MONOTONICITY RESULTS FOR ARITHMETIC MEANS OF CONCAVE AND CONVEX FUNCTIONS

HUAN-NAN SHI, TIE-QUAN XU, AND FENG QI

ABSTRACT. By majorization approaches, some known results on monotonicity of the arithmetic means of convex and concave functions are proved and generalized once again.

1. Introduction

Let f be a strictly increasing convex (or concave) function in $(0,1]$. In order to improve and generalize Alzer's inequality in [1], J.-Ch. Kuang in [7] verified that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) > \int_0^1 f(x)\,\mathrm{d}x. \tag{1}$$

The left inequality in (1) reveals a monotonic property for the arithmetic means of convex (or concave) function f.

In [9], F. Qi generalized the left inequality in (1) and obtained the following

Theorem A ([9]). *Let f be a strictly increasing convex (or concave) function in $(0,1]$. Then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right)$ is decreasing in n and k and has a lower bound $\int_0^1 f(x)\,\mathrm{d}x$, that is,*

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f\left(\frac{i}{n+k+1}\right) > \int_0^1 f(x)\,\mathrm{d}x \tag{2}$$

where k is a nonnegative integer and n a natural number.

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As a generalization of Theorem A, F. Qi and B.-N. Guo in [11], among other things, obtained the following

Theorem B ([11]). *Let f be an increasing convex (or concave) function on $[0,1]$ and $\{a_i\}_{i\in\mathbb{N}}$ an increasing positive sequence such that $\bigl\{i\bigl(\frac{a_i}{a_{i+1}}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases (or $\bigl\{i\bigl(\frac{a_{i+1}}{a_i}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases). Then the sequence $\bigl\{\frac{1}{n}\sum_{i=1}^n f\bigl(\frac{a_i}{a_n}\bigr)\bigr\}_{n\in\mathbb{N}}$ is decreasing and*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) \ge \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right) \ge \int_0^1 f(x)\,\mathrm{d}x. \tag{3}$$

As a subsequence of [9, 11], Ch.-P. Chen, F. Qi, P. Cerone and S. S. Dragomir proved in [3] the follwoing two theorems.

Theorem C ([3]). *Let f be an increasing convex (or concave) function on $[0,1]$. Then the sequence $\bigl\{\frac{1}{n}\sum_{i=1}^n f\bigl(\frac{i}{n}\bigr)\bigr\}_{n\in\mathbb{N}}$ decreases and $\bigl\{\frac{1}{n}\sum_{i=0}^{n-1} f\bigl(\frac{i}{n}\bigr)\bigr\}_{n\in\mathbb{N}}$ increases, and*

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \ge \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) \ge \int_0^1 f(x)\,\mathrm{d}x \\ \ge \frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n+1}\right) \ge \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right). \end{aligned}\tag{4}$$

Theorem D ([3]). *Let f be an increasing convex (or concave) function on $[0,1)$ and $\{a_i\}_{i\in\mathbb{N}}$ a positive increasing sequence such that the sequence $\bigl\{i\bigl(\frac{a_{i+1}}{a_i}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases (or $\bigl\{i\bigl(\frac{a_i}{a_{i+1}}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases). Then the sequence $\bigl\{\frac{1}{n}\sum_{i=0}^{n-1} f\bigl(\frac{a_i}{a_n}\bigr)\bigr\}_{n\in\mathbb{N}}$ is increasing and*

$$\int_0^1 f(x)\,\mathrm{d}x \ge \frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{a_i}{a_{n+1}}\right) \ge \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_n}\right), \tag{5}$$

where $a_0 = 0$.

In recent years, some further generalizations and applications about inequalities (1), (2), (4), (5) and (3) have been obtained in [2, 4, 6, 9, 10, 12, 13, 16] and the references therein.

In this paper, the first aim is to prove once again the left hand side inequality in (3) and the right hand side inequality in (5) by majorization approaches. The second aim is to generalize inequalities (2) and (4). As applications, Alzer's inequality in [1] and Minc-Sathre's inequality in [8] are improved partially.

2. Lemmas

The following notations and the first six lemmas can be looked up in the books [17, 18] for details.

Let $\boldsymbol{x}=(x_1,\dots,x_n)$ and $\boldsymbol{y}=(y_1,\dots,y_n)$ be two real n-tuples.

(1) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\dots,n-1$ and $\sum_{i=1}^n x_i=\sum_{i=1}^n y_i$, where $x_{[1]}\ge\dots\ge x_{[n]}$ and $y_{[1]}\ge\dots\ge y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(2) $\boldsymbol{x}$ is said to be weakly sub-majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec_w\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]}\le\sum_{i=1}^k y_{[i]}$ for $k=1,2,\dots,n$.

(3) $\boldsymbol{x}$ is said to be weakly sup-majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec^w\boldsymbol{y}$) if $\sum_{i=1}^k x_{(i)}\ge\sum_{i=1}^k y_{(i)}$ for $k=1,2,\dots,n$, where $x_{(1)}\le x_{(2)}\le\dots\le x_{(n)}$ and $y_{(1)}\le y_{(2)}\le\dots\le y_{(n)}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in an increasing order.

(4) $\boldsymbol{x}\ge\boldsymbol{y}$ means $x_i\ge y_i$ for all $i=1,2,\dots,n$. A multi-variable function φ is said to be increasing if $\boldsymbol{x}\ge\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\ge\varphi(\boldsymbol{y})$.

