A COMPLETELY MONOTONIC FUNCTION INVOLVING DIVIDED DIFFERENCE OF PSI FUNCTION AND AN EQUIVALENT INEQUALITY INVOLVING SUM

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ABSTRACT. In this paper, a function involving the divided difference of the psi function is proved to be completely monotonic, a class of inequalities involving sum are founded, and an equivalent relation between the complete monotonicity and one of the class of inequalities is established.

1. Introduction

It is well known that the classical Euler's gamma function $\Gamma(x)$ is one of the most important special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called psi or digamma function, and the derivatives $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ of the psi function $\psi(x)$ are known as the polygamma or multigamma functions.

Recall [10] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $0 \le (-1)^n f^{(n)}(x) < \infty$ for $x \in I$ and $n \ge 0$, and that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \le (-1)^k [\ln f(x)]^{(k)} < \infty$ for all $k \in \mathbb{N}$ on I. The set of the completely monotonic functions on I is denoted by $\mathcal{C}[I]$ and the set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. It is proved in [3, 10, 13, 15] that $\mathcal{L}[I] \subset \mathcal{C}[I]$. The well known Bernstein's Theorem [16, p. 161] states that $f \in \mathcal{C}[(0,\infty)]$ if and only if $f(x) = \int_0^\infty e^{-xs}\,\mathrm{d}\mu(s)$, where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. This expresses that $f \in \mathcal{C}[(0,\infty)]$ is a Laplace transform of the measure μ. In [3, Theorem 1.1] and [8, 13] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions. In the recent past, numerous functions, which are defined in terms of gamma, polygamma, and other special functions, are proved to be (logarithmically) completely monotonic and this fact is used to derive many interesting new inequalities (See, for examples, [1, 3, 4, 5, 6, 8, 10, 13, 14, 15] and the references therein).

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The Kershaw's inequality [9] states that

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s} \tag{1}$$

for $0<s<1$ and $x\ge 1$, which improved the corresponding result in [7]. In [4, 6, 11, 14], in order to obtain the best upper and lower bounds for double inequality (1), the monotonicity and convexity properties of the function

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x \tag{2}$$

in $x\in(-\alpha,\infty)$ is verified, where s and t are nonnegative numbers and $\alpha=\min\{s,t\}$. In [4, 6, 11, 14], in order to prove the monotonicity and convexity of (2), the function

$$\delta_{s,t}(x)=\begin{cases}\dfrac{\psi(x+t)-\psi(x+s)}{t-s}-\dfrac{2x+s+t+1}{2(x+s)(x+t)}, & s\ne t\\ \psi'(x+s)-\dfrac{1}{x+s}-\dfrac{1}{2(x+s)^2}, & s=t\end{cases} \tag{3}$$

in $x\in(-\alpha,\infty)$, involving the divided difference of the psi function, is derived. Recently, the positivity of the function $\delta_{0,0}(x)=\psi'(x)-1/x-1/2x^2$ was proved in [2, 5, 12] respectively.

The first aim of this paper is to prove the complete monotonicity property of the function $\delta_{s,t}(x)$.

Theorem 1. *Let s and t be nonnegative numbers and $\alpha=\min\{s,t\}$. Then the functions $\delta_{s,t}(x)$ for $|t-s|<1$ and $-\delta_{s,t}(x)$ for $|t-s|>1$ are completely monotonic in $x\in(-\alpha,\infty)$.*

The second aim of this paper is to found a class of inequalities involving sum.

Theorem 2. *Let k be a nonnegative integer and $\theta>0$ a constant.*

If $a>0$ and $b>0$, then

$$\sum_{i=0}^{k}\frac{1}{(a+\theta)^{i+1}(b+\theta)^{k-i+1}}+\sum_{i=0}^{k}\frac{1}{a^{i+1}b^{k-i+1}}>2\sum_{i=0}^{k}\frac{1}{(a+\theta)^{i+1}b^{k-i+1}} \tag{4}$$

holds for $b-a>-\theta$ and reveres for $b-a<-\theta$.

