-2)x\Big)^{p/(1-p)-1}-\frac{p}{(1-p)^2}\Big(1+x\Big)^{-p/(1-p)-2}. \end{aligned}$$

We now define for $x \geq 0$,

$$g(x) = (1/p-2)^{2(1-p)/(1-2p)}(1+x)^{(2-p)/(1-2p)} - (1+(1/p-2)x).$$

It is easy to see that $g(x) \geq 0$ implies $f''(x) \geq 0$. Note that $(2-p)/(1-2p) \geq 1$ so that

$$\begin{aligned} g'(x) &= (1/p-2)^{2(1-p)/(1-2p)}(2-p)/(1-2p)(1+x)^{(2-p)/(1-2p)-1} - (1/p-2) \\ &\geq (1/p-2)^{2(1-p)/(1-2p)} - (1/p-2) \geq 0, \end{aligned}$$

where the last inequality above follows from $2(1-p)/(1-2p) \geq 1$ and $0 < p \leq 1/3$ so that $1/p - 2 \geq 1$. It follows from this that $f''(x) \geq 0$ and as one checks easily that $f'(0) = 0$, which implies $f'(x) \geq 0$ so that $f(x) \geq f(0) = 0$ which is just what we want to prove.

4. Proof of Theorem 1.4

We follow our strategy in the previous section to look for a sequence $\mathbf{w}$ of positive terms with $w_n^{-2}n^{-1}$ decreasing to 0, such that for any integer $n \geq 1$,

$$(w_1 + \cdots + w_n)^{-2} \leq C\Big(\frac{w_n^{-2}}{n} - \frac{w_{n+1}^{-2}}{(n+1)}\Big), \tag{4.1}$$

then it is easy to see that Theorem 1.4 with $\sqrt{7}/3$ replaced by $C^{-1/2}$ follows from this.

We now define our sequence $\mathbf{w}$ to be

$$w_{n+1} = \frac{n+\alpha}{n}w_n, \quad n \geq 1. \tag{4.2}$$

Here $\alpha > -1/2$ is our parameter so that we hope to optimize the constant C in (4.1) later by choosing a suitable α. Similar to our treatment in the previous section, we let $w_1 = 1$ here and note that $w_n > 0$ for all n and that

$$\sum_{i=1}^{n} w_i = \frac{n+\alpha}{1+\alpha}w_n. \tag{4.3}$$

Moreover, one can easily check that

$$\frac{w_n^{-2}}{n} = O(n^{-1-2\alpha}),$$

so that $w_n^{-2}n^{-1}$ decreases to 0 as n approaches infinity.

We now combine (4.2)-(4.3) to recast inequality (4.1) as

$$\frac{(1+\alpha)^2}{C} \leq \frac{(n+\alpha)^2}{n} - \frac{n^2}{n+1}.$$

Or equivalently,

$$(1+2\alpha)n^2 + (2\alpha+\alpha^2)n + \alpha^2 \geq \frac{(1+\alpha)^2}{C}n(n+1). \tag{4.4}$$

We point out here that the choice $\alpha = 0$ will lead to $C \geq 2$ in order for inequality (4.4) to hold for all $n \geq 1$, which corresponds Theorem 1.2 with $\sqrt{7}/3$ replaced by $2^{-1/2}$. The choice $\alpha = 1$ will lead to $C \geq 4/3$ in order for inequality (4.4) to hold for all $n \geq 1$, which corresponds Theorem 1.2 with

$\sqrt{7}/3$ replaced by $\sqrt{3}/2$. We now choose $\alpha = 1/2$ here with $C = 9/7$ and one checks readily that inequality (4.4) holds for such choices and this leads to the desired constant $\sqrt{7}/3$ in Theorem 1.2.

5. Another look at Inequality (1.3)

In this section we return to the consideration of inequality (1.3) via our approach in Section 2, which boils down to a construction of a sequence $\mathbf{w}$ of positive terms with w_n^{p-1}/λ_n^p decreasing to 0, such that for any integer $n \geq 1$, inequality (2.3) is satisfied. Certainly here the choice for $\mathbf{w}$ may not be unique and in fact in the case $\alpha = 0$, Bennett asked in [1] (see the paragraph below Lemma 4.11) for other sequences, not multiples of Knopp's, that satisfy (2.3). He also mentioned that the obvious choice, $w_n = n^{-1/p}$, does not work.

We point out here even though the choice $w_n = n^{-1/p}$ does not satisfy (2.3) when $\alpha = 0$ for all $p > 1$, as one can see by considering inequality (2.3) for the case $n = 1$ with $p \to 1^+$, it nevertheless works for $p \geq 3$, which we now show by first rewriting (2.3) in our case as

$$\Big(\sum_{i=1}^{n} i^{-1/p}\Big)^{p-1} < \Big(\frac{p}{p-1}\Big)^{p} n^{p}\Big(n^{-(p-1)/p} - (n+1)^{-(p-1)/p}\Big). \tag{5.1}$$

We note that the case $n = 1$ of (5.1) follows from the case $\alpha = 0$ of the following inequality,

$$1 - 2^{-(p-1)/p-\alpha} > \Big(1 - \frac{1}{(\alpha+1)p}\Big)^{p}, \quad 0 \leq \alpha \leq 1/p. \tag{5.2}$$