Lemma 1 ([17, p. 7])**.** *Let $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ and $\delta=\sum_{i=1}^n(y_i-x_i)$. If $\boldsymbol{x}\prec_w\boldsymbol{y}$, then*

$$\Bigg(\boldsymbol{x},\underbrace{\frac{\delta}{n},\dots,\frac{\delta}{n}}_{n}\Bigg)\prec\Bigg(\boldsymbol{y},\underbrace{0,\dots,0}_{n}\Bigg). \tag{6}$$

Lemma 2 ([17, p. 5])**.** *Let $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$ and $\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^m$.*

(1) *If $\boldsymbol{x}\prec_w\boldsymbol{y}$ and $\boldsymbol{u}\prec_w\boldsymbol{v}$, then $(\boldsymbol{x},\boldsymbol{u})\prec_w(\boldsymbol{y},\boldsymbol{v})$;*

(2) *If $\boldsymbol{x}\prec^w\boldsymbol{y}$ and $\boldsymbol{u}\prec^w\boldsymbol{v}$, then $(\boldsymbol{x},\boldsymbol{u})\prec^w(\boldsymbol{y},\boldsymbol{v})$.*

Lemma 3 ([17, p. 5])**.** *Let $\boldsymbol{x}\in\mathbb{R}^n$ and $\bar{\boldsymbol{x}}=\frac{1}{n}\sum_{i=1}^n x_i$. Then $(\bar{\boldsymbol{x}},\dots,\bar{\boldsymbol{x}})\prec\boldsymbol{x}$.*

Lemma 4 ([17, pp. 48–49])**.** *Let $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$, $I\subset\mathbb{R}$ be an interval and $g:I\to\mathbb{R}$.*

(1) *$\boldsymbol{x}\prec\boldsymbol{y}$ if and only if*

$$\sum_{i=1}^n g(x_i)\le\sum_{i=1}^n g(y_i) \tag{7}$$

holds for all convex functions g,

(2) *$\boldsymbol{x}\prec\boldsymbol{y}$ if and only if* (7) *reverses for all concave functions g,*

(3) *$\boldsymbol{x}\prec_w\boldsymbol{y}$ if and only if* (7) *holds for all increasing convex functions g,*

(4) *$\boldsymbol{x}\prec^w\boldsymbol{y}$ if and only if* (7) *reverses for all increasing concave functions g.*

In order to prove our main results, we need the following four lemmas which can be showed by majorization approaches.

Lemma 5. *Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive and increasing sequence,*

$$\boldsymbol{x}=\left(\underbrace{\frac{a_1}{a_{n+1}},\dots,\frac{a_1}{a_{n+1}}}_{n},\underbrace{\frac{a_2}{a_{n+1}},\dots,\frac{a_2}{a_{n+1}}}_{n},\dots,\underbrace{\frac{a_{n+1}}{a_{n+1}},\dots,\frac{a_{n+1}}{a_{n+1}}}_{n}\right),$$

and

$$\boldsymbol{y}=\left(\underbrace{\frac{a_1}{a_n},\dots,\frac{a_1}{a_n}}_{n+1},\underbrace{\frac{a_2}{a_n},\dots,\frac{a_2}{a_n}}_{n+1},\dots,\underbrace{\frac{a_n}{a_n},\dots,\frac{a_n}{a_n}}_{n+1}\right).$$

(1) *If $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}_{i\in\mathbb{N}}$ decreases, then $\boldsymbol{x}\prec_w\boldsymbol{y}$;*
(2) *If $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ increases, then $\boldsymbol{x}\prec^w\boldsymbol{y}$.*

Proof. Let

$$\boldsymbol{u}_i=\left(\underbrace{\frac{a_i}{a_{n+1}},\dots,\frac{a_i}{a_{n+1}}}_{n}\right)\qquad\text{and}\qquad\boldsymbol{v}_i=\left(\underbrace{\frac{a_{i-1}}{a_n},\dots,\frac{a_{i-1}}{a_n}}_{i-1},\underbrace{\frac{a_i}{a_n},\dots,\frac{a_i}{a_n}}_{n-i+1}\right)$$

for $i=1,2,\dots,n+1$. The first conclusion in Lemma 2 tells us that, in order to prove $\boldsymbol{x}\prec_w\boldsymbol{y}$, it is sufficient to prove $\boldsymbol{u}_i\prec_w\boldsymbol{v}_i$. If $1\le k\le n-i+1$, then

$$\sum_{j=1}^k u_{i_{[j]}}=k\frac{a_i}{a_{n+1}}\le k\frac{a_i}{a_n}=\sum_{j=1}^k v_{i_{[j]}}.$$

If $n-i+1<k\le n$, then

$$\begin{aligned}\sum_{j=1}^k u_{i_{[j]}}=k\frac{a_i}{a_{n+1}}\le(n-i+1)\frac{a_i}{a_n}+[k-(n-i+1)]\frac{a_{i-1}}{a_n}=\sum_{j=1}^k v_{i_{[j]}},\\ k\frac{a_n}{a_{n+1}}\le(n-i+1)+[k-(n-i+1)]\frac{a_{i-1}}{a_i},\\ k\left(\frac{a_n}{a_{n+1}}-\frac{a_{i-1}}{a_i}\right)\le(n-i+1)\left(1-\frac{a_{i-1}}{a_i}\right).\end{aligned}\tag{8}$$

If $\frac{a_n}{a_{n+1}}-\frac{a_{i-1}}{a_i}\le 0$, since $\{a_n\}_{n\in\mathbb{N}}$ is positive and increasing, then $1-\frac{a_{i-1}}{a_i}\ge 0$ and inequality (8) holds. If $\frac{a_n}{a_{n+1}}-\frac{a_{i-1}}{a_i}>0$, since $\bigl\{i\bigl(\frac{a_i}{a_{i+1}}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ decreases, then

$$\begin{aligned}
k\left(\frac{a_n}{a_{n+1}}-\frac{a_{i-1}}{a_i}\right)&\le n\left(\frac{a_n}{a_{n+1}}-\frac{a_{i-1}}{a_i}\right)=n\left(\frac{a_n}{a_{n+1}}-1\right)-n\left(\frac{a_{i-1}}{a_i}-1\right)\\
&\le(i-1)\left(\frac{a_{i-1}}{a_i}-1\right)-n\left(\frac{a_{i-1}}{a_i}-1\right)=(n-i+1)\left(1-\frac{a_{i-1}}{a_i}\right),
\end{aligned}$$

and inequality (8) holds also.