If $a<-\theta$ and $b<-\theta$, then inequalities

$$\sum_{i=0}^{2k}\frac{1}{(a+\theta)^{i+1}(b+\theta)^{2k-i+1}}+\sum_{i=0}^{2k}\frac{1}{a^{i+1}b^{2k-i+1}}>2\sum_{i=0}^{2k}\frac{1}{(a+\theta)^{i+1}b^{2k-i+1}} \tag{5}$$

and

$$\sum_{i=0}^{2k+1}\frac{1}{(a+\theta)^{i+1}(b+\theta)^{2k-i+2}}+\sum_{i=0}^{2k+1}\frac{1}{a^{i+1}b^{2k-i+2}}<2\sum_{i=0}^{2k+1}\frac{1}{(a+\theta)^{i+1}b^{2k-i+2}} \tag{6}$$

hold for $b-a>-\theta$ and reverse for $b-a<-\theta$.

If $-\theta<a<0$ and $-\theta<b<0$, then inequality (5) holds and inequality (6) is valid for $a+b+\theta>0$ and is reversed for $a+b+\theta<0$.

If $a<-\theta$ and $b>0$, then inequality (5) holds and inequality (6) is valid for $a+b+\theta>0$ and is reversed for $a+b+\theta<0$.

If $a > 0$ and $b < -\theta$, then inequality (5) *is reversed and inequality* (6) *holds for $a + b + \theta < 0$ and reverses for $a + b + \theta > 0$.*

If $b = a - \theta$, then inequalities (4), (5) *and* (6) *become equalities.*

Remark 1. The following shadows in Figure 1 describes the domains on the plane aOb such that Theorem 2 being valid.

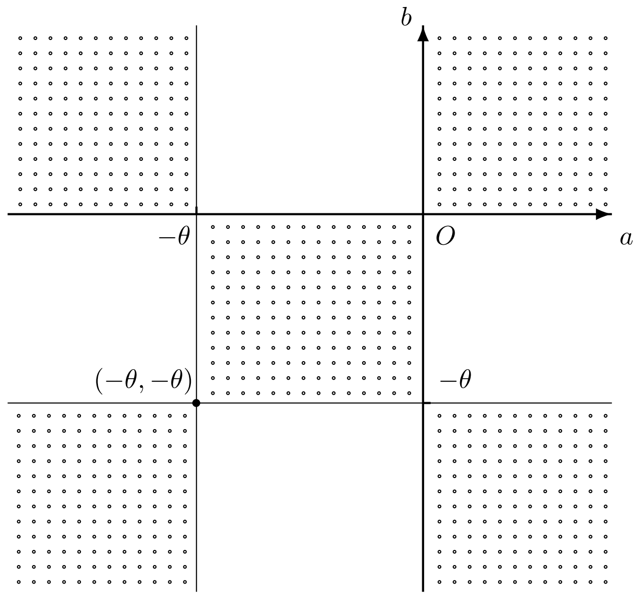

FIGURE 1.

Remark 2. Hinted by Theorem 2, the following open problem is posed: Let k be a nonnegative integer and θ a positive constant, discuss the validity of the inequality

$$\frac{(a+b)^k}{a^{k+1}b^{k+1}} + \frac{(a+b+2\theta)^k}{(a+\theta)^{k+1}(b+\theta)^{k+1}} > \frac{2(a+b+\theta)^k}{(a+\theta)^{k+1}b^{k+1}} \tag{7}$$

or its reverse for $a, b \notin \{0, -\theta\}$.

The final aim of this article is to establish an equivalent relationship between Theorem 1 and inequality (4) for positive numbers a and b.

Theorem 3. *Inequality* (4) *for positive numbers a and b is equivalent to Theorem 1.*

2. Proofs of theorems

The basic tool of this paper is the following lemma.