To show (5.2), we see by Taylor expansion, that for $p \geq 2, x < 0$,

$$(1+x)^p < 1 + px + \frac{p(p-1)x^2}{2}.$$

Apply the above inequality with $x = -1/(\alpha p + p)$, we obtain for $p \geq 3$,

$$\Big(1 - \frac{1}{(\alpha+1)p}\Big)^{p} < 1 - \frac{1}{(\alpha+1)} + \frac{(p-1)}{2(\alpha+1)^2 p}.$$

Hence inequality (5.2) will follow from

$$1 - \frac{p-1}{2(\alpha+1)p} - 2^{-(p-1)/p}\frac{(\alpha+1)}{2^{\alpha}} > 0.$$

It is easy to see that when $p \geq 3$, the function $\alpha \mapsto (1+\alpha)2^{-\alpha}$ is an increasing function of α for $0 \leq \alpha \leq 1/p$. It follows from this that for $0 \leq \alpha \leq 1/p$,

$$1 - \frac{p-1}{2(\alpha+1)p} - 2^{-(p-1)/p}\frac{(\alpha+1)}{2^{\alpha}} > 1 - \frac{p-1}{2p} - 2^{-(p-1)/p}\frac{(1/p+1)}{2^{1/p}} = 0,$$

and from which inequality (5.2) follows.

Now, to show (5.1) holds for all $n \geq 2, p \geq 3$, we first note that for $p > 1$,

$$\sum_{i=1}^{n} i^{-1/p} < 1 + \int_1^n x^{-1/p} dx = \frac{p}{p-1} n^{1-1/p} - \frac{1}{p-1}.$$

On the other hand, by Hadamard's inequality, which asserts that for a continuous convex function $f(x)$ on $[a, b]$,

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

we have for $p > 1$,

$$n^{-(p-1)/p} - (n+1)^{-(p-1)/p} = \frac{p-1}{p}\int_n^{n+1} x^{-1-1/p^*} dx \geq \frac{p-1}{p}(n+1/2)^{-1-1/p^*}.$$

Hence inequality (5.1) will follow from the following inequality for $n \geq 2$,

$$\frac{p}{p-1}n^{1-1/p}-\frac{1}{p-1} \leq p^*n^{1/p^*}\Big(1+\frac{1}{2n}\Big)^{-(1+p^*)/p}.$$

It is easy to see that for $p>1$,

$$\Big(1+\frac{1}{2n}\Big)^{-(1+p^*)/p} \geq 1-\frac{1+p^*}{p}\frac{1}{2n}.$$

Hence it suffices to show

$$\frac{p}{p-1}n^{1-1/p}-\frac{1}{p-1} \leq p^*n^{1/p^*}\Big(1-\frac{1+p^*}{p}\frac{1}{2n}\Big),$$

or equivalently,

$$\Big(1+\frac{1}{2p-2}\Big)^{p} \leq n.$$

It's easy to check that the right-hand expression above is a decreasing function of $p \geq 3$ and is equal to $5^3/4^3 < 2$ when $p=3$. Hence it follows that (5.1) holds for all $n \geq 2, p \geq 3$.

We consider lastly inequality (2.3) for other values of α and we take $w_n = n^{\alpha-1/p}$ for $n \geq 1$ so that we can rewrite (2.3) as

$$\Big(\sum_{i=1}^n i^{\alpha-1/p}\Big)^{p-1} < \Big(\frac{(\alpha+1)p}{(\alpha+1)p-1}\Big)^p\Big(\sum_{i=1}^n i^{\alpha}\Big)^p\Big(n^{-(p-1)/p-\alpha}-(n+1)^{-(p-1)/p-\alpha}\Big). \tag{5.3}$$

We end our discussion here by considering the case $1 \leq \alpha \leq 1+1/p$ and we apply Lemma 2.1 to obtain

$$\begin{aligned}\sum_{i=1}^n i^{\alpha-1/p} &\leq \frac{\alpha-1/p}{\alpha-1/p+1}\frac{n^{\alpha-1/p}(n+1)^{\alpha-1/p}}{(n+1)^{\alpha-1/p}-n^{\alpha-1/p}}=\frac{1}{\alpha-1/p+1}\Big(\int_n^{n+1}x^{-\alpha+1/p-1}dx\Big)^{-1},\\ \sum_{i=1}^n i^{\alpha} &\geq \frac{\alpha}{\alpha+1}\frac{n^{\alpha}(n+1)^{\alpha}}{(n+1)^{\alpha}-n^{\alpha}}=\frac{1}{\alpha+1}\Big(\int_n^{n+1}x^{-\alpha-1}dx\Big)^{-1}\end{aligned}$$

We further write

$$n^{-(p-1)/p-\alpha}-(n+1)^{-(p-1)/p-\alpha}=(\alpha-1/p+1)\int_n^{n+1}x^{-\alpha+1/p-2}dx,$$

so that inequality (5.3) will follow from

$$\int_n^{n+1}x^{-\alpha-1}dx < \Big(\int_n^{n+1}x^{-\alpha+1/p-1}dx\Big)^{1-1/p}\Big(\int_n^{n+1}x^{-\alpha+1/p-2}dx\Big)^{1/p}.$$

One can easily see that the above inequality holds by Hölder's inequality and it follows that inequality (5.3) holds for $p>1, 1 \leq \alpha \leq 1+1/p$. This provides another proof of inequality (1.3) for $p>1, 1 \leq \alpha \leq 1+1/p$.

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