ON OSTROWSKI TYPE INEQUALITIES FOR STIELTJES INTEGRALS WITH ABSOLUTELY CONTINUOUS INTEGRANDS AND INTEGRATORS OF BOUNDED VARIATION

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ABSTRACT. Some Ostrowski type inequalities are given for the Stieltjes integral where the integrand is absolutely continuous while the integrator is of bounded variation. The case when $|f'|$ is convex is explored. Applications for the midpoint rule and a generalised trapezoid type rule are also presented.

1. INTRODUCTION

The following result is known in the literature as Ostrowski's inequality:

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)|\le M$ for all $t\in(a,b)$. Then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M \tag{1.1}$$

for all $x\in(a,b)$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

The above result has been naturally extended for absolutely continuous functions and Lebesgue $p-$norms of the derivative f' in [11] – [13] and can be stated as:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{1.2}$$

$$\le\begin{cases}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f'\|_\infty & \text{if } f'\in L_\infty[a,b];\\[2ex] \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{x-a}{b-a}\right)^{p+1}+\left(\frac{b-x}{b-a}\right)^{p+1}\right](b-a)^{\frac{1}{q}}\|f'\|_q & \\ \qquad\text{if } f'\in L_p[a,b],\ \frac{1}{p}+\frac{1}{q}=1,\ p>1; & \\[2ex] \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\|f'\|_1, & \end{cases}$$

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where $\|\cdot\|_r$ $(r \in [1,\infty])$ *are the usual Lebesgue norms on* $L_r[a,b]$, *i.e.,*

$$\|g\|_\infty := ess \sup_{t\in[a,b]} |g(t)| \quad and \quad \|g\|_r := \left(\int_a^b |g(t)|^r dt\right)^{\frac{1}{r}}, \quad r \in [1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{1/p}}$ *and* $\frac{1}{2}$ *respectively are sharp in the sense mentioned above.*

They can also be obtained, in a slightly different form, as particular cases of some results established by A.M. Fink in [14] for $n-$time differentiable functions.

For other Ostrowski type inequalities concerning Lipschitzian and $r-H-$Hölder type functions, see [8] and [10].

The cases of bounded variation functions and monotonic functions were considered in [4] and [7] while the case of convex functions was studied in [3].

In an effort to obtain an Ostrowski type inequality for the Stieltjes integral, which obviously contains the weighted integrals case, S.S. Dragomir established in [5] the following result:

Theorem 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a function of bounded variation and* $u:[a,b]\to\mathbb{R}$ *a function of* $r-H-$*Hölder type, i.e.,*

$$|u(x)-u(y)| \le H|x-y|^r \quad \text{for any } x,y\in[a,b], \tag{1.3}$$

where $r\in(0,1]$ *and* $H>0$ *are given. Then, for any* $x\in[a,b]$,

$$\begin{aligned} &\left|[u(b)-u(x)]f(x) - \int_a^b f(t)\,du(t)\right| \\ &\le H\left[(x-a)^r \bigvee_a^x(f) + (b-x)^r \bigvee_x^b(f)\right] \\ &\le H\times\begin{cases} [(x-a)^r+(b-x)^r]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]; \\ [(x-a)^{qr}+(b-x)^{qr}]^{\frac{1}{q}}\left[\left(\bigvee_a^x(f)\right)^p+\left(\bigvee_x^b(f)\right)^p\right]^{\frac{1}{p}} \\ \qquad\qquad \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(f), \end{cases} \end{aligned} \tag{1.4}$$

where $\bigvee_c^d(f)$ *denotes the total variation of* f *on the interval* $[c,d]$.

The dual case was considered in [6] and can be stated as follows:

Theorem 3. *Let* $u:[a,b]\to\mathbb{R}$ *be a function of bounded variation on* $[a,b]$ *and* $f:[a,b]\to\mathbb{R}$ *a function of* $r-H-$*Hölder type. Then*

$$\begin{aligned} &\left|[u(b)-u(a)]f(x) - \int_a^b f(t)\,du(t)\right| \\ &\qquad\qquad \le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r \bigvee_a^b(u) \end{aligned} \tag{1.5}$$

for any $x\in[a,b]$.

For other results concerning inequalities for Stieltjes integrals, see [1], [15] and [16].

The aim of the present paper is to continue the study of Ostrowski type inequalities for Stieltjes integrals $\int_a^b f(t)\,du(t)$ where the function f, the *integrand*, is assumed to be absolutely continuous while the *integrator* u, is of bounded variation. Applications to the midpoint rule and for a generalised trapezoid rule are also pointed out.