Lemma 1. *Let $f(x)$ be defined in an infinite interval I. If $\lim_{x\to\infty} f(x) = 0$ and $f(x) - f(x+\varepsilon) > 0$ for any given $\varepsilon > 0$, then $f(x) > 0$ in I.*

Proof. By induction, for any $x \in I$, we have

$$f(x) > f(x+\varepsilon) > f(x+2\varepsilon) > \cdots > f(x+k\varepsilon) \to 0$$

as $k \to \infty$. The proof of Lemma 1 is complete. □

2.1. **Proof of Theorem 1.**

2.1.1. *The case* $\delta_{s,s}(x)$. We only need to prove the function $\delta_{0,0}(x)=\psi'(x)-1/x-1/2x^2$ being completely monotonic in $(0,\infty)$.

Successive differentiation of the function $\delta_{0,0}(x)$ with respect to $x>0$ yields

$$\delta_{0,0}^{(k)}(x)=\psi^{(k+1)}(x)+\frac{(-1)^{k+1}k!}{x^{k+1}}+\frac{(-1)^{k+1}(k+1)!}{2x^{k+2}} \tag{8}$$

for nonnegative integer k.

It is well known that the polygamma functions $\psi^{(n)}(x)$ can be expressed as

$$\psi^{(n)}(x)=(-1)^{n+1}\int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}t \tag{9}$$

for $n\in\mathbb{N}$, and

$$\frac{1}{x^r}=\frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\,\mathrm{d}t \tag{10}$$

for $x>0$ and $r>0$.

Applying formulas (9) and (10) in (8) yields

$$(-1)^k\delta_{0,0}^{(k)}(x)=\int_0^\infty\left(\frac{t}{1-e^{-t}}-1-\frac{t}{2}\right)t^ke^{-xt}\,\mathrm{d}t>0. \tag{11}$$

Thus, the function $\delta_{0,0}(x)$ is completely monotonic in $(0,\infty)$.

2.1.2. *The case* $\delta_{s,t}(x)$ *for* $s\ne t$. The function $\delta_{s,t}(x)$ can be rewritten as

$$\delta_{s,t}(x)=\frac{1}{t-s}\int_s^t\psi'(x+u)\,\mathrm{d}u-\frac{1}{2}\left[\left(1-\frac{1}{t-s}\right)\frac{1}{x+t}+\left(1+\frac{1}{t-s}\right)\frac{1}{x+s}\right], \tag{12}$$

then, for nonnegative integer k,

$$\begin{aligned}\delta_{s,t}^{(k)}(x)&=\frac{1}{t-s}\int_s^t\psi^{(k+1)}(x+u)\,\mathrm{d}u\\&\quad-\frac{(-1)^kk!}{2}\left[\left(1-\frac{1}{t-s}\right)\frac{1}{(x+t)^{k+1}}+\left(1+\frac{1}{t-s}\right)\frac{1}{(x+s)^{k+1}}\right].\end{aligned} \tag{13}$$

Since $\lim_{x\to\infty}\delta_{s,t}^{(k)}(x)=0$ from (9), by Lemma 1, to show $(-1)^k\delta_{s,t}^{(k)}(x)\gtrless 0$, it is sufficient to verify $(-1)^k[\delta_{s,t}^{(k)}(x)-\delta_{s,t}^{(k)}(x+1)]\gtrless 0$.

Taking the logarithm of the difference equation $\Gamma(x+1)=x\Gamma(x)$ and consecutive differentiating yields

$$\psi^{(i-1)}(x+1)=\psi^{(i-1)}(x)+\frac{(-1)^{i-1}(i-1)!}{x^i} \tag{14}$$

for $i\in\mathbb{N}$ and $x>0$.

In the following, for our own convenience and simplicity, denote $p=x+s>0$ and $q=x+t>0$, when no confusion appears in the context.