Let

$$\boldsymbol{u}_i=\Biggl(\underbrace{\frac{a_i}{a_{n+1}},\dots,\frac{a_i}{a_{n+1}}}_{n-i+1},\underbrace{\frac{a_{i+1}}{a_{n+1}},\dots,\frac{a_{i+1}}{a_{n+1}}}_{i}\Biggr)\quad\text{and}\quad\boldsymbol{v}_i=\Biggl(\underbrace{\frac{a_i}{a_n},\dots,\frac{a_i}{a_n}}_{n+1}\Biggr)$$

for $i=1,2,\dots,n$. The second conclusion in Lemma 2 shows that, in order to prove $\boldsymbol{x}\prec^w\boldsymbol{y}$, it is sufficient to prove $\boldsymbol{u}_i\prec^w\boldsymbol{v}_i$. If $1\le k\le n-i+1$, then

$$\sum_{j=1}^k v_{i_{[j]}}=k\frac{a_i}{a_n}\le k\frac{a_i}{a_{n+1}}=\sum_{j=1}^k u_{i_{[j]}}.$$

If $n-i+1<k\le n+1$, then

$$\begin{gathered}
\sum_{j=1}^k v_{i_{[j]}}\ge\sum_{j=1}^k u_{i_{[j]}},\\
k\frac{a_i}{a_n}\ge(n-i+1)\frac{a_i}{a_{n+1}}+[k-(n-i+1)]\frac{a_{i+1}}{a_{n+1}},\\
k\frac{a_{n+1}}{a_n}\ge(n-i+1)+[k-(n-i+1)]\frac{a_{i+1}}{a_i},\\
k\left(\frac{a_{n+1}}{a_n}-\frac{a_{i+1}}{a_i}\right)\ge(n-i+1)\left(1-\frac{a_{i+1}}{a_i}\right).
\end{gathered}\tag{9}$$

If $\frac{a_{n+1}}{a_n}-\frac{a_{i+1}}{a_i}\ge 0$, since $\{a_n\}_{n\in\mathbb{N}}$ be positive and increasing, then $1-\frac{a_{i+1}}{a_i}\le 0$ and inequality (9) holds. If $\frac{a_{n+1}}{a_n}-\frac{a_{i+1}}{a_i}<0$, since $\bigl\{i\bigl(\frac{a_{i+1}}{a_i}-1\bigr)\bigr\}_{i\in\mathbb{N}}$ increases, then

$n\big(\frac{a_{n+1}}{a_n}-1\big)\ge i\big(\frac{a_{i+1}}{a_i}-1\big)$, hence

$$\begin{aligned}
k\left(\frac{a_{n+1}}{a_n}-\frac{a_{i+1}}{a_i}\right)&\ge(n+1)\left(\frac{a_{n+1}}{a_n}-\frac{a_{i+1}}{a_i}\right)\\
&=(n+1)\left(\frac{a_{n+1}}{a_n}-1\right)-(n+1)\left(\frac{a_{i+1}}{a_i}-1\right)\\
&\ge n\left(\frac{a_{n+1}}{a_n}-1\right)-(n+1)\left(\frac{a_{i+1}}{a_i}-1\right)\\
&\ge i\left(\frac{a_{i+1}}{a_i}-1\right)-(n+1)\left(\frac{a_{i+1}}{a_i}-1\right)\\
&=(n-i+1)\left(1-\frac{a_{i+1}}{a_i}\right),
\end{aligned}$$

and then inequality (9) holds. The proof of Lemma 5 is complete. □

By the same method as in the proof of the second conclusion in Lemma 5, we obtain the following

Lemma 6. *Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive increasing sequence,*

$$\boldsymbol{u}=\Bigg(\underbrace{\frac{a_1}{a_{n+2}},\dots,\frac{a_1}{a_{n+2}}}_{n},\dots,\underbrace{\frac{a_{n+1}}{a_{n+2}},\dots,\frac{a_{n+1}}{a_{n+2}}}_{n}\Bigg)$$

and

$$\boldsymbol{v}=\Bigg(\underbrace{\frac{a_1}{a_{n+1}},\dots,\frac{a_1}{a_{n+1}}}_{n+1},\dots,\underbrace{\frac{a_n}{a_{n+1}},\dots,\frac{a_n}{a_n}}_{n+1}\Bigg).$$

If $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then $\boldsymbol{v}\prec^w\boldsymbol{u}$.