2. General Bounds for Absolutely Continuous Functions

The following representation result is of interest:

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$ and $u : [a,b] \to \mathbb{R}$ such that the Stieltjes integrals*

$$\int_a^b f(t)\,du(t) \qquad \text{and} \qquad \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t)$$

exist for each $x \in [a,b]$. Then

$$f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t) = \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t) \tag{2.1}$$

or, equivalently,

$$\int_a^b u(t)\,df(t) - u(b)[f(b)-f(x)] - u(a)[f(x)-f(a)] = \int_a^b (x-t)\left(\int_0^1 f'[\lambda t + (1-\lambda)x]\,d\lambda\right) du(t) \tag{2.2}$$

for each $x \in [a,b]$.

Proof. Since f is absolutely continuous on $[a,b]$, hence, for any $x,t \in [a,b]$ with $x \neq t$, one has

$$\frac{f(x)-f(t)}{x-t} = \frac{\int_t^x f'(u)\,du}{x-t} = \int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda$$

giving the equality (see also [9]):

$$f(x) = f(t) + (x-t)\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda \tag{2.3}$$

for any $x,t \in [a,b]$.

Integrating the identity (2.3) we deduce

$$f(x)\int_a^b du(t) = \int_a^b f(t)\,du(t) + \int_a^b (x-t)\left(\int_0^1 f'[(1-\lambda)x + \lambda t]\,d\lambda\right) du(t),$$

which is exactly the desired inequality (2.1).

Now, on utilising the integration by parts formula for the Stieltjes integral, we have

$$\begin{aligned} & f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t) \\ & = f(x)[u(b)-u(a)]-\left[f(b)u(b)-f(a)u(a)-\int_a^b u(t)\,df(t)\right] \\ & = \int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)] \end{aligned}$$

and the representation (2.2) is also obtained. □

For an absolutely continuous function $f:[a,b]\to\mathbb{R}$, let us denote by $\mu(f;x,t):=\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right|$, where $(t,x)\in[a,b]^2$. It is obvious that, by the Hölder inequality, we have

$$\mu(f;x,t)\le\begin{cases}\|f'\|_{[t,x],\infty} & \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[t,x],p} & \text{if } f'\in L_p[a,b],\ p\ge 1,\end{cases} \tag{2.4}$$

where

$$\|f'\|_{[t,x],\infty}:=\sup_{\substack{u\in[t,x] \\ (u\in[x,t])}}|f'(u)|,$$

$$\|f'\|_{[t,x],p}:=\left|\int_t^x |f'(u)|^p\,du\right|^{\frac{1}{p}},\quad p\ge 1$$

and $t,x\in[a,b]$.

We can also state the following result of Ostrowski type for the Stieltjes integral:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $u:[a,b]\to\mathbb{R}$ a function of bounded variation on $[a,b]$. Then*

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right|\le M(x), \tag{2.5}$$

and, equivalently

$$\left|\int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)]\right|\le M(x), \tag{2.6}$$

where $M(x)=M_1(x)+M_2(x)$ and

$$M_1(x):=\bigvee_a^x(u)\sup_{t\in[a,x]}[(x-t)\mu(f;x,t)],$$

$$M_2(x):=\bigvee_x^b(u)\sup_{t\in[x,b]}[(t-x)\mu(f;x,t)],$$

for $x\in[a,b]$.

Remark 1. *Using the notations in Theorem 4, we have*

$$M_1(x) \le (x-a)\bigvee_a^x(u)\sup_{t\in[a,x]}\mu(f;x,t)$$
$$\le (x-a)\bigvee_a^x(u)\cdot\begin{cases}\|f'\|_{[a,x],\infty} & \text{if } f'\in L_\infty[a,b]\,;\\[2ex] \|f'\|_{[a,x],p} & \text{if } f'\in L_p[a,b]\,,\ p\ge 1,\end{cases}$$

$$M_2(x) \le (b-x)\bigvee_x^b(u)\sup_{t\in[x,b]}\mu(f;x,t)$$
$$\le (b-x)\bigvee_x^b(u)\cdot\begin{cases}\|f'\|_{[x,b],\infty} & \text{if } f'\in L_\infty[a,b]\,;\\[2ex] \|f'\|_{[x,b],p} & \text{if } f'\in L_p[a,b]\,,\ p\ge 1,\end{cases}$$

for any $x\in[a,b]$.