By using formulas (14) and (10), it is obtained that

$$\begin{aligned}&(-1)^k[\delta_{s,t}^{(k)}(x)-\delta_{s,t}^{(k)}(x+1)]\\&=\frac{(-1)^k}{t-s}\int_s^t[\psi^{(k+1)}(x+u)-\psi^{(k+1)}(x+u+1)]\,\mathrm{d}u\\&\quad-\frac{k!}{2}\left\{\left[1-\frac{1}{t-s}\right]\left[\frac{1}{q^{k+1}}-\frac{1}{(q+1)^{k+1}}\right]\right.\end{aligned}$$

$$
\begin{aligned}
&+\left[1+\frac{1}{t-s}\right]\left[\frac{1}{p^{k+1}}-\frac{1}{(p+1)^{k+1}}\right]\Biggr\}\\
&=\frac{(k+1)!}{t-s}\int_s^t\frac{1}{(x+u)^{k+2}}\,\mathrm{d}u-\frac{k!}{2}\Biggl\{\left[1-\frac{1}{t-s}\right]\left[\frac{1}{q^{k+1}}-\frac{1}{(q+1)^{k+1}}\right]\\
&+\left[1-\frac{1}{t-s}\right]\left[\frac{1}{p^{k+1}}-\frac{1}{(p+1)^{k+1}}\right]\Biggr\}\\
&=\frac{k!}{t-s}\left[\frac{1}{p^{k+1}}-\frac{1}{q^{k+1}}\right]-\frac{k!}{2}\Biggl\{\left[1-\frac{1}{t-s}\right]\left[\frac{1}{q^{k+1}}-\frac{1}{(q+1)^{k+1}}\right]\\
&+\left[1-\frac{1}{t-s}\right]\left[\frac{1}{p^{k+1}}-\frac{1}{(p+1)^{k+1}}\right]\Biggr\}\\
&=\frac{k!}{2}\Biggl\{\left(1+\frac{1}{q-p}\right)\frac{1}{(p+1)^{k+1}}+\left(\frac{1}{q-p}-1\right)\frac{1}{p^{k+1}}\\
&+\left(1-\frac{1}{q-p}\right)\frac{1}{(q+1)^{k+1}}-\left(1+\frac{1}{q-p}\right)\frac{1}{q^{k+1}}\Biggr\}\\
&=\frac{1}{2}\int_0^\infty z^k\Biggl[\left(1+\frac{1}{q-p}\right)e^{-(p+1)z}+\left(\frac{1}{q-p}-1\right)e^{-pz}\\
&+\left(1-\frac{1}{q-p}\right)e^{-(q+1)z}-\left(1+\frac{1}{q-p}\right)e^{-qz}\Biggr]\,\mathrm{d}z\\
&=\frac{1}{2}\int_0^\infty z^{k+1}(e^z+1)\bigl[e^{-(p+1)z}+e^{-(q+1)z}\bigr]\\
&\times\left[\frac{1}{(q-p)z}\frac{e^{(q-p)z}-1}{e^{(q-p)z}+1}-\frac{1}{z}\frac{e^z-1}{e^z+1}\right]\mathrm{d}z\\
&=\frac{1}{2}\int_0^\infty z^{k+1}(e^z+1)\bigl[e^{-(p+1)z}+e^{-(q+1)z}\bigr]\\
&\times\left[\frac{1}{(q-p)z}\tanh\frac{(q-p)z}{2}-\frac{1}{z}\tanh\frac{z}{2}\right]\mathrm{d}z.
\end{aligned}
\tag{15}
$$

Since the function $[\tanh(y/2)]/y$ is even in $(-\infty,\infty)$ and decreasing in $(0,\infty)$, then

$$
\frac{1}{(q-p)z}\tanh\frac{(q-p)z}{2}\gtrless\frac{1}{z}\tanh\frac{z}{2}
$$

is valid for $|q-p|=|t-s|\lessgtr 1$. This reveals that $(-1)^k[\delta_{s,t}^{(k)}(x)-\delta_{s,t}^{(k)}(x+1)]\gtrless 0$ for $|t-s|\lessgtr 1$, which implies $(-1)^k\delta_{s,t}^{(k)}(x)\gtrless 0$ for $|t-s|\lessgtr 1$.

In conclusion, the function $\delta_{s,t}(x)$ is completely monotonic if $|t-s|<1$ and $-\delta_{s,t}(x)$ is completely monotonic if $|t-s|>1$. The proof of Theorem 1 is complete.