Lemma 7. *Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive increasing sequence,*

$$\boldsymbol{x}=\Bigg(\underbrace{\frac{a_0}{a_n},\dots,\frac{a_0}{a_n}}_{n+1},\underbrace{\frac{a_1}{a_n},\dots,\frac{a_1}{a_n}}_{n+1},\dots,\underbrace{\frac{a_{n-1}}{a_n},\dots,\frac{a_{n-1}}{a_n}}_{n+1}\Bigg)$$

and

$$\boldsymbol{y}=\Bigg(\underbrace{\frac{a_0}{a_{n+1}},\dots,\frac{a_0}{a_{n+1}}}_{n},\underbrace{\frac{a_1}{a_{n+1}},\dots,\frac{a_1}{a_{n+1}}}_{n},\dots,\underbrace{\frac{a_n}{a_{n+1}},\dots,\frac{a_n}{a_{n+1}}}_{n}\Bigg)$$

with assumption $a_0=0$.

(1) *If* $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ *decreases, then* $\boldsymbol{x}\prec_w \boldsymbol{y}$*;*
(2) *If* $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}_{i\in\mathbb{N}}$ *increases, then* $\boldsymbol{x}\prec^w \boldsymbol{y}$*.*

Proof. Since $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ decreases, then

$$i\left(\frac{a_{i+1}}{a_i}-1\right)\ge (i+1)\left(\frac{a_{i+2}}{a_i}-1\right)\ge i\left(\frac{a_{i+2}}{a_i}-1\right),$$

therefore, the sequence $\left\{\frac{a_{i+1}}{a_i}\right\}_{i\in\mathbb{N}}$ decreases.

Let

$$\boldsymbol{u}_i=\left(\underbrace{\frac{a_i}{a_n},\dots,\frac{a_i}{a_n}}_{n+1}\right)\qquad\text{and}\qquad \boldsymbol{v}_i=\left(\underbrace{\frac{a_i}{a_{n+1}},\dots,\frac{a_i}{a_{n+1}}}_{n-i},\underbrace{\frac{a_{i+1}}{a_{n+1}},\dots,\frac{a_{i+1}}{a_{n+1}}}_{i+1}\right)$$

for $i=0,1,\dots,n-1$. Considering the first conclusion in Lemma 2, it is easy to see that, in order to prove the first conclusion in Lemma 7, it suffices to show $\boldsymbol{u}_i\prec_w \boldsymbol{v}_i$.

From the assumption that $a_0=0$, it follows easily that $\boldsymbol{u}_0\prec_w \boldsymbol{v}_0$.

For $i\ge 1$ and $1\le k\le i+1$, since $\left\{\frac{a_{i+1}}{a_i}\right\}_{i\in\mathbb{N}}$ decreases, then $\frac{a_{i+1}}{a_i}\ge\frac{a_{n+1}}{a_n}$ and

$$\sum_{j=1}^{k}u_{i_{[j]}}=k\frac{a_i}{a_n}\le k\frac{a_{i+1}}{a_{n+1}}=\sum_{j=1}^{k}v_{i_{[j]}}.$$

For $i\ge 1$ and $i+1<k\le n+1$,

$$\sum_{j=1}^{k}u_{i_{[j]}}\le\sum_{j=1}^{k}v_{i_{[j]}},$$
$$k\frac{a_i}{a_n}\le (i+1)\frac{a_{i+1}}{a_{n+1}}+(k-i-1)\frac{a_i}{a_{n+1}},$$
$$k\left(\frac{a_{n+1}}{a_n}-1\right)\le (i+1)\left(\frac{a_{i+1}}{a_i}-1\right). \tag{10}$$

From $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ and $\left\{\frac{a_{i+1}}{a_i}\right\}_{i\in\mathbb{N}}$ being decreasing, it follows that

$$n\left(\frac{a_{n+1}}{a_n}-1\right)\le i\left(\frac{a_{i+1}}{a_i}-1\right)$$

and

$$\frac{a_{n+1}}{a_n}-1\le\frac{a_{i+1}}{a_i}-1.$$

Addition on both sides of above two inequalities yields

$$(n+1)\left(\frac{a_{n+1}}{a_n}-1\right)\le (i+1)\left(\frac{a_{i+1}}{a_i}-1\right).$$

Combining this with

$$k\left(\frac{a_{n+1}}{a_n}-1\right)\le(n+1)\left(\frac{a_{n+1}}{a_n}-1\right)$$

leads to (10).

By similar argument as above, since $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}_{i\in\mathbb{N}}$ increases, so does $\left\{\frac{a_i}{a_{i+1}}\right\}_{i\in\mathbb{N}}$.

Let

$$\boldsymbol{u}_i=\Bigg(\underbrace{\frac{a_{i-1}}{a_n},\dots,\frac{a_{i-1}}{a_n}}_{i},\underbrace{\frac{a_i}{a_n},\dots,\frac{a_i}{a_n}}_{n-i}\Bigg)\quad\text{and}\quad\boldsymbol{v}_i=\Bigg(\underbrace{\frac{a_i}{a_{n+1}},\dots,\frac{a_i}{a_{n+1}}}_{n}\Bigg)$$

for $i=0,1,\dots,n$. From the first conclusion in Lemma 2, it is sufficient to show $\boldsymbol{v}_i\prec^w\boldsymbol{u}_i$.

Since $a_0=0$, it is clear that $\boldsymbol{v}_0\prec^w\boldsymbol{u}_0$.