Proof. We use the fact that, if $p,v:[c,d]\to\mathbb{R}$ are such that p is continuous and v is of bounded variation, then the Stieltjes integral $\int_c^d p(t)\,dv(t)$ exists and

$$\left|\int_c^d p(x)\,dv(x)\right| \le \sup_{x\in[c,d]}|p(x)|\bigvee_c^d(v)\,.$$

Utilising the representation (2.1) we have

$$\begin{aligned}
&\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|\\
&=\left|\int_a^x (x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right.\\
&\qquad\left.+\int_x^b (x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right|\\
&\le\left|\int_a^x (x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right|\\
&\qquad+\left|\int_x^b (x-t)\left(\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right)du(t)\right|\\
&\le\bigvee_a^x(u)\sup_{t\in[a,x]}[(x-t)\mu(f;x,t)]+\bigvee_x^b(u)\sup_{t\in[x,b]}[(t-x)\mu(f;x,t)]\\
&\le M_1(x)+M_2(x)=:M(x)\,.
\end{aligned}$$

The other inequalities for M_1 and M_2 are obvious from the inequality (2.4) and the details are omitted. □

Remark 2. *Hence, if we denote by* $\|f'\|_{[c,d],p}$ *the* p *norm on the interval* $[c,d]$, *where* $1 \le p \le \infty$, *then for* $f' \in L_p[a,b]$, *we have*

$$\left| f(x)[u(b)-u(a)] - \int_a^b f(t)\,du(t) \right| \le (x-a)\bigvee_a^x(u)\,\|f'\|_{[a,x],p} + (b-x)\bigvee_x^b(u)\,\|f'\|_{[x,b],p} =: N(x), \tag{2.7}$$

where $p \in [1,\infty]$ *and* $x \in [a,b]$.

Obviously one can derive many upper bounds for the function $N(x)$ defined above. We intend to present in the following only a few that are simple and perhaps of interest for applications.

Estimate 1:

$$N(x) \le \left[(x-a)\bigvee_a^x(u) + (b-x)\bigvee_x^b(u)\right]\|f'\|_{[a,b],p} \tag{2.8}$$
$$\le \|f'\|_{[a,b],p}\cdot\begin{cases} \max\{x-a,b-x\}\left[\bigvee_a^x(u)+\bigvee_x^b(u)\right]; \\ \left[(x-a)^\alpha+(b-x)^\alpha\right]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{\beta}} \\ \qquad\text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (b-a)\max\left\{\bigvee_a^x(u),\bigvee_x^b(u)\right\} \end{cases}$$
$$= \|f'\|_{[a,b],p}\cdot\begin{cases} \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u); \\ \left[(x-a)^\alpha+(b-x)^\alpha\right]^{\frac{1}{\alpha}}\left[\left(\bigvee_a^x(u)\right)^\beta+\left(\bigvee_x^b(u)\right)^\beta\right]^{\frac{1}{\beta}} \\ \qquad\text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (b-a)\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right] \end{cases}$$

for any $x \in [a,b]$.

Estimate 2:

$$N(x) \le \max\{x-a,b-x\}\left[\bigvee_a^x(u)\,\|f'\|_{[a,x],p} + \bigvee_x^b(u)\,\|f'\|_{[x,b],p}\right]$$
$$= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\bigvee_a^x(u)\,\|f'\|_{[a,x],p} + \bigvee_x^b(u)\,\|f'\|_{[x,b],p}\right]$$

$$
\begin{aligned}
&\leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right] \\
&\quad\times \begin{cases}
\max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} \bigvee_a^b(u); \\
\left[\|f'\|_{[a,x],p}^p+\|f'\|_{[x,b],p}^p\right]^{\frac{1}{p}} \left[\left(\bigvee_a^x(u)\right)^q+\left(\bigvee_x^b(u)\right)^q\right]^{\frac{1}{q}} \\
\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right]
\end{cases} \\
&= \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right] \\
&\quad\times \begin{cases}
\max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\} \bigvee_a^b(u); \\
\|f'\|_{[a,b],p} \left[\left(\bigvee_a^x(u)\right)^q+\left(\bigvee_x^b(u)\right)^q\right]^{\frac{1}{q}} \\
\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right]
\end{cases}
\end{aligned}
$$

for any $x \in [a,b]$.

Estimate 3:

$$
\begin{aligned}
N(x) &\leq \max\left\{\bigvee_a^x(u), \bigvee_x^b(u)\right\}\left[(x-a)\|f'\|_{[a,x],p}+(b-x)\|f'\|_{[x,b],p}\right] \\
&= \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]\left[(x-a)\|f'\|_{[a,x],p}+(b-x)\|f'\|_{[x,b],p}\right] \\
&\leq \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right] \\
&\quad\times \begin{cases}
\max\left\{\|f'\|_{[a,x],p}, \|f'\|_{[x,b],p}\right\}(b-a); \\
\left[(x-a)^q+(b-x)^q\right]^{\frac{1}{q}}\|f'\|_{[a,b],p} \\
\qquad\qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left[\|f'\|_{[a,x],p}+\|f'\|_{[x,b],p}\right]
\end{cases}
\end{aligned}
$$

for each $x \in [a,b]$.