2.2. **Proof of Theorem 2.** For real numbers $p,q\notin\{0,-1\}$, denote the function in the pair of braces in the ninth and tenth lines of (15) by $T_k(p,q)$. Then

$$
\begin{aligned}
T_k(p,q)&=\left(\frac{1}{q-p}-1\right)\Biggl\{\left[\frac{1}{p^{k+1}}-\frac{1}{q^{k+1}}\right]+\left[\frac{1}{(p+1)^{k+1}}-\frac{1}{(q+1)^{k+1}}\right]\Biggr\}\\
&+2\left[\frac{1}{(p+1)^{k+1}}-\frac{1}{q^{k+1}}\right]\\
&=[1-(q-p)]\Biggl[\frac{1}{(p+1)^{k+1}(q+1)^{k+1}}\sum_{i=0}^k(q+1)^i(p+1)^{k-i}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{p^{k+1}q^{k+1}}\sum_{i=0}^{k}q^ip^{k-i}\Bigg]+\frac{2(q-p-1)}{(p+1)^{k+1}q^{k+1}}\sum_{i=0}^{k}q^i(p+1)^{k-i}\\
&=(p-q+1)\Bigg\{\frac{1}{(p+1)^{k+1}(q+1)^{k+1}}\sum_{i=0}^{k}(p+1)^{k-i}(q+1)^i\\
&+\frac{1}{p^{k+1}q^{k+1}}\sum_{i=0}^{k}p^{k-i}q^i-\frac{2}{(p+1)^{k+1}q^{k+1}}\sum_{i=0}^{k}(p+1)^{k-i}q^i\Bigg\}\\
&=[1-(q-p)]\Bigg\{\sum_{i=0}^{k}\frac{1}{(p+1)^{i+1}(q+1)^{k-i+1}}\\
&+\sum_{i=0}^{k}\frac{1}{p^{i+1}q^{k-i+1}}-2\sum_{i=0}^{k}\frac{1}{(p+1)^{i+1}q^{k-i+1}}\Bigg\}.
\end{aligned}
\tag{16}
$$

2.2.1. For $p>0$ and $q>0$, formula (15) tells us that $T_k(p,q)\gtrless 0$ for $|q-p|\lessgtr 1$. Considering (16), it is concluded that if $p>0$ and $q>0$ then inequality

$$
\sum_{i=0}^{k}\frac{1}{(p+1)^{i+1}(q+1)^{k-i+1}}+\sum_{i=0}^{k}\frac{1}{p^{i+1}q^{k-i+1}}>2\sum_{i=0}^{k}\frac{1}{(p+1)^{i+1}q^{k-i+1}} \tag{17}
$$

holds for $q-p>-1$ and reveres for $q-p<-1$.

2.2.2. For $p<-1$ and $q<-1$, utilizing (10) reveals

$$
\begin{aligned}
T_k(p,q)&=(-1)^{k+1}\Bigg\{\left(1+\frac{1}{q-p}\right)\frac{1}{[-(p+1)]^{k+1}}+\left(\frac{1}{q-p}-1\right)\frac{1}{(-p)^{k+1}}\\
&+\left(1-\frac{1}{q-p}\right)\frac{1}{[-(q+1)]^{k+1}}-\left(1+\frac{1}{q-p}\right)\frac{1}{(-q)^{k+1}}\Bigg\}\\
&=\frac{(-1)^{k+1}}{k!}\int_0^\infty z^k\Bigg\{e^{(p+1)z}-e^{pz}+\frac{e^{pz}-e^{qz}}{q-p}\\
&+e^{(q+1)z}-e^{qz}+\frac{e^{(p+1)z}-e^{(q+1)z}}{q-p}\Bigg\}\,\mathrm{d}z\\
&=\frac{(-1)^{k+1}}{k!}\int_0^\infty z^{k+1}e^{pz}(1+e^z)\big[1+e^{(q-p)z}\big]\\
&\times\left[\frac{1}{z}\frac{e^z-1}{e^z+1}-\frac{1}{(q-p)z}\frac{e^{(q-p)z}-1}{e^{(q-p)z}+1}\right]\mathrm{d}z\\
&=\frac{(-1)^{k}}{k!}\int_0^\infty z^{k+1}e^{pz}(1+e^z)\big[1+e^{(q-p)z}\big]\\
&\times\left[\frac{1}{(q-p)z}\tanh\frac{(q-p)z}{2}-\frac{1}{z}\tanh\frac{z}{2}\right]\mathrm{d}z.
\end{aligned}
$$