For $i\ge1$ and $1\le k\le i$, since $\left\{\frac{a_i}{a_{i+1}}\right\}$ increases, then $\frac{a_{i-1}}{a_i}\le\frac{a_n}{a_{n+1}}$ and

$$\sum_{j=1}^{k}v_{i_{[j]}}=k\frac{a_i}{a_{n+1}}\ge k\frac{a_{i-1}}{a_n}=\sum_{j=1}^{k}u_{i_{[j]}}.$$

For $i\ge1$ and $i+1\le k\le n$,

$$\begin{aligned}\sum_{j=1}^{k}v_{i_{[j]}}&\ge\sum_{j=1}^{k}u_{i_{[j]}},\\ k\frac{a_i}{a_{n+1}}&\ge i\frac{a_{i-1}}{a_n}+(k-i)\frac{a_i}{a_n},\\ k\frac{a_n}{a_{n+1}}&\ge i\frac{a_{i-1}}{a_i}+(k-i),\\ k\left(\frac{a_n}{a_{n+1}}-1\right)&\ge i\left(\frac{a_{i-1}}{a_i}-1\right).\end{aligned}\tag{11}$$

From the increasingly monotonic property of $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ and $\left\{\frac{a_i}{a_{i+1}}\right\}_{i\in\mathbb{N}}$, it is deduced that

$$n\left(\frac{a_n}{a_n+1}-1\right)\ge(i-1)\left(\frac{a_{i-1}}{a_i}-1\right)$$

and

$$\left(\frac{a_n}{a_{n+1}}-1\right)\ge\left(\frac{a_{i-1}}{a_i}-1\right).$$

Adding these two inequalities on both sides gives

$$(n+1)\left(\frac{a_n}{a_{n+1}}-1\right)\ge i\left(\frac{a_{i-1}}{a_i}-1\right).\tag{12}$$

Substituting

$$k\left(\frac{a_n}{a_{n+1}}-1\right)\geq(n+1)\left(\frac{a_n}{a_{n+1}}-1\right)$$

into (12) leads to (11). The proof of Lemma 7 is complete. □

By the same method as above, the following is obtained.

Lemma 8. *Let $\{a_i\}_{i\in\mathbb{N}}$ be a positive increasing sequence,*

$$\boldsymbol{u}=\left(\underbrace{\frac{a_0}{a_{n-1}},\dots,\frac{a_0}{a_{n-1}}}_{n+1},\underbrace{\frac{a_1}{a_{n-1}},\dots,\frac{a_1}{a_{n-1}}}_{n+1},\dots,\underbrace{\frac{a_{n-1}}{a_{n-1}},\dots,\frac{a_{n-1}}{a_{n-1}}}_{n+1}\right)$$

and

$$\boldsymbol{v}=\left(\underbrace{\frac{a_0}{a_n},\dots,\frac{a_0}{a_n}}_{n},\underbrace{\frac{a_1}{a_n},\dots,\frac{a_1}{a_n}}_{n},\dots,\underbrace{\frac{a_n}{a_n},\dots,\frac{a_n}{a_n}}_{n}\right).$$

If $\big\{i\big(\frac{a_{i-1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then $\boldsymbol{v}\prec^w\boldsymbol{u}$.

3. Main results and their proofs

In the following, we are in a position to state our main results and give proofs of them.

Theorem 1. *Let f be an increasing function on $(0,1]$ and $\{a_i\}_{i\in\mathbb{N}}$ a positive increasing sequence.*

(1) *If f is convex (or concave) and $\big\{i\big(\frac{a_i}{a_{i+1}}-1\big)\big\}_{i\in\mathbb{N}}$ decreases (or $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ increases), then*

$$\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{a_i}{a_n}\right)\geq\frac{1}{n+1}\sum_{i=1}^{n+1}f\left(\frac{a_i}{a_{n+1}}\right).\tag{13}$$

(2) *If f is concave and $\big\{i\big(\frac{a_{i+1}}{a_i}-1\big)\big\}_{i\in\mathbb{N}}$ increases, then*

$$\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{a_i}{a_{n+1}}\right)\geq\frac{1}{n+1}\sum_{i=1}^{n+1}f\left(\frac{a_i}{a_{n+2}}\right).\tag{14}$$

Proof. Inequality (13) follows from combining the third and fourth conclusions in Lemma 4 with the first and second conclusions in Lemma 5 respectively. Inequality (14) can be deduced from combining the fourth conclusion in Lemma 4 with Lemma 6. □

Remark 1. The first conclusion in Theorem 1 is the same as the left hand side inequality in (3) of Theorem B. However, we recover it by a majorization method here.

Remark 2. We claim that inequality (13) can be deduced from (14). In fact, since the sequence $\{a_i\}_{i\in\mathbb{N}}$ be a positive increasing sequence, then $\frac{a_i}{a_n} \ge \frac{a_i}{a_{n+1}}$, and, utilizing the increasingly monotonicity of f,

$$\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) \ge \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right). \tag{15}$$

Replacing $n+1$ by n in (14) leads to

$$n\sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) \ge (n-1)\sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right). \tag{16}$$

Combining (15) and (16) yields

$$n\sum_{i=1}^{n-1} f\left(\frac{a_i}{a_n}\right) + nf\left(\frac{a_n}{a_n}\right) + \sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) \\ \ge (n-1)\sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right) + nf\left(\frac{a_{n+1}}{a_{n+1}}\right) + \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right),$$

which can be rewritten as

$$(n+1)\sum_{i=1}^{n} f\left(\frac{a_i}{a_n}\right) \ge n\sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right),$$

which is equivalent to (13).

The following two corollaries show that inequality (14) is better than (13).