In practical applications, the midpoint rule, that results for $x = \frac{a+b}{2}$, is of obvious interest due to its simpler form.

Corollary 1. *With the assumptions in Theorem 4, we have the inequalities:*

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \tag{2.9}$$
$$\leq \frac{1}{2}(b-a)\left[\bigvee_a^{\frac{a+b}{2}}(u)\,\|f'\|_{[a,\frac{a+b}{2}],p}+\bigvee_{\frac{a+b}{2}}^{b}(u)\,\|f'\|_{[\frac{a+b}{2},b],p}\right]$$
$$\leq \frac{1}{2}(b-a)\begin{cases}\max\left\{\|f'\|_{[a,\frac{a+b}{2}],p},\|f'\|_{[\frac{a+b}{2},b],p}\right\}\bigvee_a^b(u); \\ \left[\|f'\|^{\alpha}_{[a,\frac{a+b}{2}],p}+\|f'\|^{\alpha}_{[\frac{a+b}{2},b],p}\right]^{\frac{1}{\alpha}} \\ \qquad\times\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{\beta}+\left(\bigvee_{\frac{a+b}{2}}^{b}(u)\right)^{\beta}\right]^{\frac{1}{\beta}} \\ \qquad\qquad \text{if } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^{b}(u)\right|\right] \\ \qquad\times\left[\|f'\|_{[a,\frac{a+b}{2}],p}+\|f'\|_{[\frac{a+b}{2},b],p}\right],\end{cases}$$

where $p\in[1,\infty]$.

From the above, it is obvious that we can get some appealing inequalities as follows:

$$\left|[u(b)-u(a)]f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \tag{2.10}$$
$$\leq \frac{1}{2}(b-a)\begin{cases}\|f'\|_{[a,b],\infty}\bigvee_a^b(u), \quad \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}(u)\right)^{q}\right]^{\frac{1}{q}} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p[a,b]; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^{b}(u)\right|\right]\|f'\|_{[a,b],1}.\end{cases}$$

Remark 3. *Similar inequalities can be obtained for the generalised trapezoid rule. We only state here the following simple results:*

$$\left|\int_a^b u(t)\,df(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right|$$
$$\leq \frac{1}{2}(b-a)\begin{cases}\|f'\|_{[a,b],\infty}\bigvee_a^b(u), \quad \text{if } f'\in L_\infty[a,b]; \\ \|f'\|_{[a,b],p}\left[\left(\bigvee_a^{\frac{a+b}{2}}(u)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}(u)\right)^{q}\right]^{\frac{1}{q}} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ f'\in L_p[a,b]; \\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(u)-\bigvee_{\frac{a+b}{2}}^{b}(u)\right|\right]\|f'\|_{[a,b],1}\end{cases}$$

provided that u *is of bounded variation and* f *is absolutely continuous on* $[a,b]$.

3. Bounds in the Case of $|f'|$ a Convex Function

Some of the above results can be improved provided that a convexity assumption for $|f'|$ is in place:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$, $u : [a,b] \to \mathbb{R}$ a function of bounded variation on $[a,b]$ and $x \in [a,b]$. If $|f'|$ is convex on $[a,x]$ and $[x,b]$ (and the intervals can be reduced at a single point), then*

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \tag{3.1}$$
$$\leq \frac{1}{2}\left[\bigvee_a^x(u)\sup_{t\in[a,x]}\{(x-t)|f'(t)|\}+\bigvee_x^b(u)\sup_{t\in[x,b]}\{(t-x)|f'(t)|\}\right]$$
$$+\frac{1}{2}|f'(x)|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\leq \frac{1}{2}\left[(x-a)\bigvee_a^x(u)\|f'\|_{[a,x],\infty}+(b-x)\bigvee_x^b(u)\|f'\|_{[x,b],\infty}\right]$$
$$+\frac{1}{2}|f'(x)|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right],$$

for any $x \in [a,b]$.