Thus, $(-1)^kT_k(p,q)\gtrless 0$ for $|q-p|\lessgtr 1$. Combing with (16), it is deduced that if $p<-1$ and $q<-1$ then inequalities

$$
\sum_{i=0}^{2k}\frac{1}{(p+1)^{i+1}(q+1)^{2k-i+1}}+\sum_{i=0}^{2k}\frac{1}{p^{i+1}q^{2k-i+1}}>2\sum_{i=0}^{2k}\frac{1}{(p+1)^{i+1}q^{2k-i+1}} \tag{18}
$$

and

$$\sum_{i=0}^{2k+1}\frac{1}{(p+1)^{i+1}(q+1)^{2k-i+2}}+\sum_{i=0}^{2k+1}\frac{1}{p^{i+1}q^{2k-i+2}}<2\sum_{i=0}^{2k+1}\frac{1}{(p+1)^{i+1}q^{2k-i+2}} \tag{19}$$

hold for $q-p>-1$ and reverse for $q-p<-1$.

2.2.3. For $-1<p<0$ and $-1<q<0$, by using (10), we have

$$\begin{aligned}T_k(p,q)&=\left(1+\frac{1}{q-p}\right)\frac{1}{(p+1)^{k+1}}+\left(\frac{1}{q-p}-1\right)\frac{(-1)^{k+1}}{(-p)^{k+1}}\\&\quad+\left(1-\frac{1}{q-p}\right)\frac{1}{(q+1)^{k+1}}-\left(1+\frac{1}{q-p}\right)\frac{(-1)^{k+1}}{(-q)^{k+1}}\\&=\frac{1}{k!}\int_0^\infty z^k(e^{qz}+e^{pz})\left[\frac{1}{q-p}\frac{e^{(q-p)z}-1}{e^{(q-p)z}+1}+1\right]\left[e^{-(p+q+1)z}+(-1)^k\right]\,\mathrm{d}z.\end{aligned}$$

If k is even, then $T_k(p,q)>0$ and inequality (18) holds for all $-1<p<0$ and $-1<q<0$. If k is odd, then $T_k(p,q)\gtrless 0$ for $p+q+1\lessgtr 0$, and inequality (19) is valid for $p+q+1>0$ and is reversed for $p+q+1<0$.

2.2.4. For $p<-1$ and $q>0$, using (10) shows

$$\begin{aligned}T_k(p,q)&=\left(1+\frac{1}{q-p}\right)\frac{(-1)^{k+1}}{[-(p+1)]^{k+1}}+\left(\frac{1}{q-p}-1\right)\frac{(-1)^{k+1}}{(-p)^{k+1}}\\&\quad+\left(1-\frac{1}{q-p}\right)\frac{1}{(q+1)^{k+1}}-\left(1+\frac{1}{q-p}\right)\frac{1}{q^{k+1}}\\&=\frac{1}{k!}\int_0^\infty z^k(1+e^z)e^{pz}\left(\frac{e^z-1}{e^z+1}+\frac{1}{q-p}\right)\left[(-1)^{k+1}-e^{-(p+q+1)z}\right]\,\mathrm{d}z.\end{aligned}$$

If k is even, then $T_k(p,q)<0$ and inequality (18) holds for all $p<-1$ and $q>0$ by (16). If k is odd, then $T_k(p,q)\gtrless 0$ for $p+q+1\gtrless 0$, and inequality (19) holds for $p+q+1>0$ and reverses for $p+q+1<0$.