Corollary 1. *For $n \in \mathbb{N}$,*

$$\frac{n+1}{n+2} \le \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{17}$$

Proof. Let $f(t) = \ln t$ in $(0,1]$, an increasing concave function in $(0,1]$. Taking $a_i = i$, then it is clear that $\{a_i\}_{i\in\mathbb{N}}$ is a positive increasing sequence such that $\left\{i\left(\frac{a_{i+1}}{a_i} - 1\right)\right\}_{i\in\mathbb{N}}$ increases. Applying this to the second conclusion in Theorem 1 reveals

$$\frac{1}{n}\sum_{i=1}^{n}[\ln i - \ln(n+1)] \ge \frac{1}{n+1}\sum_{i=1}^{n+1}[\ln i - \ln(n+2)],$$

which can be rewritten as the form of inequality (17). □

Remark 3. Inequality (17), a refinement of the left hand side inequality of Minc-Sathre's inequality in [1, 8]

$$\frac{n}{n+1}<\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}<1,$$

has been generalized in [5, 6, 10, 13, 14, 15] and the references therein.

Corollary 2. *Let $n\in\mathbb{N}$ and $0<r\le 1$. Then*

$$\frac{n+1}{n+2}\le\left(\frac{\frac{1}{n}\sum_{i=1}^{n}i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1}i^r}\right)^{1/r}. \tag{18}$$

Proof. Let $f(t)=t^r$ in $(0,1]$, an increasing concave function in $(0,1]$. Taking $a_i=i$, then $\{a_i\}_{i\in\mathbb{N}}$ is a positive and increasing sequence such that $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ increases. Applying this to the second conclusion in Theorem 1 leads to

$$\frac{1}{n(n+1)^r}\sum_{i=1}^{n}i^r\ge\frac{1}{(n+1)(n+2)^r}\sum_{i=1}^{n+1}i^r,$$

which can be rearranged as (18). □

Remark 4. Let $n\in\mathbb{N}$ and $r>0$. Alzer's inequality [1] states that

$$\frac{n}{n+1}<\left(\frac{1}{n}\sum_{i=1}^{n}i^r\Bigg/\frac{1}{n+1}\sum_{i=1}^{n+1}i^r\right)^{1/r}<\frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{19}$$

When $0<r\le 1$, (18) improves (19).

Remark 5. The right hand side inequality in (3) is not valid in general. A counterexample is given as follows. Let $f(t)=t^2$ in $[0,1]$. Taking $a_i=2^i$, then $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}=-\frac{i}{2}$ increases. However, when $n\ge 4$,

$$\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{a_i}{a_n}\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{2^{n-i}}\right)^2<\frac{1}{n}\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i=\frac{4}{3n}\le\frac{1}{3}=\int_0^1 f(x)\,\mathrm{d}x.$$

Theorem 2. *Let f be an increasing function on $[0,1]$ and $\{a_i\}_{i\in\mathbb{N}}$ a positive increasing sequence.*

(1) *If f is a convex (or concave) function and $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ decreases (or $\left\{i\left(\frac{a_i}{a_{i+1}}-1\right)\right\}_{i\in\mathbb{N}}$ increases), then*

$$\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{a_i}{a_n}\right)\le\frac{1}{n+1}\sum_{i=1}^{n+1}f\left(\frac{a_i}{a_{n+1}}\right). \tag{20}$$

(2) *If f is a concave function and $\left\{i\left(\frac{a_{i-1}}{a_i}-1\right)\right\}_{i\in\mathbb{N}}$ increases, then*

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_{n-1}}\right) \le \frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \tag{21}$$

with assumption $a_0 = 0$.

Proof. Inequality (20) follows from combining the third and fourth conclusions in Lemma 4 with the first and second conclusions in Lemma 7 respectively. Inequality (21) follows from combining the fourth conclusion in Lemma 4 with Lemma 8. □

Remark 6. In the first conclusion of Theorem 2, the condition that $\left\{i\left(\frac{a_{i+1}}{a_i}-1\right)\right\}_{n\in\mathbb{N}}$ decreases can be weakened to that $\left\{i\left(\frac{a_i}{a_{i-1}}-1\right)\right\}_{n\in\mathbb{N}}$ decreases. In this case, the assumption $a_0 = 0$ can be broadened to $a_0 \ge 0$. However, when $a_0 = 0$ the sequence $\left\{i\left(\frac{a_i}{a_{i-1}}-1\right)\right\}_{i\in\mathbb{N}}$ decreases only if $i \ge 2$, and when $a_0 > 0$ the sequence $\left\{i\left(\frac{a_i}{a_{i-1}}-1\right)\right\}_{i\in\mathbb{N}}$ decreases only if $i \ge 1$.

Remark 7. The first conclusion in Theorem 2 is the right hand side inequality in (5) of Theorem D.

Remark 8. It is claimed that inequality (20) can be deduced from (21). Indeed, since $\{a_i\}_{i\in\mathbb{N}}$ is a positive increasing sequence, then $\frac{a_i}{a_n} \ge \frac{a_i}{a_{n+1}}$ and, from f being increasing,

$$\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \ge \sum_{i=1}^{n} f\left(\frac{a_i}{a_{n+1}}\right). \tag{22}$$

Replacing n by $n+1$ in (21) yields

$$(n+2)\sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right) \le (n+1)\sum_{i=0}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right),$$

which can be rewritten as

$$\begin{aligned}(n+1)\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_n}\right) + (n+1)f\left(\frac{a_n}{a_n}\right) + \sum_{i=0}^{n} f\left(\frac{a_i}{a_n}\right)\\ \le n\sum_{i=0}^{n} f\left(\frac{a_i}{a_{n+1}}\right) + nf\left(\frac{a_{n+1}}{a_{n+1}}\right) + \sum_{i=0}^{n+1} f\left(\frac{a_i}{a_{n+1}}\right).\end{aligned} \tag{23}$$

Combining (23) with (22) shows

$$(n+1)\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_n}\right) \ge n\sum_{i=0}^{n} f\left(\frac{a_i}{a_{n+1}}\right), \tag{24}$$

which is equivalent to (20).