Proof. As in the proof of Theorem 4, we have

$$\left|f(x)[u(b)-u(a)]-\int_a^b f(t)\,du(t)\right|$$
$$\leq \sup_{t\in[a,x]}\left[(x-t)\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right|\right]\bigvee_a^x(u)$$
$$+\sup_{t\in[x,b]}\left[(t-x)\left|\int_0^1 f'[\lambda t+(1-\lambda)x]\,d\lambda\right|\right]\bigvee_x^b(u)$$
$$\leq \sup_{t\in[a,x]}\left[(x-t)\int_0^1 |f'[\lambda t+(1-\lambda)x]|\,d\lambda\right]\bigvee_a^x(u)$$
$$+\sup_{t\in[x,b]}\left[(t-x)\int_0^1 |f'[\lambda t+(1-\lambda)x]|\,d\lambda\right]\bigvee_x^b(u)$$
$$\leq \sup_{t\in[a,x]}\left[(x-t)\frac{|f'(t)|+|f'(x)|}{2}\right]\bigvee_a^x(u)$$
$$+\sup_{t\in[x,b]}\left[(t-x)\frac{|f'(t)|+|f'(x)|}{2}\right]\bigvee_x^b(u)$$

$$\leq \frac{1}{2}\left[\sup_{t\in[a,x]}\left\{(x-t)\left|f'(t)\right|\right\}\cdot\bigvee_a^x(u)+\sup_{t\in[x,b]}\left\{(t-x)\left|f'(t)\right|\right\}\cdot\bigvee_x^b(u)\right]$$
$$+\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

which proves the first inequality in (3.1).

The second inequality in (3.1) is obvious using properties of sup and the theorem is completely proved. □

The midpoint inequality is of interest in applications and provides a much simpler inequality:

Corollary 2. *If f and u are as above and $|f'|$ is convex on $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, then*

$$\left|[u(b)-u(a)]\,f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,du(t)\right| \tag{3.2}$$
$$\leq \frac{1}{4}(b-a)\left[\left\|f'\right\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_a^{\frac{a+b}{2}}(u)+\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^b(u)\right]$$
$$+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_a^b(u)$$
$$\leq \frac{1}{4}(b-a)\bigvee_a^b(u)\left[\left\|f'\right\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right].$$

Remark 4. *If we denote, from the second inequality in (3.1),*

$$L_1(x):=\frac{1}{2}\left[(x-a)\left\|f'\right\|_{[a,x],\infty}\bigvee_a^x(u)+(b-x)\left\|f'\right\|_{[x,b],\infty}\bigvee_x^b(u)\right]$$

and

$$L_2(x):=\frac{1}{2}\left|f'(x)\right|\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

for $x\in[a,b]$, then we can point out various upper bounds for the functions L_1 and L_2 on $[a,b]$.

For instance, we have

$$L_1(x)\leq\frac{1}{2}\left\|f'\right\|_{[a,b],\infty}\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$

and by (3.1) we can state the following inequality of interest:

$$\left|[u(b)-u(a)]f(x)-\int_a^b f(t)\,du(t)\right| \tag{3.3}$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases}$$

for each $x\in[a,b]$.

Remark 5. *A similar result to (3.3) can be stated for the generalised trapezoid rule, out of which we would like to note the following one that is of particular interest:*

$$\left|\int_a^b u(t)\,df(t)-u(b)[f(b)-f(x)]-u(a)[f(x)-f(a)]\right| \tag{3.4}$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\left[(x-a)\bigvee_a^x(u)+(b-x)\bigvee_x^b(u)\right]$$
$$\leq \frac{1}{2}\left[\|f'\|_{[a,b],\infty}+|f'(x)|\right]\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]\bigvee_a^b(u)\\ \left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right](b-a)\end{cases}$$

for each $x\in[a,b]$.

As in Corollary 2, the case $x=\frac{a+b}{2}$ in (3.4) provides the simple result

$$\left|\int_a^b u(t)\,df(t)-u(b)\left[f(b)-f\left(\frac{a+b}{2}\right)\right]-u(a)\left[f\left(\frac{a+b}{2}\right)-f(a)\right]\right| \tag{3.5}$$
$$\leq \frac{1}{4}(b-a)\left[\|f'\|_{\left[a,\frac{a+b}{2}\right],\infty}\bigvee_a^{\frac{a+b}{2}}(u)+\|f'\|_{\left[\frac{a+b}{2},b\right],\infty}\bigvee_{\frac{a+b}{2}}^b(u)\right]$$
$$+\frac{1}{4}(b-a)\left|f'\left(\frac{a+b}{2}\right)\right|\bigvee_a^b(u)$$
$$\leq \frac{1}{4}(b-a)\bigvee_a^b(u)\left[\|f'\|_{[a,b],\infty}+\left|f'\left(\frac{a+b}{2}\right)\right|\right].$$

Remark 6. *Similar inequalities may be stated if one assumes either that* $|f'|$ *is quasi-convex or that* $|f'|$ *is log-convex on* $[a,x]$ *and* $[x,b]$. *The details are left to the interested readers.*

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