2.2.5. For $p>0$ and $q<-1$, by using (10), we obtain

$$\begin{aligned}T_k(p,q)&=\left(1+\frac{1}{q-p}\right)\frac{1}{(p+1)^{k+1}}+\left(\frac{1}{q-p}-1\right)\frac{1}{p^{k+1}}\\&\quad+\left(1-\frac{1}{q-p}\right)\frac{(-1)^{k+1}}{[-(q+1)]^{k+1}}-\left(1+\frac{1}{q-p}\right)\frac{(-1)^{k+1}}{(-q)^{k+1}}\\&=\frac{1}{k!}\int_0^\infty z^ke^{qz}(1+e^z)\left[\frac{e^z-1}{e^z+1}-\frac{1}{q-p}\right]\left[(-1)^{k+1}-e^{-(p+q+1)z}\right]\,\mathrm{d}z.\end{aligned}$$

If k is even, then $T_k(p,q)<0$ and inequality (18) is reversed for all $p>0$ and $q<-1$. If k is odd, $T_k(p,q)\lesseqgtr 0$ for $p+q+1\lesseqgtr 0$, and then inequality (19) holds for $p+q+1<0$ and reverses for $p+q+1>0$.

2.2.6. Letting $\theta>0$ and substituting $p=a/\theta$ and $q=b/\theta$ into (17) leads to

$$\begin{aligned}\sum_{i=0}^{k}\frac{1}{(a/\theta+1)^{i+1}(b/\theta+1)^{k-i+1}}&+\sum_{i=0}^{k}\frac{1}{(a/\theta)^{i+1}(b/\theta)^{k-i+1}}\\&>2\sum_{i=0}^{k}\frac{1}{(a/\theta+1)^{i+1}(b/\theta)^{k-i+1}},\end{aligned} \tag{20}$$

which is equivalent to (4).

Similarly, inequalities (5) and (6) can be deduced by applying $p = a/\theta$ and $q = b/\theta$ into (18) and (19).

2.2.7. Substituting $q = p - 1$ into the first two lines in (16) leads immediately to $T_k(p, p-1) = 0$, then inequalities (17), (18) and (19) become equalities. Hence, if $b = a - \theta$ then inequalities (4), (5) and (6) also become equalities. The proof of Theorem 2 is complete.

2.3. **Proof of Theorem 3.** For $a > 0$ and $b > 0$, suppose inequality (4) holds for $b - a > -\theta$ and reveres for $b - a < -\theta$. If assume $a = p\theta$ and $b = q\theta$ in (4), then simplifying yields that inequality (17) holds for $q - p > -1$ and reveres for $q - p < -1$, which is equivalent to $T_k(p,q) \gtreqless 0$ for $|q - p| \lesseqgtr 1$. This means that $(-1)^k[\delta_{s,t}^{(k)}(x) - \delta_{s,t}^{(k)}(x+1)] \gtreqless 0$ for $|t - s| \lesseqgtr 1$, which implies $(-1)^k \delta_{s,t}^{(k)}(x) \gtreqless 0$ for $|t - s| \lesseqgtr 1$. Theorem 1 follows.

Conversely, if Theorem 1 is valid, then $(-1)^k \delta_{s,t}^{(k)}(x) \gtreqless 0$ for $|t - s| \lesseqgtr 1$, which implies the function $(-1)^k \delta_{s,t}^{(k-1)}(x)$ is increasing/decreasing for $|t - s| \lesseqgtr 1$, and

$$(-1)^k \delta_{s,t}^{(k-1)}(x) - (-1)^k \delta_{s,t}^{(k-1)}(x+1) = (-1)^k \left[\delta_{s,t}^{(k-1)}(x) - \delta_{s,t}^{(k-1)}(x+1)\right] \lesseqgtr 0$$

for $|t - s| \lesseqgtr 1$, which means that $(-1)^k[\delta_{s,t}^{(k)}(x) - \delta_{s,t}^{(k)}(x+1)] \gtreqless 0$ for $|t - s| \lesseqgtr 1$. Further, combining this with (15) and (16) leads to (17), and then (20). Inequality (4) is proved. The proof of Theorem 3 is complete.

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