Corollary 3. *For $n \in \mathbb{N}$ and $0 < r \le 1$,*

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n-1} i^r}{\frac{1}{n+1}\sum_{i=1}^{n} i^r}\right)^{1/r} \le \frac{n-1}{n}. \tag{25}$$

Proof. Let $f(t) = t^r$ in $(0,1]$, an increasing concave function in $(0,1]$. Taking $a_i = i$, then $\{a_i\}_{i\in\mathbb{N}}$ is a positive increasing sequence such that $\left\{i\left(\frac{a_{i-1}}{a_i} - 1\right)\right\}_{i\in\mathbb{N}}$ increases. Applying these to the second conclusion in Theorem 2 produces

$$\frac{1}{n(n-1)^r}\sum_{i=1}^{n-1} i^r \le \frac{1}{(n+1)n^r}\sum_{i=1}^{n} i^r,$$

which can be rearranged into (25). □

Remark 9. Let $n \in \mathbb{N}$ and $r > 0$, Corollary 1 in [3] verified

$$\left(\frac{\frac{1}{n}\sum_{i=1}^{n-1} i^r}{\frac{1}{n+1}\sum_{i=1}^{n} i^r}\right)^{1/r} \le \frac{n}{n+1}. \tag{26}$$

When $0 < r \le 1$, inequality (25) refines (26).

Remark 10. The left hand side inequality in (5) does not hold. The following is a counterexample.

Let $f(x) = x^2$ in $[0,1]$. Taking $a_i = 1 - \frac{1}{2^i}$, then $\left\{i\left(\frac{a_{i+1}}{a_i} - 1\right)\right\} = \frac{i}{2(2^i-1)}$ decreases. Let $g(x) = \frac{x}{2^x-1}$, it is easy to verify that $g'(x) < 0$ and $g(x)$ strictly decreases in $(0,\infty)$. Hence, when $n \ge 5$,

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{a_i}{a_n}\right) &= \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{1-1/2^i}{1-1/2^n}\right)^2 > \frac{1}{n}\sum_{i=1}^{n-1}\left(1-\frac{1}{2^i}\right)^2 \\
&> \frac{1}{n}\sum_{i=1}^{n-1}\left(1-\frac{2}{2^i}\right) = \frac{n-1}{n} - \frac{2}{n}\sum_{i=1}^{n-1}\frac{1}{2^i} > \frac{n-1}{n} - \frac{2}{n}\sum_{i=1}^{\infty}\frac{1}{2^i} \\
&= \frac{n-3}{n} = 1 - \frac{3}{n} \ge \frac{2}{5} > \frac{1}{3} = \int_0^1 f(x)\,\mathrm{d}x.
\end{aligned}$$

This leads to a contradiction.

Theorem 3. *Let f be an increasing function in $[0,1]$.*

(1) *If f is convex and $k > -1$, then*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i+k}{n+k}\right) \ge \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i+k}{n+k+1}\right). \tag{27}$$

(2) *If f is concave and $k \ge 0$, then*

$$\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i+k}{n+k+1}\right) \ge \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i+k}{n+k+2}\right). \tag{28}$$

Proof. Let

$$\boldsymbol{x} = \left(\underbrace{\frac{k+1}{k+n+1},\dots,\frac{k+1}{k+n+1}}_{n},\dots,\underbrace{\frac{k+n+1}{k+n+1},\dots,\frac{k+n+1}{k+n+1}}_{n}\right)$$

and

$$\boldsymbol{y} = \left(\underbrace{\frac{k+1}{k+n},\dots,\frac{k+1}{k+n}}_{n+1},\underbrace{\frac{k+2}{k+n},\dots,\frac{k+2}{k+n}}_{n+1},\dots,\underbrace{\frac{k+n}{k+n},\dots,\frac{k+n}{k+n}}_{n+1}\right).$$

Taking $a_i = k+i$ in Lemma 5 yields $\boldsymbol{x} \prec_w \boldsymbol{y}$ for $k > -1$. By the third conclusion in Lemma 4, the first conclusion in Theorem 3 is proved.

Let

$$\boldsymbol{u} = \left(\underbrace{\frac{k+1}{k+n+2},\dots,\frac{k+1}{k+n+2}}_{n},\dots,\underbrace{\frac{k+n+1}{k+n+2},\dots,\frac{k+n+1}{k+n+2}}_{n}\right)$$

and

$$\boldsymbol{v} = \left(\underbrace{\frac{k+1}{k+n+1},\dots,\frac{k+1}{k+n+1}}_{n+1},\dots,\underbrace{\frac{k+n}{k+n+1},\dots,\frac{k+n}{k+n+1}}_{n+1}\right).$$

Applying $a_i = k+i$ to Lemma 6 leads to $\boldsymbol{v} \prec^w \boldsymbol{u}$ for $k \ge 0$. By the fourth conclusion in Lemma 4, the second conclusion in Theorem 3 is proved. □

Remark 11. As argued above, the second conclusion in Theorem 3 implies the first conclusion in Theorem 3. Hence Theorem 3 extends the first inequality in (2) where k is requested to be a nonnegative integer.

Theorem 4. *Let f be an increasing function in $[0,1]$.*

(1) *If f is convex and $0 \le k \le 1$, then*

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i+k}{n+k}\right) \ge \frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i+k}{n+k+1}\right). \tag{29}$$

(2) *If f is concave, then*

$$\frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n-1}\right) \le \frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n}\right). \tag{30}$$

Proof. Let

$$\boldsymbol{x} = \left(\underbrace{\frac{k}{k+n},\dots,\frac{k}{k+n}}_{n+1},\dots,\underbrace{\frac{k+n-1}{k+n},\dots,\frac{k+n-1}{k+n}}_{n+1}\right)$$

and

$$\boldsymbol{y} = \left(\underbrace{\frac{k}{k+n+1},\dots,\frac{k}{k+n+1}}_{n},\dots,\underbrace{\frac{k+n}{k+n+1},\dots,\frac{k+n}{k+n+1}}_{n}\right).$$

Taking $a_i = k+i$, from Remark 6 and Theorem 2, the first conclusion of Theorem 4 is proved.

Let

$$\boldsymbol{u} = \left(\underbrace{0,\dots,0}_{n+1},\underbrace{\frac{1}{n-1},\dots,\frac{1}{n-1}}_{n+1},\dots,\underbrace{\frac{n-1}{n-1},\dots,\frac{n-1}{n-1}}_{n+1}\right)$$

and

$$\boldsymbol{v} = \left(\underbrace{0,\dots,0}_{n},\underbrace{\frac{1}{n},\dots,\frac{1}{n}}_{n},\dots,\underbrace{\frac{n}{n},\dots,\frac{n}{n}}_{n}\right).$$

Taking $a_i = i$ in Lemma 6 gives $\boldsymbol{v} \prec_w \boldsymbol{u}$. Then the second conclusion of this theorem follows from the fourth conclusion in Lemma 4. The proof of Theorem 4 is complete. □

Remark 12. By Remark 2, the second conclusion of Theorem 4 can be reduced to the very right hand side inequality in (4). Hence, Theorem 4 extends the very right hand side inequality in (4).

Theorem 5. *Let f be an increasing function on $[0,1]$ and $k>-1$. Then*

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{k+i}{n+k}\right) &\geq \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{k+i}{n+k+1}\right)\\ &+ f\left(\frac{k+1}{2(n+k)(n+k+1)}\right) - f(0) \geq 2f\left(\frac{n+2k+1}{4(n+k)}\right) - f(0) \geq 0. \quad (31)\end{aligned}$$

Proof. For $\boldsymbol{x}$ and $\boldsymbol{y}$ defined in Lemma 7, we have $\boldsymbol{x} \prec_w \boldsymbol{y}$ for $k>-1$. Now let

$$\begin{aligned}\delta = \sum_{i=1}^{n(n+1)} (y_i - x_i) &= \sum\nolimits_{i=1}^{n(n+1)} y_i - \sum_{i=1}^{n(n+1)} x_i\\ &= \frac{n(n+1)(n+2k+1)}{2(n+k)} - \frac{(n+2k+2)n(n+1)}{2(n+k+1)} = \frac{n(n+1)(k+1)}{2(n+k)(n+k+1)}.\end{aligned}$$

From Lemma 1 and Lemma 3, it is obtained that

$$\begin{aligned}&\Bigg(\underbrace{\frac{n+2k+1}{4(k+n)},\dots,\frac{n+2k+1}{4(k+n)}}_{2n(n+1)},\Bigg)\\ &\qquad \prec \Bigg(\underbrace{\frac{k+1}{k+n+1},\dots,\frac{k+1}{k+n+1}}_{n},\underbrace{\frac{k+n+1}{k+n+1},\dots,\frac{k+n+1}{k+n+1}}_{n},\\ &\qquad\qquad \dots,\underbrace{\frac{k+1}{2(k+n)(n+k+1)},\dots,\frac{k+1}{2(k+n)(n+k+1)}}_{n(n+1)}\Bigg)\\ &\qquad\qquad \prec \Bigg(\underbrace{\frac{k+1}{k+n},\dots,\frac{k+1}{k+n}}_{n+1},\dots,\underbrace{\frac{k+n}{k+n},\dots,\frac{k+n}{k+n}}_{n+1},\underbrace{0,\dots,0}_{n(n+1)}\Bigg). \quad (32)\end{aligned}$$

Since f is an increasing convex function in $[0,1]$, then, by Lemma 1 and from (32),

$$\begin{aligned}&(n+1)\sum_{i=1}^{n} f\left(\frac{k+i}{n+k}\right) + n(n+1)f(0)\\ &\geq n\sum_{i=1}^{n+1} f\left(\frac{k+i}{n+k+1}\right) + n(n+1)f\left(\frac{k+1}{2(n+k)(n+k+1)}\right)\\ &\geq 2n(n+1)f\left(\frac{n+2k+1}{4(n+k)}\right).\end{aligned}$$

Therefore, inequality (31) is deduced. □

Remark 13. Since f is an increasing convex function, then

$$f\left(\frac{k+1}{2(n+k)(n+k+1)}\right)-f(0)\geq 0.$$

Thus, the condition that f is an increasing convex function and $k>-1$ shows that the first inequality in (31) strengthens the first inequality in (4).

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(H.-N. Shi) Department of Electronic information, Teacher's College, Beijing Union University, Beijing City, 100011, China

E-mail address: `shihuannan@yahoo.com.cn, sfthuannan@buu.com.cn`

(T.-Q. Xu) Department of Mathematics, Qingdao Vocational and Technical College, Qingdao City, Shandong Province, 266071, China

E-mail address: `xutiequan2003@163.com`

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `qifeng@hpu.edu.cn, fengqi618@member.ams.org, qifeng618@hotmail.com, qifeng618@msn.com, 316020821@qq.com`

URL: `http://rgmia.vu.edu.au/qi.html`